On the Diophantine equation $n^x + 13^y = z^2$ where $n \equiv 2 \pmod{39}$ and $n+1$ is not a square number

NONGLUK VIRIYAPONG, CHOKCHAI VIRIYAPONG*
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

Abstract: The purpose of the present article is to prove that the Diophantine equation $n^x + 13^y = z^2$ has exactly one solution $(n, x, y, z) = (2, 3, 0, 3)$ where x, y and z are non-negative integers and n is a positive integer with $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number. In particular, $(3, 0, 3)$ is a unique solution (x, y, z) in non-negative integers of the Diophantine equation $2^x + 13^y = z^2$.

Key-Words: Diophantine equation, congruence

Received: February 3, 2021. Revised: August 12, 2021. Accepted: August 26, 2021. Published: September 7, 2021.

1 Introduction

In 2004, the Catalan's conjecture, which was posed by Catalan [1] in 1844, was proved by Mihailescu [2]. After that, many mathematicians investigated the non-negative solutions (x, y, z) of Diophantine equations of the type $a^x + b^y = z^2$, using the Catalan's conjecture, where a and b are fixed. In 2007, Acu [3] proved that the Diophantine equation $2^x + 5^y = z^2$ has only two solutions $(x, y, z) = (3, 0, 3), (2, 1, 3)$ where x, y and z are non-negative integers. In 2011, Suvarnamani [4] gave some non-negative solutions of Diophantine equation $2^x + p^y = z^2$ when p is an odd prime number. In the same year, Suvarnamani, Singta and Chotchaisthit [5] proved that the Diophantine equations $4^x+7^y = z^2$ and $4^x+11^y = z^2$ have no solution in non-negative integer. In 2013, Sroysang [6] presented a completed proof that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions (x, y, z) for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. In the same year, Chotchaisthit [7] showed that $(3, 0, 3)$ is the only non-negative integer solution (x, y, z) of the Diophantine equation $2^x + 11^y = z^2$. In 2014, Sroysang [8] proved that the Diophantine equation $8^x + 13^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1, 0, 3)$. In 2019, Mahesh and Sinari [9] gave that all the solutions of the Diophantine equation $2^x + p^y = z^2$ for any odd prime p greater than 3 and x are y not both positive odd integers together. In 2021, Tangjai and Chubthaisong showed that Diophantine equation $3^x + p^y = z^2$ has a unique non-negative solution $(p, x, y, z) = (2, 0, 3, 3)$ when p is prime such that $p \equiv 2 \pmod 3$ and y is not divisible by 4.

In this paper, we find all non-negative solutions of the Diophantine equation $n^x + 13^y = z^2$ where $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number.

2 Preliminaries

In this section, we shall recall some basic properties of congruences, which are an important and useful tool for this work, see in [11] and [12].

Definition 1. *Let a and b be two integers such that $b \neq 0$. We say that a divides b, and write $a \mid b$, if $b = ac$ for some integer c*

Definition 2. *Let a, b and m be three integers such that $m \geqslant 1$. We say that a is congruent to b modulo m, and write $a \equiv b \pmod m$, if $n \mid (b - a)$.*

Remark 1. *If a is an integer, then there is a unique integer r such that $a \equiv r \pmod m$ and $0 \leqslant r < m$.*

Lemma 1. *Let a, b, c, d and m be five integers such that $m \geqslant 1$. Then the following statements hold.*

1. *$a \equiv a \pmod m$.*
2. *If $a \equiv b \pmod m$, then $b \equiv a \pmod m$.*
3. *If $a \equiv b \pmod m$ and $b \equiv c \pmod m$, then $a \equiv c \pmod m$.*
4. *If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $a + c \equiv b + d \pmod m$ and $ac \equiv bd \pmod m$.*
5. *If $a \equiv b \pmod m$, then $a^k \equiv b^k \pmod m$ for all integer $k \geqslant 0$.*

Lemma 2. *If n is an integer such that $n \equiv 2 \pmod{39}$, then $n \equiv 2 \pmod 3$ and $n \equiv 2 \pmod{13}$.*

Proof. Assume that n is an integer such that $n \equiv 2 \pmod{39}$. Then $n - 2 = 39t$ for some integer t. Thus $n - 2 = 3(13t)$ and $n - 2 = 13(3t)$. Hence $n \equiv 2 \pmod 3$ and $n \equiv 2 \pmod{13}$. ☐

Now, we shall recall the Catalan's conjecture in [1].

