Modified Sumudu Decomposition Method for Solving Lane-Emden-Fowler Type Systems

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Abstract: In this paper, the Sumudu decomposition method (SDM) is modified and applied to solve systems of singular equations of the Lane-Emden-Fowler type. The proposed method is based on the application of Sumudu transform and Adomian decomposition method. Some illustrative examples are given to demonstrate the efficiency of the proposed technique. The results show that the modified method is simple and effective.

Key-Words: Lane-Emden type, Emden-Fowler type, Sumudu transform, Adomian decomposition method

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1 Introduction

Many problems in the fields of mathematical physics and astrophysics are modeled by the linear and nonlinear singular differential equation of the form [1]:

$$y''(x) + \frac{k}{x}y'(x) + f(x, y(x)) = h(x), \quad x > 0, \quad (1)$$

where $k > 0$ is called the shape factor. Equation (1) is known as the Lane-Emden-Fowler type. The equation of this type has been used to model several phenomena in mathematical physics. For $k = 2, h(x) = 0$ and $f(x, y(x)) = y^m(x)$, equation (1), is the standard Lane-Emden equation of the first kind which is a fundamental equation to study of stellar structure. For $k = 2, h(x) = 0$ and $f(x, y(x)) = p(x)g(y)$, equation (1), become the Emden-Fowler type equation that arises in the modeling of several phenomena, such as fluid mechanics, population evolution and chemically reacting systems.

The singularity behavior that occurs at $x = 0$ is the main difficulty of equation (1). In recent years, many researchers are attempted to develop analytic and approximate methods to solve the linear and nonlinear Lane-Emden-Fowler type equations. Several methods for solving Lane-Emden-Fowler type equations and system of Lane-Emden-Fowler type equations such as Adomian decomposition method (ADM) [2]-[4], the variational iteration method (VIM) [5],[6], the Homotopy analysis method (HAM) [7] and the Homotopy pertubation method (HPM) [8] have been proposed. Although such methods have been successfully applied, there are still some difficulties about nonlinear, especially some difficulties and complexities have appeared for computation of nonlinear term. For example, in Adomian decomposition method, calculating Adomian polynomial to handle the nonlinear terms is difficult.

Integral transform method is most useful technique for solving many different type of differential equations. Sumudu transform was introduced and further applied to several ODEs and as well as PDEs. For example, in [9] Kiliçman et al. applied this transform to solve the system of differential equations.

In this work, we develop a computational method for solving linear and nonlinear systems of Lane-Emden-Fowler type equations. The proposed method is a combination of the two powerful techniques Adomian decomposition method and Sumudu transform method. The Sumudu transform was used to avoid integration of some difficult functions while Adomian polynomials were used to decompose the nonlinear terms of the differential equations.

2 Preliminaries of Sumudu Transform

Sumudu transform is an integral transform, which was first introduced by Watugala [10] and applied it to solve differential equations and control engineering problem.

Definition 1 *Consider a set* $\mathbb{A}$ *define as [10]*

$$\mathbb{A} = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j} \right. \\ \left. \textit{if } t \in (-1)^j \times [0, \infty) \right\}.$$

For all real $t \geq 0$*, the Sumudu transform of function*

$f(t) \in \mathbb{A}$, denoted by $\mathbb{S}[f(t)] = F(u)$, is defined as

$$F(u) = \mathbb{S}[f(t)] := \int_0^\infty e^{-t} f(ut)dt, \quad u \in (-\tau_1, \tau_2). \tag{2}$$

Function $f(t)$ in (2) is called inverse Sumudu transform of $F(u)$ and is denoted by $f(t) = \mathbb{S}^{-1}[F(u)]$.

Some properties of the Sumudu transform are as follow:

1. $\mathbb{S}[1] = 1$
2. $\mathbb{S}[t] = u$
3. $\mathbb{S}[t^n] = \dfrac{u^n}{n!}; \quad n = 1, 2, \ldots$

Theorem 1 *Let f be in $\mathbb{A}$, $n \geq 1$ and $F(u)$ be the Sumudu transform of function $f(t)$. The Sumudu transform of the n^{th} derivative of $f(t)$ is given by [11]*

$$\mathbb{S}[f^{(n)}(t)](u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}$$

For further detail and properties about Sumudu transform can found in [11].

3 Sumudu decomposition method for Lane-Emden-Fowler type systems

In this section, we will present the use of the concepts of Sumudu decomposition method.

To demonstrate the key ideas of the method, we consider the following system of Lane-Emden-Fowler type

$$\begin{aligned} y_i''(x) + \frac{\alpha_i}{x} y_i'(x) =& f_i(x, y_1(x), y_2(x), \ldots, y_m(x)) \\ &+ h_i(x), \ i = 1, 2, \ldots, m \end{aligned} \tag{3}$$

subject to the initial conditions

$$y_i(0) = \beta_i, \ y_i'(0) = 0, \tag{4}$$

where $\alpha_i > 0$ and β_i are constants.

