A Note on Interior Bases of Semigroups

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Abstract: - The concept of two-sided bases of a semigroup S was introduced by Fabrici in 2009. In this paper, we introduce the concept of interior bases of a semigroup S which is based on the result of interior ideals generated by a nonempty subset of the semigroup S. Then, we study some results of a semigroup S containing interior bases and characterize when a nonempty subset of a semigroup S is an interior base of S.

Key-Words: - semigroup, interior ideal, interior base, quasi-order

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1 Introduction and Preliminaries

The concepts of left bases and right bases of a semigroup were introduced by Tamura [1]. Later, Fabrici [2] introduced and studied the structure of a semigroup containing one-sided bases, namely, two-sided bases.

Definition 1.1. [2] *A subset A of a semigroup S is called a two-sided base of S if it satisfies the following conditions:*

(i) $S = A \cup SA \cup AS \cup SAS$;

(ii) if B is a subset of A such that $S = B \cup SB \cup BS \cup SBS$, then $B = A$.

When a nonempty subset of a semigroup S is a two-sided base of S, the author provided a characterization. Afterward, Fabrici and Kepka proved in [3] that there is a relationship between a semigroup's bases and maximal ideals.

Changphas and Sammaprab [4] have extended the conclusions obtained by Fabrici [2] to ordered semigroups. Next, based on the result of bi-ideals generated by a nonempty subset of a semigroup S called bi-bases, was introduced by Kummoon and Changphas [5]. In 2021, Jantanan, Budpan and Loesna [6] studied some properties of ordered Γ-semihypergroups containing two-sided bases. Recently, the concept of bi-bases was introduced and discussed in semihypergroups and ordered Γ-semihypergroups by [7] and [8], respectively.

In 1976, Lajos [9] gave the concept of interior ideals of semigroups.

Definition 1.2. [9] *A subsemigroup I of a semigroup S is called an interior ideal of S if $SIS \subseteq I$.*

This concept generalizes the concept of two-sided ideals in semigroups. Then, the concept of interior ideals has been studied in other algebraic structures, for example, [10, 11, 12, 13, 14, 15].

Let $\{I_i \mid i \in \Lambda\}$ be a family of interior ideals of a semigroup S. It is known that if $\bigcap_{i\in\Lambda} I_i \neq \emptyset$, then $\bigcap_{i\in\Lambda} I_i$ is also an interior ideal of S. Furthermore, for any nonempty subset A of S, we denote $(A)_I$ as the smallest interior ideal of S containing A. For any $a \in S$, we denote $(a)_I = (\{a\})_I$. The form of $(A)_I$ is shown in the following lemma.

Lemma 1.3. *If A is a nonempty subset of a semigroup S, then $(A)_I = A \cup AA \cup SAS$.*

Proof. Put $N = A \cup AA \cup SAS$. Obviously, $A \subseteq N$. Next, we consider

$$\begin{aligned} NN &= (A \cup AA \cup SAS)(A \cup AA \cup SAS) \\ &\subseteq AA \cup SAS \\ &\subseteq A \cup AA \cup SAS = N \end{aligned}$$

and

$$\begin{aligned} SNS &= S(A \cup AA \cup SAS)S \\ &\subseteq SAS \subseteq A \cup AA \cup SAS = N. \end{aligned}$$

Hence, N is an interior ideal of S. Let K be any interior ideal of S containing A. It follows that $N = A \cup AA \cup SAS \subseteq K \cup KK \cup SKS \subseteq K$. Therefore, N is the smallest interior ideal of S containing A, that is, $(A)_I = N = A \cup AA \cup SAS$. □

In a particular case of Lemma 1.3, if $A = \{a\}$ then we have the following corollary.

Corollary 1.4. *If S is a semigroup and $a \in S$, then $(a)_I = a \cup aa \cup SaS$.*

Since the interior ideal is a kind of ideals that is popularly studied in semigroups, then it is important to investigate based on the result of interior ideals of semigroups. The purpose of this paper to introduce the concept of interior bases in a semigroup S which derive from interior ideals generated by a nonempty subset of the semigroup S. Then, we discuss the structure of a semigroup S containing interior bases.