Theorem 1 (Catalan's conjecture)**.** $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *of the Diophantine equation* $a^x - b^y = 1$ *where a, b, x and y are integers with* $\min\{a, b, x, y\} > 1$.

Next, we shall give two lemmas that follow from the Catalan's conjecture.

Lemma 3. [6] *The Diophantine equation* $1 + 13^y = z^2$ *has no non-negative integer solution where y and z are non-negative integers.*

Lemma 4. *Let* n *be a positive number such that* $n+1$ *is not a square. Then* $(n, x, z) = (2, 3, 3)$ *is a unique solution for the Diophantine equation* $n^x + 1 = z^2$ *where x and z are non-negative integers.*

Proof. Let x and z be non-negative integers such that

$$n^x + 1 = z^2. \quad (1)$$

If $n = 1$, then $z^2 = 2$. It is impossible. Now, $n > 1$. Next, we will divide the number x into three cases.

Case 1: $x = 0$. Then $z^2 = 2$. It is impossible.

Case 2: $x = 1$. Then $z^2 = n + 1$ which is a contradiction with $n + 1$ is not a square.

Case 3: $x > 1$. Then $z^2 = n^x + 1 \geqslant n + 1 > 2$, and so $z > 1$. By Theorem 1, $(n, x, z) = (2, 3, 3)$ is only solution for the Diophantine equation (1).

Therefore, $(n, x, z) = (2, 3, 3)$ is a unique solution for the Diophantine equation $n^x + 1 = z^2$ where x and z are non-negative integers. ☐

3 Main Results

In this section, we begin by introducing three lemmas which will be useful in our work.

Lemma 5. *If* x *is an odd positive integer, then*

$$2^x \equiv 2, 5, 6, 7, 8, 11 \pmod{13}.$$

Proof. We will establish by induction that

$$2^{2n-1} \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$$

for all $n \in \mathbb{N}$. If $n = 1$, we have

$$2^1 \equiv 2 \pmod{13}.$$

Thus the statement is true for $n = 1$.
Assume that it is true for $n = k$. Then

$$2^{2k-1} \equiv 2, 5, 6, 7, 8, 11 \pmod{13}.$$

Case 1: $2^{2k-1} \equiv 2 \pmod{13}$.
Then, we obtain $2^{2k+1} \equiv 8 \pmod{13}$.

Case 2: $2^{2k-1} \equiv 5 \pmod{13}$.
Then, we obtain $2^{2k+1} \equiv 20 \pmod{13}$, and so $2^{2k+1} \equiv 7 \pmod{13}$.

Case 3: $2^{2k-1} \equiv 6 \pmod{13}$.
Then we get $2^{2k+1} \equiv 24 \pmod{13}$, and so $2^{2k+1} \equiv 11 \pmod{13}$.

Case 4: $2^{2k-1} \equiv 7 \pmod{13}$.
Then, we obtain $2^{2k+1} \equiv 28 \pmod{13}$, and so $2^{2k+1} \equiv 2 \pmod{13}$.

Case 5: $2^{2k-1} \equiv 8 \pmod{13}$.
Then, we obtain $2^{2k+1} \equiv 32 \pmod{13}$, and so $2^{2k+1} \equiv 6 \pmod{13}$.

Case 6: $2^{2k-1} \equiv 11 \pmod{13}$.
Then, we obtain $2^{2k+1} \equiv 44 \pmod{13}$, and so $2^{2k+1} \equiv 5 \pmod{13}$.

Hence $2^{2k+1} \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$. Thus the statement is true for $n = k + 1$. This proof is complete. ☐

Lemma 6. *If* z *is an integer, then* $z^2 \equiv 0, 1 \pmod 3$.

Proof. Let z be an integer. Then $z \equiv r \pmod 3$ for some $r \in \{0, 1, 2\}$.

Case 1: $z \equiv 0 \pmod 3$. Then $z^2 \equiv 0 \pmod 3$.

Case 2: $z \equiv 1 \pmod 3$. Then $z^2 \equiv 1 \pmod 3$.

Case 3: $z \equiv 2 \pmod 3$. Then $z^2 \equiv 4 \pmod 3$. and so $z^2 \equiv 1 \pmod{13}$.