In order to solve this system by using modified Sumudu decomposition method, we first apply the Sumudu transform on both sides of (3), we get

$$\begin{aligned} \mathbb{S}[y_i''(x) + \frac{\alpha_i}{x} y_i'(x)] =& \mathbb{S}[f_i(x, y_1(x), y_2(x), \ldots, y_m(x))] \\ &+ \mathbb{S}[h_i(x)]. \end{aligned}$$

Using the properties of the Sumudu transform, we get

$$\begin{aligned} \frac{\mathbb{S}[y_i(x)]}{u^2} - \frac{y_i(0)}{u^2} - \frac{y_i'(0)}{u} =& \mathbb{S}[F_i] + \mathbb{S}[h_i(x)] \\ &- \mathbb{S}[\frac{\alpha_i}{x} y_i'(x)] \end{aligned} \tag{5}$$

where $F_i = f_i(x, y_1(x), y_2(x), \ldots, y_m(x))$.

Simplifying (5), we obtain

$$\begin{aligned} \mathbb{S}[y_i(x)] =& y_i(0) + u y_i'(0) + u^2 \mathbb{S}[F_i] + u^2 \mathbb{S}[h_i(x)] \\ &- u^2 \mathbb{S}[\frac{\alpha_i}{x} y_i'(x)]. \end{aligned} \tag{6}$$

The inverse Sumudu transform of (6) and the initial condition (4) yields

$$\begin{aligned} y_i(x) =& \beta_i + \mathbb{S}^{-1}[u^2 \mathbb{S}[F_i]] + \mathbb{S}^{-1}[u^2 \mathbb{S}[h_i(x)]] \\ &- \mathbb{S}^{-1}[u^2 \mathbb{S}[\frac{\alpha_i}{x} y_i'(x)]]. \end{aligned} \tag{7}$$

We decompose F_i into two parts:

$$F_i = L_i + N_i, \ i = 1, 2, \ldots, m \tag{8}$$

where $L_i = L_i(x, y_1, y_2, \ldots, y_m)$ and $N_i = N_i(y_1, y_2, \ldots, y_m)$ denote the linear and the nonlinear term, respectively.

Applying the Adomian decomposition method, we suppose that the solution $y_i(x)$ can be expressed by an infinite series

$$y_i(x) = \sum_{n=0}^{\infty} y_{in}(x), \ i = 1, 2, \ldots, m \tag{9}$$

and the nonlinear term $N_i(y_1, y_2, \ldots, y_m)$ can be expressed by an infinite series of the Adomian polynomials A_{in} in the following form

$$N_i = \sum_{n=0}^{\infty} A_{in}, \ i = 1, 2, \ldots, m \tag{10}$$

where Adomian polynomials A_{in} can be calculated by the following formula:

$$A_{in} = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N_i \Big(\sum_{n=0}^{\infty} \lambda^n y_{1n}, \ldots, \sum_{n=0}^{\infty} \lambda^n y_{mn} \Big) \right]_{\lambda=0} \tag{11}$$

Substituting (8) and (9) into (7) , we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} y_{in} =& \beta_i + \mathbb{S}^{-1}\left[u^2 \mathbb{S}[h_i(x)]\right] - \mathbb{S}^{-1}\left[u^2 \mathbb{S}[\frac{\alpha_i}{x} y_i'(x)]\right] \\ &+ \mathbb{S}^{-1}\left[u^2 \mathbb{S}\left[L_i + \sum_{n=0}^{\infty} A_{in}\right]\right]. \end{aligned}$$

The recursive relation for SDM is given by

$$\left.\begin{aligned} y_{i0} =& \ \beta_i + \mathbb{S}^{-1}\left[u^2 \mathbb{S}[h_i(x)]\right] = \Phi \\ y_{i,n+1} =& -\mathbb{S}^{-1}[u^2 \mathbb{S}[\frac{\alpha_i}{x} y_{in}'(x)]] \\ &+ \mathbb{S}^{-1}[u^2 \mathbb{S}[L_i(\overline{y}_n) + A_{in}]] \end{aligned}\right\} \tag{12}$$

where $L_i(\overline{y}_n) = L_i\big(y_{1n}, y_{2n}, \ldots, y_{mn}\big)$ for $n = 0, 1, 2, \ldots$.

The modified Sumudu decomposition method introduces a slight variation to the recursive relation (12) that will lead to the determination of the components of y_{in} in a faster and easier way.