2 Main Results

In this section, we give the definition of an interior base of a semigroup S and characterize when a nonempty subset of S is an interior base by using the quasi-order on S defined by the principal interior ideals of S.

Definition 2.1. *Let S be a semigroup. A subset A of S is called an interior base of S if it satisfies the following two conditions:*

(i) $S = (A)_I$ (i.e., $S = A \cup AA \cup SAS$);

(ii) if B is a subset of A such that $S = (B)_I$, then $A = B$.

Example 2.2. *Let $S = \{a, b, c, d\}$ be a semigroup with the binary operation $\cdot$ on S defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

By routine calculations, we obtain that $\{c, d\}$ is an interior base of S.

Example 2.3. *Consider $S = \{a, b, c, d\}$ is a semigroup with the multiplication $\cdot$ on S defined by:*

$\cdot$	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	c	d
d	a	b	d	c

Let $A = \{c, d\}$. We have that $S = A \cup AA \cup SAS$, but A is not an interior base of S because there exists a subset $\{c\}$ of A such that $S = \{c\} \cup \{c\}\{c\} \cup S\{c\}S$, and importantly, $\{c\} \neq A$. Moreover, we can show that $\{c\}$ and $\{d\}$ are interior bases of S.

Lemma 2.4. *Let I be an interior base of a semigroup S and $a, b \in I$. If $a \in bb \cup SbS$, then $a = b$.*

Proof. Assume that $a \in bb \cup SbS$, and suppose that $a \neq b$. Put $A := I \setminus \{a\}$. Thus, $A \subset I$. Since $a \neq b$, $b \in A$. Next, we will show that $(A)_I = S$. Clearly, $(A)_I \subseteq S$. Let $x \in S$. Then, by $(I)_I = S$, it follows that $x \in I \cup II \cup SIS$. So, there are three cases to consider:

Case 1: $x \in I$.

Subcase 1.1: $x \neq a$. Then,

$$x \in I \setminus \{a\} = A \subseteq (A)_I.$$

Subcase 1.2: $x = a$. By assumption,

$$x = a \in bb \cup SbS \subseteq AA \cup SAS \subseteq (A)_I.$$

Case 2: $x \in II$. Then, $x = b_1b_2$ for some $b_1, b_2 \in I$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned} x = b_1b_1 &\in (bb \cup SbS)(bb \cup SbS) \\ &= bbbb \cup bbSbS \cup SbSbb \cup SbSSbS \\ &\subseteq AAAA \cup AASAS \cup SASAA \cup SASSAS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. Then, $b_1 \in I \setminus \{a\} = A$. By assumption,

$$\begin{aligned} x = b_1b_1 &\in A(bb \cup SbS) \\ &= Abb \cup ASbS \\ &\subseteq AAA \cup ASAS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. Then, $b_2 \in A$. By assumption,

$$\begin{aligned} x = b_1b_1 &\in (bb \cup SbS)A \\ &= bbA \cup SbSA \\ &\subseteq AAA \cup SASA \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. Then, $b_1, b_2 \in I \setminus \{a\} = A$. It follows that,

$$x = b_1b_1 \in AA \subseteq (A)_I.$$

Case 3: $x \in SIS$. Then, $x = sb_3t$ for some $b_3 \in I$ and $s, t \in S$.

Subcase 3.1: $b_3 \neq a$. Then, $b_3 \in I \setminus \{a\} = A$. It turns out that

$$x = sb_3t \in SAS \subseteq (A)_I.$$

Subcase 3.2: $b_3 = a$. By assumption,

$$\begin{aligned} x = sb_3t &\in s(bb \cup SbS)t \\ &\subseteq SAAS \cup SSASS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

This implies that $(A)_I = S$. This is a contradiction. Therefore, $a = b$. □

Lemma 2.5. *Let I be an interior base of a semigroup S and $a, b, c \in I$. If $a \in bc \cup SbcS$, then $a = b$ or $a = c$.*