Hence $z^2 \equiv 0, 1 \pmod{13}$. ☐

Lemma 7. *If* z *is an integer, then*

$$z^2 \equiv 0, 1, 3, 4, 9, 10, 12 \pmod{13}.$$

Proof. Let z be an integer. Then $z \equiv r \pmod{13}$ for some $r \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Case 1: $z \equiv 0 \pmod{13}$. Then $z^2 \equiv 0 \pmod{13}$.

Case 2: $z \equiv 1 \pmod{13}$. Then $z^2 \equiv 1 \pmod{13}$.

Case 3: $z \equiv 2 \pmod{13}$. Then $z^2 \equiv 4 \pmod{13}$.

Case 4; $z \equiv 3 \pmod{13}$. Then $z^2 \equiv 9 \pmod{13}$.

Case 5: $z \equiv 4 \pmod{13}$. Then $z^2 \equiv 16 \pmod{13}$, and so $z^2 \equiv 3 \pmod{13}$.

Case 6: $z \equiv 5 \pmod{13}$. Then $z^2 \equiv 25 \pmod{13}$, and so $z^2 \equiv 12 \pmod{13}$.

Case 7: $z \equiv 6 \pmod{13}$. Then $z^2 \equiv 36 \pmod{13}$, and so $z^2 \equiv 10 \pmod{13}$.

Case 8: $z \equiv 7 \pmod{13}$. Then $z^2 \equiv 49 \pmod{13}$, and so $z^2 \equiv 10 \pmod{13}$.

Case 9: $z \equiv 8 \pmod{13}$. Then $z^2 \equiv 64 \pmod{13}$, and so $z^2 \equiv 12 \pmod{13}$.

Case 10: $z \equiv 9 \pmod{13}$. Then we obtain $z^2 \equiv 81 \pmod{13}$, and so $z^2 \equiv 3 \pmod{13}$.

Case 11: $z \equiv 10 \pmod{13}$. Then we obtain $z^2 \equiv 100 \pmod{13}$, and so $z^2 \equiv 9 \pmod{13}$.

Case 12: $z \equiv 11 \pmod{13}$. Then we obtain $z^2 \equiv 121 \pmod{13}$, and so $z^2 \equiv 4 \pmod{13}$.

Case 13: $z \equiv 12 \pmod{13}$. Then we obtain $z^2 \equiv 144 \pmod{13}$, and so $z^2 \equiv 1 \pmod{13}$.

Hence $z^2 \equiv 0, 1, 3, 4, 9, 10, 12 \pmod{13}$. □

Now, we shall give our main result.

Theorem 2. *Let n be a positive number such that $n \equiv 2 \pmod{39}$ and $n+1$ is not a square. Then $n^x + 13^y = z^2$ has a unique solution $(n, x, y, z) = (2, 3, 0, 3)$ where x, y, and z are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that

$$n^x + 13^y = z^2. \tag{2}$$

If $y = 0$, then, by Lemma 4, $(n, x, y, z) = (2, 3, 0, 3)$ is a solution of the equation (2). Now, we assume that $y \geqslant 1$. By Lemma 3, we have $x \geqslant 1$. We will divide the number x into two cases.

Case 1: x is even. Since $n \equiv 2 \pmod 3$, $n \equiv -1 \pmod 3$. Then $n^x \equiv 1 \pmod 3$. Since $13^y \equiv 1 \pmod 3$, by (2), we obtain that $z^2 \equiv 2 \pmod 3$. It is a contradiction with Lemma 6.

Case 2: x is odd. Since $n \equiv 2 \pmod 3$, by Lemma 5, we obtain

$$n^x \equiv 2, 5, 6, 7, 8, 11 \pmod{13}.$$

Since $13^y \equiv 0 \pmod{13}$, by (2), we obtain that $z^2 \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$. It is a contradiction with Lemma 7.

Therefore, $(2, 3, 0, 3)$ is a unique solution (n, x, y, z) in non-negative integers of the Diophantine equation $n^x + 13^y = z^2$. □

Next, we shall give some special cases of Theorem 2 when $n = 2, 41, 119$.