We recall the modified Sumudu decomposition method was established based on the assumption that the function

$$\Phi = \sum_{n=0}^{\infty} a_{in}x^n - p\sum_{n=0}^{\infty} a_{in}x^n + \beta_i + \mathbb{S}^{-1}\left[u^2\mathbb{S}[h_i(x)]\right], \tag{13}$$

where p is an artificial parameter and a_{in} are unknown coefficients that can be divide into two parts. We set

$$\Phi = \varphi_1 + \varphi_2, \tag{14}$$

where

$$\varphi_1 = \beta_i + \sum_{n=0}^{\infty} a_{in}x^n,$$
$$\varphi_2 = -p\sum_{n=0}^{\infty} a_{in}x^n + \mathbb{S}^{-1}\left[u^2\mathbb{S}[h_i(x)]\right].$$

Accordingly, a slight variation was proposed only on the components y_{i0} and y_{i1}. The suggestion was that only the part φ_1 be assigned to the zeroth component y_{i0}, whereas the remaining part φ_2 be combined with the other terms given in (12) to define y_{i1}. Consequently, the modified recursive relation:

$$\left.\begin{aligned} y_{i0} =& \beta_i + \sum_{n=0}^{\infty} a_{in}x^n \\ y_{i1} =& -p\sum_{n=0}^{\infty} a_{in}x^n + \mathbb{S}^{-1}\left[u^2\mathbb{S}[h_i(x)]\right] \\ &+ \mathbb{S}^{-1}\left[u^2\mathbb{S}\left[-\frac{\alpha_i}{x}y_{i0}'(x) + L_i(\overline{y}_0) + A_{i0}\right]\right] \\ y_{i,n+1} =& \mathbb{S}^{-1}\left[u^2\mathbb{S}\left[-\frac{\alpha_i}{x}y_{in}'(x) + L_i(\overline{y}_n) + A_{in}\right]\right] \end{aligned}\right\} \tag{15}$$

To find the coefficients a_{in} and b_{in} and to avoid computation of Adomian polynomial A_{in} for $n = 1, 2, \ldots$, we put $y_{i1} = 0$ for $i = 1, 2, \ldots, m$. This implies that $y_{in} = 0$ for $n > 1$. Setting $p = 1$, yields the solution will be obtained as $y_i(x) = y_{i0}$.

4 Applications

In this section, to demonstrate the applicability and validity of the proposed method, we have applied it to linear and nonlinear systems of Lane-Emden-Fowler type equations with singular behavior at $x = 0$ and the result obtained will be compared with the exact solution.

Example 4.1 *First, we consider the following system of linear homogeneous equations of Lane-Emden type with $\alpha_1 = 3$ and $\alpha_2 = 2$ [1]:*

$$\left.\begin{aligned} y''(x) + \frac{3}{x}y'(x) = 4(y(x) + v(x)), \\ v''(x) + \frac{2}{x}v'(x) = -3(y(x) + v(x)), \end{aligned}\right\} \tag{16}$$

with the following initial conditions

$$\left.\begin{aligned} y(0) = 1, \quad y'(0) = 0, \\ v(0) = 1, \quad v'(0) = 0, \end{aligned}\right\} \tag{17}$$

which has the exact solution

$$(y(x), v(x)) = (1 + x^2, 1 - x^2).$$

To solve this problem by the proposed method, we apply the modified Sumudu decomposition method. From (16), we have

$$h_1(x) = h_2(x) = 0,\ L_1 = 4(y + v),$$
$$L_2 = -3(y + v),\ N_1 = N_2 = 0.$$

By using the recursive relation (15) and the initial conditions, we get $a_0 = b_0 = 0$ and $a_1 = b_1 = 0$. Then, we obtain

$$y_0 = 1 + \sum_{n=2}^{\infty} a_n x^n \quad \text{and} \quad v_0 = 1 + \sum_{n=2}^{\infty} b_n x^n.$$

In view of (15), y_1, v_1 are given by

$$\begin{aligned} y_1 =& -p\sum_{n=2}^{\infty} a_n x^n - 3\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}y_0']\right] \\ &+ 4\mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0 + v_0]\right] \\ v_1 =& -p\sum_{n=2}^{\infty} b_n x^n - 2\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}v_0']\right] \\ &- 3\mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0 + v_0]\right] \end{aligned}$$