Proof. Assume that $a \in bc \cup SbcS$, and suppose that $a \neq b$ and $a \neq c$. Setting $A := I \setminus \{a\}$. Then, $A \subset I$. Since $a \neq b$ and $a \neq c$, we have $b, c \in A$. We will show that $(A)_I = S$. Clearly, $(A)_I \subseteq S$. On the other hand, let $x \in S$, By $(I)_I = S$, $x \in I \cup II \cup SIS$. We consider three cases:
Case 1: $x \in I$.
Subcase 1.1: $x \neq a$. Then,

$$x \in I \setminus \{a\} = A \subseteq (A)_I.$$

Subcase 1.2: $x = a$. By assumption,

$$\begin{aligned} x = a &\in bc \cup SbcS \\ &\subseteq AA \cup SAAS \\ &\subseteq AA \cup SAS \subseteq (A)_I. \end{aligned}$$

Case 2: $x \in II$. Then, $x = b_1b_2$ for some $b_1, b_2 \in I$.
Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned} x = b_1b_2 &\in (bc \cup SbcS)(bc \cup SbcS) \\ &= bcbc \cup bcSbcS \cup SbcSbc \cup SbcSSbcS \\ &\subseteq AAAA \cup AASAAS \cup SAASAA \\ &\quad \cup SAASSAAS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. Then, $b_1 \in I \setminus \{a\} = A$. By assumption,

$$\begin{aligned} x = b_1b_2 &\in A(bc \cup SbcS) \\ &= Abc \cup ASbcS \\ &\subseteq AAA \cup ASAAS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. Then, $b_2 \in I \setminus \{a\} = A$. By assumption,

$$\begin{aligned} x = b_1b_2 &\in (bc \cup SbcS)A \\ &= bcA \cup SbcSA \\ &\subseteq AAA \cup SAASA \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. Then $b_1, b_2 \in I \setminus \{a\} = A$. Thus,

$$x = b_1b_2 \in AA \subseteq (A)_I.$$

Case 3: $x \in SIS$. Then, $x = sb_3t$ for some $b_3 \in I$ and $s, t \in S$.
Subcase 3.1: $b_3 \neq a$. Then, $b_3 \in I \setminus \{a\} = A$. Thus,

$$x = sb_3t \in SAS \subseteq (A)_I.$$

Subcase 3.2: $b_3 = a$. By assumption,

$$\begin{aligned} x = sb_3t &\in s(bb \cup SbS)t \\ &\subseteq SAAS \cup SSASS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

This implies that $(A)_I = S$. This is a contradiction. Therefore, $a = b$ or $a = c$. ☐

Now, we need the quasi-order defined as follows to provide a characterization when a nonempty subset of a semigroup is interior base of the semigroup.

Definition 2.6. *Let S be a semigroup. Define a quasi-order on S by for every $a, b \in S$,*

$$a \leqslant b :\Leftrightarrow (a)_I \subseteq (b)_I.$$

Note that the relation $\leqslant$ defined above need not to be a partial order as shown by the following example.

Example 2.7. *From Example 2.3, we see that $(c)_I \subseteq (d)_I$ (i.e., $c \leqslant d$) and $(d)_I \subseteq (c)_I$ (i.e., $d \leqslant c$), but $c \neq d$. Hence, $\leqslant$ is not a partial order on S.*

Lemma 2.8. *Let I be an interior base of a semigroup S. If $a, b \in I$ such that $a \neq b$, then neither $a \leqslant b$, nor $b \leqslant a$.*

Proof. Assume that $a, b \in I$ such that $a \neq b$. Suppose that $a \leqslant b$. Then, $a \in (a)_I \subseteq (b)_I$. Since $a \neq b$, we have $a \in bb \cup SbS$. By Lemma 2.4, $a = b$. This is a contradiction. For the case $b \leqslant a$, we can be proved similarly. ☐

Lemma 2.9. *Let I be a interior base of a semigroup S. For every $a, b, c \in I$ and every $s, t \in S$, then the following statements hold:*

(i) if $a \in bc \cup bcbc \cup SbcS$, then $a = b$ or $a = c$;

(ii) if $a \in sbt \cup sbtsbt \cup SsbtS$, then $a = b$.

