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Solutions of Certain Fractional Partial Differential Equations

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Abstract: In this paper we find certain solutions of some fractional partial differential equations. Tensor product of Banachspacesisusedwhereseparationofvariablesdoesnotwork.

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1 Introduction

In [1], a new definition called $\alpha-$conformable fractional derivative was introduced:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$ let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists, then it is called the $\alpha-$**conformable fractional derivative** of f at x.

For $x = 0$, if f is $\alpha-$differentiable on $(0,r)$ for some $r > 0$, and $\lim\limits_{x \to 0^+} D^{\alpha} f(x)$ exists then we define

$$D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(x).$$

One can easily see that **Conformable derivative satisfies:**

1. $D^{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, for all $a, b \in \mathbb{R}$.

2. $D^{\alpha}.(\lambda) = 0$, for all constant functions $f(t) = \lambda$. Further, for $\alpha \in (0,1]$ and f, g are $\alpha-$differentiable at a point t, we have:

3. $D^{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.

4. $D^{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$, $g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

1. $D^{\alpha}(t^p) = p\, t^{p-\alpha}$.
2. $D^{\alpha}(\sin \frac{1}{\alpha} t^{\alpha}) = \cos \frac{1}{\alpha} t^{\alpha}$.
3. $D^{\alpha}(\cos \frac{1}{\alpha} t^{\alpha}) = -\sin \frac{1}{\alpha} t^{\alpha}$.
4. $D^{\alpha}(e^{\frac{1}{\alpha} t^{\alpha}}) = e^{\frac{1}{\alpha} t^{\alpha}}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $D^{\frac{1}{2}}(f)(0) = 1$. This is not the case for the known classical fractional derivatives, since $D^1(f)(0)$ does not exist.

For more on fractional calculus and its applications we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Many deferential equations can be transformed to fractional form and can have many applications in many branches of science. The main technique to solve partial differential equations is separation of variables and using Fourier series. So, Fractional Fourier series was introduced in [10]. Such concept proved to be very fruitful in solving fractional partial differential equations. But in many equations, separation of variables is not applicable. In this case theory of tensor product of Banach spaces gives us some types of solutions called atomic solutions.

In this paper we will use separation of variables to solve fractional Laplace type equation, and tensor product to find atomic solution of some fractional partial differential equations.

2 Atomic solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$.
The operator $T : X^* \to Y$, defined by

$$T(x^*) = x^*(x)y$$

is a bounded one rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products.

Atoms are used in theory of best approximation in Banach spaces, [].

It is a known result, [], and we need it in our paper that: If the sum of two atoms is an atom, then either the first components are dependent or the second are dependent. For more on tensor products of Banach spaces we refer to [].

Let us write: $D_x^\alpha u$ to mean the partial $\alpha-$derivative of u with respect to x. Further we write $D_x^{2\alpha} u$ to mean $D_x^\alpha D_x^\alpha u$. Similarly for derivatives with respect to y.

Our main object in this section is to find an atomic solution of the equation

$$\begin{aligned} D_x^\alpha D_x^\alpha u(x,y) + D_y^\beta u(x,y) &= D_x^\alpha D_y^\beta D_y^\beta u(x,y) \quad (1) \\ u(0,0) = 1 \quad , &\quad D_x^\alpha D_y^\beta u(0,0) = 1 \end{aligned}$$

These conditions can be formulated in the form:

$$P(0) = Q(0) = 1 \text{ and } P^{(\alpha)}(0) = Q^{(\beta)}(0) = 1 \quad (*)$$

This is a linear fractional partial differential equation. But separation of variables very difficult to work. Hence we go for atomic solution.

Procedure.
Let $u(x,y) = P(x)Q(y)$ be the suggested form of the atomic solution.

Substitute in (1) to get

$$P^{(2\alpha)} \otimes Q + P \otimes Q^{(\beta)} = P^{(\alpha)} \otimes Q^{(2\beta)} \quad (2)$$

As functions, equation (2) can be written in the form

$$P^{(2\alpha)}(x)Q(y) + P(x)Q^{(\beta)}(y) = P^{(\alpha)}(y)Q^{(2\beta)}(y) \quad (3)$$

In equation (2) we have the case of: the sum of two atoms is an atom. Hence, [18], we have two cases:

Case(i) :
$$P^{(2\alpha)} = P = P^{(\alpha)}.$$

Case(ii) :
$$Q^{(\beta)} = Q = Q^{(2\beta)}$$

Let us discuss **case (i)**. $P^{(2\alpha)} = P$. This is a linear fractional differential equation. Hence using the result in [], we get

$$P(x) = c_1 e^{\frac{x^\alpha}{\alpha}} + c_2 e^{\frac{-x^\alpha}{\alpha}}$$

Using the conditions in $(*)$, we get $c_1 = 1$, and $c_2 = 0$.
Thus

$$P(x) = e^{\frac{x^\alpha}{\alpha}}$$

As for $P = P^{(\alpha)}$, it gives the same solution

$$P(x) = e^{\frac{x^\alpha}{\alpha}}.$$

Similarly $P^{(2\alpha)} = P^{(\alpha)}$, it gives

$$P(x) = e^{\frac{x^\alpha}{\alpha}}.$$

Now, we go back to equation (3).
The equation is true for any value of t.
So substitute $t = 0$ we get

$$Q + Q^{(\beta)} = Q^{(2\beta)} \quad (4)$$

This is a linear fraction differential equation. Hence using the result in [14], we get

$$Q(y) = c_1 e^{r_1 \frac{y^\beta}{\beta}} + c_2 e^{r_2 \frac{y^\beta}{\beta}}$$

where $r_1 = \frac{1+\sqrt{5}}{2}$, and $r_2 = \frac{1-\sqrt{5}}{2}$.