Now we find a_i, b_i, in such a way that $y_1 = 0$ and $v_1 = 0$. If $y_1 = v_1 = 0$ then the all components $y_i = v_i = 0$ for $i = 1, 2, \ldots$, and the solution will be obtained as $y(x) = y_0$ and $v(x) = v_0$. Therefore

$$\begin{aligned} y_1 =& -p(a_2x^2 + a_3x^3 + \ldots) \\ &- 3\left(a_2x^2 + \frac{a_3}{2}x^3 + \frac{a_4}{3}x^4 + \ldots\right) \\ &+ 4\left(x^2 + \frac{(a_2 + b_2)}{4\cdot 3}x^4 + \frac{(a_3 + b_3)}{5\cdot 4}x^5 + \ldots\right) \\ =& 0 \end{aligned}$$

$$\begin{aligned}v_1 =& -p(b_2x^2+b_3x^3+\ldots)\\ &-2\left(b_2x^2+\frac{b_3}{2}x^3+\frac{b_4}{3}x^4+\ldots\right)\\ &-3\left(x^2+\frac{(a_2+b_2)}{4\cdot 3}x^4+\frac{(a_3+b_3)}{5\cdot 4}x^5+\ldots\right)\\ =&0.\end{aligned}$$

By equating the coefficients of x^n and setting $p=1$, we have

$$\begin{aligned}-a_2-3a_2+4 =&0,\\ -a_3-\frac{3}{2}a_3 =&0,\\ -a_4-a_4+\frac{(a_2+b_2)}{3} =&0,\\ -pa_5-\frac{3}{4}a_5+\frac{(a_3+b_3)}{5} =&0,\\ &\vdots\end{aligned}$$

and

$$\begin{aligned}-b_2-2b_2-3 =&0\\ -b_3-b_3 =&0\\ -b_4-\frac{2}{3}b_4-\frac{(a_2+b_2)}{4} =&0\\ -pb_5-\frac{3}{4}b_5-\frac{3(a_3+b_3)}{5\cdot 4} =&0\\ &\vdots\end{aligned}$$

Solving the above algebraic equations, we have

$$\begin{aligned}&a_2=1,\ a_k=0,\ k=3,4,\ldots\\ &b_2=-1,\ b_k=0,\ k=3,4,\ldots\end{aligned}$$

Thus the solution of (16) is obtained as

$$\begin{aligned}y(x)=y_0 =&1+x^2\\ v(x)=v_0 =&1-x^2\end{aligned}$$

which is the same as the exact solution. □

Example 4.2 *Consider the following system of linear homogeneous equations of Emden-Fowler type with $\alpha_1=2$ and $\alpha_2=2$ [12]:*

$$\left.\begin{aligned}y''(x)+\frac{2}{x}y'(x)=y(x)+(4x^2+5)v(x),\\ v''(x)+\frac{2}{x}v'(x)=-v(x)+(4x^2+7)y(x),\end{aligned}\right\} \tag{18}$$

with the following initial conditions

$$\left.\begin{aligned}y(0)=1,\quad y'(0)=0,\\ v(0)=1,\quad v'(0)=0,\end{aligned}\right\} \tag{19}$$

which has the exact solution

$$(y(x),v(x))=(e^{x^2},e^{x^2}).$$

To solve this problem by the proposed method, we apply the modified Sumudu decomposition method. From (18), we have

$$\begin{aligned}&h_1(x)=h_2(x)=0,\ L_1=y+(4x^2+5)v,\\ &L_2=-v+(4x^2+7)y,\ N_1=N_2=0.\end{aligned}$$

By using the recursive relation (15) and the initial conditions, we get $a_0=b_0=0$ and $a_1=b_1=0$. Then, we obtain

$$y_0=1+\sum_{n=2}^{\infty}a_nx^n \quad \text{and} \quad v_0=1+\sum_{n=2}^{\infty}b_nx^n.$$

In view of (15), y_1, v_1 are given by

$$\begin{aligned}y_1 =& -p\sum_{n=2}^{\infty}a_nx^n-2\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}y_0']\right]\\ &+\mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0+4x^2v_0+5v_0]\right]\\ v_1 =& -p\sum_{n=2}^{\infty}b_nx^n-2\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}v_0']\right]\\ &+\mathbb{S}^{-1}\left[u^2\mathbb{S}[-v_0+7y_0+4x^2y_0]\right]\end{aligned}$$

Now we find a_i, b_i, in such a way that $y_1=0$ and $v_1=0$. If $y_1=v_1=0$ then the all components $y_i=v_i=0$ for $i=1,2,\ldots$, and the solution will be obtained as $y(x)=y_0$ and $v(x)=v_0$. Therefore

$$\begin{aligned}y_1 =& -p(a_2x^2+a_3x^3+\ldots)\\ &-2\left(a_2x^2+\frac{a_3}{2}x^3+\frac{a_4}{3}x^4+\ldots\right)+3x^2\\ &+\frac{(a_2+5b_2+4)}{4\cdot 3}x^4+\frac{(a_3+5b_3)}{5\cdot 4}x^5+\ldots\\ =&0\end{aligned}$$

$$\begin{aligned}v_1 =& -p(b_2x^2+b_3x^3+\ldots)\\ &-2\left(b_2x^2+\frac{b_3}{2}x^3+\frac{b_4}{3}x^4+\ldots\right)+3x^2\\ &+\frac{(7a_2-b_2+4)}{4\cdot 3}x^4+\frac{(7a_3-b_3)}{5\cdot 4}x^5+\ldots\\ =&0.\end{aligned}$$