Proof. (i) Assume that $a \in bc \cup bcbc \cup SbcS$. Suppose that $a \neq b$ and $a \neq c$. Let $A := I \setminus \{a\}$. Then, $A \subset I$. Since $a \neq b$ and $a \neq c$, we have that $b, c \in A$. We will show that $(I)_I \subseteq (A)_I$. It is sufficient to prove that $I \subseteq (A)_I$. Let $x \in I$. If $x \neq a$, then $x \in A$, and so $x \in (A)_I$. If $x = a$, then by given assumption, we have

$$\begin{aligned} x = a &\in bc \cup bcbc \cup SbcS \\ &\subseteq AA \cup AAAA \cup SAAS \\ &\subseteq AA \cup SAS \subseteq (A)_I. \end{aligned}$$

Hence, $I \subseteq (A)_I$. This implies that $(I)_I \subseteq (A)_I$. Since I is an interior base of S, $S = (I)_I \subseteq (A)_I \subseteq S$. Thus, $S = (A)_I$, which is a contradiction. Therefore, $a = b$ or $a = c$.

(ii) Assume that $a \in sbt \cup sbtsbt \cup SsbtS$, and suppose that $a \neq b$. Letting $A := I \setminus \{a\}$. Then, $A \subset I$. Since $a \neq b$, $b \in A$. We need to show that $I \subseteq (A)_I$. Let $x \in I$. If $x \neq a$, then $x \in A$, and so $x \in (A)_I$. If $x = a$, then by given assumption, we have

$$\begin{aligned} x = a &\in sbt \cup sbtsbt \cup SsbtS \\ &\subseteq SAS \cup SASSAS \cup SSASS \\ &\subseteq SAS \subseteq (A)_I. \end{aligned}$$

Thus, $I \subseteq (A)_I$, implies that $(I)_I \subseteq (A)_I$. Since I is an interior base of S, $S = (I)_I \subseteq (A)_I \subseteq S$. Hence, $S = (A)_I$. This is a contradiction. Therefore, $a = b$. □

Lemma 2.10. *Let I be an interior base of a semigroup S. Then the following statements hold:*

(i) for every $a, b, c \in I$, if $a \neq b$ and $a \neq c$, then $a \nleqslant bc$;

(ii) for every $a, b \in I$ and every $s, t \in S$, if $a \neq b$, then $a \nleqslant sbt$.

Proof. (i) Let $a, b, c \in I$ such that $a \neq b$ and $a \neq c$. Suppose that $a \leqslant bc$. Then,

$$a \in (a)_I \subseteq (bc)_I = bc \cup bcbc \cup SbcS.$$

By Lemma 2.9(i), we have that $a = b$ or $a = c$. This is a contradiction to the assumption. It follows that $a \nleqslant bc$.

(ii) Let $a, b \in I$ and $s, t \in S$ such that $a \neq b$. Suppose that $a \leqslant sbt$. We obtain that

$$a \in (a)_I \subseteq (sbt)_I = sbt \cup sbtsbt \cup SsbtS.$$

By Lemma 2.9(ii), we get that $a = b$, which is a contradiction to the assumption. Therefore, $a \nleqslant sbt$. □

Finally, we present the main result of this paper by characterizing when a nonempty subset of a semigroup S is an interior base of S.

Theorem 2.11. *Let I be a nonempty subset of a semigroup S. Then I is an interior base of S if and only if I satisfies the following conditions:*

(i) for every $x \in S$,

(i.a) there exists $a \in I$ such that $x \leqslant a$; or

(i.b) there exist $a_1, a_2 \in I$ such that $x \leqslant a_1a_2$; or

(i.c) there exists $a_3 \in I$ and there exist $s, t \in S$ such that $x \leqslant sa_3t$;

(ii) for every $a, b, c \in I$, if $a \neq b$ and $a \neq c$, then $a \nleqslant bc$;

(iii) for every $a, b \in S$ and every $s, t \in S$, if $a \neq b$, then $a \nleqslant sbt$.