Corollary 1. $(3, 0, 3)$ *is a unique solution (x, y, z) in non-negative integers of the Diophantine equation $2^x + 13^y = z^2$.*

Proof. Since $2 \equiv 2 \pmod{39}$ and $2 + 1 = 3$ is not a square, by Theorem 2, $(x, y, z) = (3, 0, 3)$ is a unique solution in non-negative integers of the Diophantine equation $2^x + 13^y = z^2$. □

Corollary 2. *The Diophantine equation $41^x + 13^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.*

Proof. Since $41 \equiv 2 \pmod{39}$, $41 + 1 = 42$ is not a square and $41 \neq 2$, by Theorem 2, the Diophantine equation $41^x + 13^y = z^2$ has no non-negative integer solution. □

Corollary 3. *The Diophantine equation $119^x + 13^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.*

Proof. Since $119 \equiv 2 \pmod{39}$, $119 + 1 = 120$ is not a square and $119 \neq 2$, by Theorem 2, the Diophantine equation $119^x + 13^y = z^2$ has no non-negative integer solution. □

Finally, we shall consider the main theorem in [8]. The Diophantine equation $8^t + 13^y = z^2$ is a special case of the Diophantine equation $2^x + 13^y = z^2$. It easy to prove the following corollary.

Corollary 4. $(1, 0, 3)$ *is a unique solution (x, y, z) for the Diophantine equation $8^t + 13^y = z^2$. where t, y and z are non-negative integers.*

Proof. Let t, y and z be non-negative integers such that $8^t + 13^y = z^2$. Set $x = 3t$. Thus $2^x + 13^y = z^2$. By Corollary 1, we have $(x, y, z) = (3, 0, 3)$. Then $t = 1$. Hence $(1, 0, 3)$ is a unique solution for the equation $8^t + 13^y = z^2$ where t, y and z are non-negative integers. □

4 Conclusion

In this article, we obtain that the Diophantine equation $n^x + 13^y = z^2$ has a unique non-negative solution $(n, x, y, z) = (2, 3, 0, 3)$ where $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square, using basic properties of congruences and the Catalan's conjecture. The main result can apply to the case where n is not a prime, see in Corrollary 3. In case the first positive number $n = 2$ satisfying the conditions of the main result, the Diophantine equation $2^x + 13^y = z^2$ has a unique non-negative solution $(x, y, z) = (3, 0, 3)$. This above result is generalization of the Sroysang' s result in [8]. In main result, we study the Diophantine equation $n^x + 13^y = z^2$ on the positive number n that $n \equiv 2 \pmod{39}$. Hence, it is still an interesting problem that all findings of the solutions of Diophantine equation $n^x + 13^y = z^2$ on the other case.

Acknowledgment

The authors are very grateful to the reviewers for valuable suggestion towards the improvement of this paper.

References:

[1] E. Catalan, Note extraite dune lettre adressee a lediteur, *J. Reine Angew. Math.*, Vol.27, 1844, pp. 192.

[2] S. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, Vol.572, 2004, pp. 167-195.

[3] D. Acu, On the Diophantine equation, *Gen. Math.*, Vol.15, No.4, 2007, pp. 145-148.

[4] A. Suvarnamani, Solutions of the Diophantine equations $2^x + p^y = z^2$, *Int. J. Math. Sci. Appl.*, Vol.1, No.3, 2011, pp. 1415-1419.

[5] A. Suvarnamani, A. Singta and S. Chotchaisthit, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.*, Vol.1, No.1, 2011, pp. 25-28.

[6] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.* Vol.84, No.2, 2013, pp. 133-137.

[7] S. Chotchaisthit, On the Diophantine equation $2^x + 11^y = z^2$, *Maejo Int. J. Sci. Technol.*, Vol.7, No.2, 2013, pp. 291-293.

[8] B. Sroysang, On the Diophantine equation $8^x + 13^y = z^2$, *Int. J. Pure Appl. Math.*, Vol.90, No.1, 2014, pp. 69-72.

[9] G. Mahesh, and V. Sinari, On the Diophantine equation $2^x + p^y = z^2$, *BOMSR*, Vol.7, No.2, 2019, pp. 36-38.

[10] W. Tangjai and C. Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2 \pmod 3$, *WSEAS Transactions on Mathematics*, Vol.20, 2021, pp. 283-287.

[11] W. J. LeVeque, *Elementary theory of numbers*, Courier Corporation, 1990.

[12] G. A. Jones and J. M. Jones, *Elementary theory of numbers*, Springer Science & Business Media, 2012.