By equating the coefficients of x^n and setting $p=1$, we have

$$\begin{aligned} -a_2-2a_2+3 =&0,\\ -a_3-a_3 =&0,\\ -a_4-\frac{2}{3}a_4+\frac{(a_2+5b_2+4)}{4\cdot 3} =&0,\\ -a_5-\frac{1}{2}a_5+\frac{(a_3+5b_3)}{5\cdot 4} =&0,\\ -a_6-\frac{2}{5}a_6+\frac{(a_4+5b_4+4b_2)}{6\cdot 5} =&0,\\ &\vdots \end{aligned}$$

and

$$\begin{aligned} -b_2-2b_2+3 =&0\\ -b_3-b_3 =&0\\ -b_4-\frac{2}{3}b_4+\frac{(7a_2-b_2+4)}{4\cdot 3} =&0\\ -b_5-\frac{1}{2}b_5+\frac{(7a_3-b_3)}{5\cdot 4} =&0\\ -b_6-\frac{2}{5}b_6+\frac{(7a_4-b_4+4a_2)}{6\cdot 5} =&0,\\ &\vdots \end{aligned}$$

Solving the above algebraic equations, we have

$$\begin{aligned} &a_2=1,\ a_4=\frac{1}{2},\ a_6=\frac{1}{6},\ \ldots\\ &b_2=1,\ b_4=\frac{1}{2},\ b_6=\frac{1}{6},\ \ldots\\ &a_{2k-1}=b_{2k-1}=0,\ k=1,2,\ldots. \end{aligned}$$

Thus the solution of (18) is obtained as

$$\begin{aligned} y(x)=y_0 =&1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots=e^{x^2}\\ v(x)=v_0 =&1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots=e^{x^2} \end{aligned}$$

which is gives exact solution of the problem. □

Example 4.3 *Consider the following non-homogeneous nonlinear system of Lane-Emden-type with $\alpha_1=\alpha_2=2$ [5]:*

$$\left.\begin{aligned} y''(x)+\frac{2}{x}y'(x) =&y^2(x)-v^2(x)-6v(x)\\ &+6x^2+6,\\ v''(x)+\frac{2}{x}v'(x) =&v^2(x)-y^2(x)+6v(x)\\ &-6x^2+6, \end{aligned}\right\} \quad (20)$$

subject to conditions

$$\left.\begin{aligned} y(0)=1,\quad y'(0)=0,\\ v(0)=-1,\quad v'(0)=0, \end{aligned}\right\} \quad (21)$$

which has the exact solution

$$(y(x),v(x))=(x^2+e^{x^2},x^2-e^{x^2}).$$

We apply the new modified Sumudu decomposition method. From (20), we have

$$\begin{aligned} &h_1(x)=6x^2+6,\quad L_1=-6v,\quad N_1=y^2-v^2,\\ &h_2(x)=-6x^2+6,\quad L_2=6v,\quad N_2=v^2-y^2. \end{aligned}$$

By using the recursive relation (15) and initial condition, we get $a_0=b_0=0$ and $a_1=b_1=0$. Then, we obtain

$$y_0=1+\sum_{n=2}^{\infty}a_nx^n \quad \text{and} \quad v_0=-1+\sum_{n=2}^{\infty}b_nx^n.$$

In view of (15), y_1, v_1 are given by

$$\begin{aligned} y_1=&-p\sum_{n=2}^{\infty}a_nx^n+6\mathbb{S}^{-1}\left[u^2\mathbb{S}[1+x^2]\right]\\ &-2\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}y_0']\right]-6\mathbb{S}^{-1}\left[u^2\mathbb{S}[v_0]\right]\\ &+\mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0^2-v_0^2]\right]\\ v_1=&-p\sum_{n=2}^{\infty}b_nx^n+6\mathbb{S}^{-1}\left[u^2\mathbb{S}[1-x^2]\right]\\ &-2\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}v_0']\right]+6\mathbb{S}^{-1}\left[u^2\mathbb{S}[v_0]\right]\\ &-\mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0^2-v_0^2]\right] \end{aligned}$$