Using the conditions in $(*)$, we get $c_1 = \frac{-\sqrt{5}}{2-\sqrt{5}}$, and $c_2 = \frac{2}{2-\sqrt{5}}$

So the atomic solution is

$$u(x,y) = e^{\frac{x^\alpha}{\alpha}}(c_1 e^{r_1 \frac{y^\beta}{\beta}} + c_2 e^{r_2 \frac{y^\beta}{\beta}})$$

The second case is discussed similarly to get

$$u(x,y) = (c_1 e^{r_1 \frac{x^\beta}{\beta}} + c_2 e^{r_2 \frac{x^\beta}{\beta}}) e^{\frac{y^\alpha}{\alpha}}.$$

3 Complete Solution of Some Equation

Consider the fractional partial differential equation

$$D_x^\alpha D_x^\alpha u(x,y) + D_y^\beta D_y^\beta u(x,y) = u(x,y) \quad (1)$$

with conditions

$$u(0,y) = u(1,y) = 0 \quad (**)$$

and

$$u(x,0) = 0, D_y^\beta (x,0) = f(x) \quad (***)$$

This is a linear fractional partial differential equations. Hence we can use fractional Fourier series and separation of variables to solve equation (1).

Procedure.
Let $u(x,y) = P(x)Q(y)$. Substitute in the differential equation (1) to get

$$P^{2\alpha}(x)Q(y) + P(x)Q^{2\beta}(y) = P(x)Q(y) \quad (2)$$

Simplifying (2) we get

$$\frac{P^{2\alpha}(x)}{P(x)} = \frac{Q(y) - Q^{2\beta}(y)}{Q(y)} \quad (3)$$

Since x and y are independent, we get

$$\frac{P^{2\alpha}(x)}{P(x)} = \frac{Q(y) - Q^{2\beta}(y)}{Q(y)} = \lambda$$

Hence, we have

$$P^{2\alpha}(x) - \lambda P(x) = 0 \quad (4)$$

and

$$Q^{2\beta}(y) - (1-\lambda)Q(y) = 0 \quad (5)$$

We deal with equation (4) first to determine the values of λ that gives non trivial solution for the problem.

There are three cases to be considered.

1. $\lambda = 0$.
So $P^{2\alpha} = 0$. Hence, $P(x) = c_1 \frac{x^\alpha}{\alpha} + c_2$.
Using conditions in $(**)$, we get $c_1 = c_2 = 0$.
Thus $\lambda = 0$ gives the trivial solution.

2. $\lambda > 0$.
So $\lambda = k^2$. So equation (4) becomes

$$P^{2\alpha}(x) - k^2 P(x) = 0.$$

Again using the result in [], we get

$$P(x) = c_1 e^{k\frac{x^\alpha}{\alpha}} + c_2 e^{-k\frac{x^\alpha}{\alpha}}$$

Using the conditions in $(**)$, we get:
$c_1 + c_2 = 0$, and $c_1 e^{k\frac{1}{\alpha}} + c_2 e^{-k\frac{1}{\alpha}} = 0$.
This gives $c_1 = c_2 = 0$.
Hence, $\lambda > 0$, gives the trivial solution.

3. $\lambda < 0$.
So $\lambda = -k^2$. Hence equation (4) becomes

$$P^{2\alpha}(x) + k^2 P(x) = 0.$$

Another use of the result in [], we get

$$P(x) = c_1 \cos k\frac{x^\alpha}{\alpha} + c_2 \sin k\frac{x^\alpha}{\alpha}$$

Using the conditions in $(**)$ we get $c_1 = 0$ and $c_2 \sin \frac{k}{\alpha} = 0$.
Thus $\frac{k}{\alpha} = n\pi$. Or $k = \alpha n\pi$.

Consequently, the value of λ that gives non trivial solution is

$$\lambda = -\alpha n\pi$$

Hence

$$P(x) = c_n \sin \alpha n\pi \frac{x^\alpha}{\alpha} = c_n \sin n\pi x^\alpha \quad (6)$$

Now, we go to equation (5). This has the form

$$Q^{2\beta}(y) - \left(1 + k^2\right)Q(y) = 0$$

Thus, using the result in [], we get

$$Q\left(y\right)=c_1e^{\sqrt{1+k^2}\frac{y^\beta}{\beta}}+c_2e^{-\sqrt{1+k^2}\frac{y^\beta}{\beta}}$$

Condition $u\left(x,0\right)=0$ implies that $c_1=-c_2$. Hence

$$Q\left(y\right)=a_n\sinh\sqrt{1+\alpha n\pi}\frac{y^\beta}{\beta}\quad (7)$$

From equations (6) and (7) our solution can be written in the form

$$u\left(x,y\right)=\sum_{n=1}^{\infty}b_n\sin n\pi x^\alpha\sinh\sqrt{1+\left(n\pi\alpha\right)^2}\frac{y^\beta}{\beta}\quad (8)$$

Remains to find the coefficients b_n.

Using the condition in $(***)$, we get

$$f\left(x\right)=\sum_{n=1}^{\infty}b_n\sqrt{1+\left(n\pi\alpha\right)^2}\sin n\pi x^\alpha$$

Hence using results in [10], we get the coefficients $b_n\sqrt{1+\left(n\pi\alpha\right)^2}$ to the fractional sin Fourier coefficients of f.

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