Proof. Assume that I is an interior base of S. Then, $S = (I)_I$. Next, we need to show that (i) holds. Let $x \in S$. Thus, $x \in (I)_I = I \cup II \cup SIS$. Now, we will consider three cases:
Case 1: $x \in I$. Then, there exists $a \in I$ such that $x = a$. This means that $(x)_I = (a)_I$, that is, $x \leqslant a$.
Case 2: $x \in II$. Then, $x = a_1a_2$ for some $a_1, a_2 \in I$. This implies that $(x)_I = (a_1a_2)_I$. Hence, $x \leqslant a_1a_2$.
Case 3: $x \in SIS$. Then, $x = sa_3t$ for some $a_3 \in I$ and $s, t \in S$. It turns out that $(x)_I = (sa_3t)_I$. So, $x \leqslant sa_3t$.
The conditions of (ii) and (iii) hold from Lemma 2.10(i) and Lemma 2.10(ii), respectively.

Conversely, assume that the conditions (i), (ii) and (iii) hold. We will show that I is an interior base of S. Obviously, $(I)_I \subseteq S$. By (i), we have that for every $x \in S$, there exists $a \in I$ such that $x \leqslant a$. This implies that $x \in (x)_I \subseteq (a)_I = a \cup aa \cup SaS \subseteq I \cup II \cup SIS = (I)_I$. Hence, $S \subseteq (I)_I$. It follows that $S = (I)_I$. Next, we show that I is a minimal subset of S such that $S = (I)_I$. Suppose that $S = (A)_I$ for some $A \subset I$. Then, there exists $x \in I \setminus A$. Since $x \in I \subseteq S = (A)_I$ and $x \notin A$, we have that $x \in AA \cup SAS$, Thus, we will consider two cases:
Case 1: $x \in AA$. Then, there exist $a_1, a_2 \in A$ such that $x = a_1a_2$. We obtain that $a_1, a_1 \in I$. Since $x \notin A$, $x \neq a_1$ and $x \neq a_2$. Since $x = a_1a_2$, $(x)_I \subseteq (a_1a_1)_I$. This implies that $x \leqslant a_1a_2$. This contradicts to the condition (ii).
Case 2: $x \in SAS$, Then, $x = sa_3t$ for some $a_3 \in A$ and $s, t \in S$. Thus, $a_3 \in I$ because $A \subset I$. Since $x \notin A$, $x \neq a_3$. Since $x = sa_3t$, $(x)_I \subseteq (sa_3t)_I$. This means that $x \leqslant sa_3t$, which is a contradiction to the condition (iii).
Therefore, I is an interior base of S, This completes the proof. □

In Example 2.2, we have that $\{c, d\}$ is an interior base of S, but it is not a subsemigroup of S. Hence, we also need find a requirement for an interior base to be a subsemigroup.

Theorem 2.12. *Let I be an interior base of a semigroup S. Then I is a subsemigroup of S if and only if I satisfies the following conditions: for any $b, c \in I$, $bc = b$ or $bc = c$.*

Proof. Assume that I is a subsemigroup of S. Let $b, c \in I$. Then, $bc \in I$. So, $bc \in bc \cup SbcS$. By Lemma 2.5, it follows that $bc = b$ or $bc = c$. The opposite direction is clear. □

3 Conclusion

This paper deals with the concept of interior bases of a semigroup S which obtained from interior ideals generated by a nonempty subset of the semigroup S. Then, we investigated the quasi-order defined by the principal interior ideals of a semigroup S. Finally, we discussed a characterization of interior bases when a nonempty subset of S is an interior ideal of S (Theorem 2.11). Moreover, we have found the condition necessary for an interior base to be a subsemigroup (Theorem 2.12). It is known that every semigroup can be considered to be a semihypergroup. In our future study, we will extend the concept of interior bases to investigate in the structure of semihypergroups.

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