To find a_i and b_i for $i\geq 1$, we put $y_1=v_1=0$. Therefore

$$\begin{aligned} y_1=&-p(a_2x^2+a_3x^3+a_4x^4+\ldots)+3x^2\\ &+\frac{x^4}{2}-2\left(a_2x^2+\frac{a_3}{2}x^3+\frac{a_4}{3}x^4+\ldots\right)\\ &-6\left(\frac{x^2}{2}+\frac{b_2}{4\cdot 3}x^4+\frac{b_3}{5\cdot 4}x^5+\ldots\right)\\ &+\frac{(a_2+b_2)}{3\cdot 2}x^4+\frac{(a_3+b_3)}{5\cdot 2}x^5+\frac{(a_4+b_4)}{5\cdot 3}x^6\\ &+\frac{(a_2^2-b_2^2)}{6\cdot 5}x^6+\ldots\\ =&\,0\\ v_1=&-p(b_2x^2+b_3x^3+b_4x^4+\ldots)+3x^2\\ &-\frac{x^4}{2}-2\left(b_2x^2+\frac{b_3}{2}x^3+\frac{b_4}{3}x^4+\ldots\right)\\ &+6\left(\frac{x^2}{2}+\frac{b_2}{4\cdot 3}x^4+\frac{b_3}{5\cdot 4}x^5+\ldots\right)\\ &-\frac{(a_2+b_2)}{3\cdot 2}x^4-\frac{(a_3+b_3)}{5\cdot 2}x^5-\frac{(a_4+b_4)}{5\cdot 3}x^6\\ &+\frac{(a_2^2-b_2^2)}{6\cdot 5}x^6+\ldots\\ =&\,0. \end{aligned}$$

By equating the coefficients of x^n and setting $p = 1$, we obtain

$$\begin{aligned} -a_2 - 2a_2 + 6 =&0, \\ -a_3 - a_3 =&0, \\ -a_4 - \frac{2}{3}a_4 - \frac{1}{2}b_2 + \frac{(a_2+b_2)}{6} + \frac{1}{2} =&0, \\ -a_5 - \frac{1}{2}a_5 - \frac{6}{5\cdot 4}b_3 + \frac{(a_3+b_3)}{5\cdot 2} =&0, \\ -a_6 - \frac{2}{5}a_6 - \frac{1}{5}b_4 + \frac{(a_4+b_4)}{5\cdot 3} + \frac{(a_2^2-b_2^2)}{6\cdot 5} =&0, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} -b_2 - 2b_2 =&0 \\ -b_3 - b_3 =&0 \\ -b_4 - \frac{2}{3}b_4 - \frac{1}{2}b_2 - \frac{(a_2+b_2)}{6} - \frac{1}{2} =&0, \\ -b_5 - \frac{1}{2}b_5 - \frac{6}{5\cdot 4}b_3 - \frac{(a_3+b_3)}{5\cdot 2} =&0, \\ -b_6 - \frac{2}{5}b_6 - \frac{1}{5}b_4 - \frac{(a_4+b_4)}{5\cdot 3} - \frac{(a_2^2-b_2^2)}{6\cdot 5} =&0, \\ &\vdots \end{aligned}$$

Solving the above algebraic equations , we have

$$\begin{aligned} &a_2 = 2,\ a_4 = \frac{1}{2},\ a_6 = \frac{1}{6},\ \ldots \\ &b_2 = 0,\ b_4 = -\frac{1}{2},\ b_6 = -\frac{1}{6},\ \ldots \\ &a_{2k-1} = b_{2k-1} = 0,\ k = 1,2,\ldots. \end{aligned}$$

Thus the solution of (20) is obtained as

$$\begin{aligned} y(x) = u_0 =& 1 + 2x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots, \\ v(x) = v_0 =& -1 - \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots, \end{aligned}$$

which converges to the exact solution

$$(y(x), v(x)) = (x^2 + e^{x^2}, x^2 - e^{x^2})$$

and is the same as the solutions obtain by A.M.Wazwaz using VIM [5]. □

Example 4.4 *We consider the following non-homogeneous nonlinear system of Lane-Emden-Fowler type equations with* $\alpha_1 = 3, \alpha_2 = 4$ *[1]:*

$$\left.\begin{aligned} y''(x) + \frac{3}{x}y'(x) = y(x)v(x) + x^4 + 7, \\ v''(x) + \frac{4}{x}v'(x) = y(x)v(x) + x^4 - 11, \end{aligned}\right\} \quad (22)$$

subject to the following initial conditions

$$\left.\begin{aligned} y(0) = 1, \quad y'(0) = 0, \\ v(0) = 1, \quad v'(0) = 0, \end{aligned}\right\} \quad (23)$$

with the exact solution

$$(y(x), v(x)) = (1 + x^2, 1 - x^2).$$

We apply the modified Sumudu decomposition method. From (22), we have

$$\begin{aligned} &h_1(x) = x^4 + 7,\ h_2(x) = x^2 - 11, \\ &N_1 = N_2 = y(x)v(x). \end{aligned}$$

By using the recursive relation (15) and the initial condition, we get $a_0 = b_0 = 0$ and $a_1 = b_1 = 0$. Then, we obtain

$$y_0 = 1 + \sum_{n=2}^{\infty} a_n x^n \quad \text{and} \quad v_0 = 1 + \sum_{n=2}^{\infty} b_n x^n$$

In view of (15), y_1, v_1 are given by

$$\begin{aligned} y_1 =& -p\sum_{n=2}^{\infty} a_n x^n + \mathbb{S}^{-1}\left[u^2\mathbb{S}[x^4+7]\right] \\ &- 3\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}y_0']\right] + \mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0v_0]\right] \\ v_1 =& -p\sum_{n=2}^{\infty} b_n x^n + \mathbb{S}^{-1}\left[u^2\mathbb{S}[x^4-11]\right] \\ &- 4\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}v_0']\right] + \mathbb{S}^{-1}\left[u^2\mathbb{S}[y_0v_0]\right]. \end{aligned}$$

To find a_i and b_i, $i \geq 1$, we put $y_1 = v_1 = 0$. Therefore

$$\begin{aligned} y_1 =& -p(a_2x^2 + a_3x^3 + \ldots) \\ &- 3\left(a_2x^2 + \frac{a_3}{2}x^3 + \frac{a_4}{3}x^4 + \ldots\right) + \frac{x^6}{6\cdot 5} + 4x^2 \\ &+ \frac{(a_2+b_2)}{4\cdot 3}x^4 + \frac{(a_3+b_3)}{5\cdot 4}x^5 + \frac{(a_4+b_4)}{6\cdot 5}x^6 \\ &+ \frac{a_2b_2}{6\cdot 5}x^6 + \ldots \\ =& 0 \\ v_1 =& -p(b_2x^2 + b_3x^3 + \ldots) \\ &- 4\left(b_2x^2 + \frac{b_3}{2}x^3 + \frac{b_4}{3}x^4 + \ldots\right) \\ &+ \frac{x^6}{6\cdot 5} - 5x^2 + \frac{(a_2+b_2)}{4\cdot 3}x^4 + \frac{(a_3+b_3)}{5\cdot 4}x^5 \\ &+ \frac{(a_4+b_4)}{6\cdot 5}x^6 + \frac{a_2b_2}{6\cdot 5}x^6 + \ldots \\ =& 0. \end{aligned}$$

By equating the coefficients of x^n and setting $p = 1$, we get

$$\begin{aligned} -4a_2 + 4 =&0, \\ -\frac{5}{2}a_3 =&0, \\ -2a_4 + \frac{(a_2 + b_2)}{4 \cdot 3} =&0, \\ -\frac{7}{4}a_5 + \frac{(a_3 + b_3)}{5 \cdot 4} =&0, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} -5b_2 - 5 =&0 \\ -3b_3 =&0 \\ -\frac{7}{3}b_4 - \frac{(a_2 + b_2)}{4 \cdot 3} =&0 \\ -2b_5 + \frac{(a_3 + b_3)}{5 \cdot 4} =&0 \\ &\vdots \end{aligned}$$

Solving the above algebraic equations, we have

$$\begin{aligned} &a_2 = 1,\ a_k = 0,\ k = 3, 4, \dots \\ &b_2 = -1,\ b_k = 0,\ k = 3, 4, \dots \end{aligned}$$

Thus the solution of (22) is obtained as

$$\begin{aligned} y(x) = y_0 =&1 + x^2 \\ v(x) = v_0 =&1 - x^2 \end{aligned}$$

which is the same as the exact solution. □

Example 4.5 *Consider the following system of non-linear homogeneous equations of Emden-Fowler type [1]:*

$$\left.\begin{aligned} v''(x) + \frac{8}{x}v'(x) = 4v(x)\ln y(x) - 18v(x), \\ y''(x) + \frac{4}{x}y'(x) = 10y(x) - 4y(x)\ln v(x), \end{aligned}\right\} \tag{24}$$

subject to the initial conditions

$$\left.\begin{aligned} v(0) = 1, \quad v'(0) = 0, \\ y(0) = 1, \quad y'(0) = 0, \end{aligned}\right\} \tag{25}$$

which has the exact solution

$$(v(x), y(x)) = (e^{-x^2}, e^{x^2}).$$

To solve problem (24)-(25), by the modified method we can use the transform

$$v(x) = e^{w(x)} \quad \text{and} \quad y(x) = e^{z(x)},$$

where $w(x)$ and $z(x)$ are unknown functions. We have transformed system as

$$\left.\begin{aligned} w''(x) + \frac{8}{x}w'(x) = 4z(x) - 18 - (w'(x))^2, \\ z''(x) + \frac{4}{x}z'(x) = 10 - 4w(x) - (z'(x))^2, \end{aligned}\right\} \tag{26}$$

with the initial conditions

$$\left.\begin{aligned} w(0) = 0, \quad w'(0) = 0, \\ z(0) = 0, \quad z'(0) = 0. \end{aligned}\right\} \tag{27}$$

We apply the new modified Sumudu decomposition method. From (26), we have $\alpha_1 = 8$ and $\alpha_2 = 4$ and

$$\begin{aligned} &h_1(x) = -18, \quad L_1 = 4z(x), \quad N_1 = -(w'(x))^2, \\ &h_2(x) = 10, \quad L_2 = -4w(x), \quad N_2 = -(z'(x))^2. \end{aligned}$$

By using the recursive relation (15) and the initial conditions, we get $a_0 = b_0 = 0$ and $a_1 = b_1 = 0$. Then, we obtain

$$w_0 = \sum_{n=2}^{\infty} a_n x^n \quad \text{and} \quad z_0 = \sum_{n=2}^{\infty} b_n x^n$$

In view of (15), w_1, z_1 are given by

$$\begin{aligned} w_1 =& - p\sum_{n=2}^{\infty} a_n x^n - 8\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}w_0']\right] \\ &+ \mathbb{S}^{-1}\left[u^2\mathbb{S}[4z_0 - 18]\right] - \mathbb{S}^{-1}\left[u^2\mathbb{S}[(w_0')^2]\right] \\ z_1 =& - p\sum_{n=2}^{\infty} b_n x^n - 4\mathbb{S}^{-1}\left[u^2\mathbb{S}[x^{-1}z_0']\right] \\ &+ \mathbb{S}^{-1}\left[u^2\mathbb{S}[10 - 4w_0]\right] - \mathbb{S}^{-1}\left[u^2\mathbb{S}[(z_0')^2]\right] \end{aligned}$$

To find a_i and $b_i, i \geq 1$, we put $z_1 = w_1 = 0$.

Therefore

$$\begin{aligned}
w_1 =& -p(a_2x^2+a_3x^3+\ldots)\\
&-8\left(a_2x^2+\frac{a_3}{2}x^3+\frac{a_4}{3}x^4+\ldots\right)\\
&-9x^2+4\left(\frac{b_2}{4\cdot 3}x^4+\frac{b_3}{5\cdot 4}x^5+\ldots\right)\\
&-\left(\frac{a_2^2}{3}x^4+\frac{3a_2a_3}{5}x^5+\ldots\right)\\
=&0\\
z_1 =& -p(b_2x^2+b_3x^3+\ldots)\\
&-4\left(b_2x^2+\frac{b_3}{2}x^3+\frac{b_4}{3}x^4+\ldots\right)\\
&+5x^2-4\left(\frac{a_2}{4\cdot 3}x^4+\frac{a_3}{5\cdot 4}x^5+\ldots\right)\\
&-\left(\frac{b_2^2}{3}x^4+\frac{3b_2b_3}{5}x^5+\ldots\right)\\
=&0.
\end{aligned}$$

By equating the coefficients of x^n and setting $p = 1$, we get

$$\begin{aligned}
-a_2-8a_2-9 =&0\\
-a_3-4a_3 =&0\\
-a_4-\frac{8}{3}a_4+\frac{(b_2-a_2^2)}{3} =&0\\
-a_5-2a_5+\frac{(b_3-3a_2a_3)}{5} =&0\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
-b_2-4b_2+5 =&0,\\
-b_3-2b_3 =&0,\\
-b_4-\frac{4}{3}b_4-\frac{(b_2^2+a_2)}{3} =&0,\\
-2b_5-\frac{(3b_2b_3+a_3)}{5} =&0,\\
&\vdots
\end{aligned}$$

Solving the above algebraic equations, we have

$$a_2=-1,\ a_i=0,\ i=3,4,\ldots$$
$$b_2=1,\ b_i=0,\ i=3,4,\ldots$$

Thus the solution of (26) is obtained as

$$w(x)=w_0=-x^2 \quad \text{and} \quad z(x)=z_0=x^2.$$

Thus the solution of (24) will be obtained as

$$v(x)=e^{w(x)}=e^{-x^2} \quad \text{and} \quad y(x)=e^{z(x)}=e^{x^2}$$

which is the same as the exact solution. □

5 Conclusion

In this article, the modification of Sumudu decomposition method has been applied to solve systems of singular initial value problems of Lane-Emden-Fowler type equations. From the obtained results, it is show that the proposed method is powerful and easy to calculate because this method yields very accurate solutions using only a few iterates and only requires the calculation of the first Adomian polynomial. The advantage of this method is that it reduce steps of calculation and give accurate results. We expect that this modified algorithm will be a promising method for investigating exact solutions to other singular IVPs of higher-order nonlinear ODEs.

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