Several results on conjugacy class sizes of some elements of finite groups

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Abstract: Let G be a finite group. For an element x of G, x^G denotes the conjugacy class of x in G. $|x^G|$ is called the size of the conjugacy class x^G. In this paper, we establish several results on conjugacy class sizes of some elements of finite groups. In addition, we give a simple and clearer proof of a known result.

Key–Words: Conjugacy class sizes; Finite groups; Conjugacy class size; Primary element; Quasi-Frobenius group.

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1 Introduction and Preliminaries

Throughout this paper, the term group always means a group of finite order. The letter G always denotes a group, and the letter p always denotes a prime. For an element $x \in G$, $o(x)$ denotes the order of x, and x^G denotes the conjugacy class of x in G. $|x^G|$ is called the size of the conjugacy class x^G. We denote by $\pi(G)$ the set of prime divisors of the order $|G|$ of a group G. For a positive integer m, $\pi(m)$ denotes the set of prime divisors of m. All further unexplained notations are standard and can be found in [1].

Let x be an element of G. We say that x is primary if $o(x)$ is a power of a prime (including the case where $x = 1$, the identity element); We say that x is biprimary if $o(x)$ is divisible by precisely two distinct primes.

We write

$$cs(G) = \{|x^G| : x \in G\},$$

$$pcs(G) = \{|x^G| : x \in G \text{ and } x \text{ is primary}\}$$

and

$$ppcs(G) = \{|x^G| : x \in G \text{ and } x \text{ is primary or biprimary}\}.$$

We say that G is a quasi-Frobenius group if $G/Z(G)$ is a Frobenius group. The inverse images in G of the kernel and a complement of $G/Z(G)$ are called the kernel and a complement of G. It is well-known that if G is a quasi-Frobenius group with abelian kernel and complement, then $cs(G) = cs(G/Z(G))$ (see [2]).

A classical topic of finite group theory is to study the influence of the conjugacy class sizes on the structure of groups. However, studying such properties only from partial information, provided by connjugacy class sizes of certain elements, can be a more complex and more interesting problem. In this paper we study the influence of the conjugacy class sizes of primary and biprimary elements on the structure of groups. In addition, we give a simple and clearer proof of a known result.

In this section we list some lemmas which will be used in the sequel. The following Lemma 1.1 is well-known.

Lemma 1.1 *Let $x \in G$. Assume that $o(x) = p_1^{m_1}...p_n^{m_n}$, where $p_1, ..., p_n$ are distinct primes and $m_1, ..., m_n$ are positive integers. Then, $x = x_1...x_n$ with $o(x_i) = p_i^{m_i}$ and $x_r x_s = x_s x_r$ for $s, r = 1, ..., n$. Furthermore, there exist integers k_i such that $x^{k_i} = x_i$ for $i = 1, \cdots, n$.*

Lemma 1.2 ([3, Lemma 2.4]) *A prime p does not divide the conjugacy class sizes of primary elements of G if and only if G has a central Sylow $p-$subgroup.*

Lemma 1.3 ([4, LEMMA 1(1)]) *Let $a, b \in G$. If $(|a^G|, |b^G|) = 1$, then $G = C_G(a)C_G(b)$.*

Lemma 1.4 ([5, Theorem 6.4.3]) *If $G = AB$, where A and B are two nilpotent subgroups of G, then G is solvable.*

Lemma 1.5 ([6, Lemma 2.1]) *Let G be a $\pi-$separable group with $\pi \subseteq \pi(G)$. Then,*

(i) $|x^G|$ is a $\pi'-$number for every primary $\pi'-$element x if and only if $G = O_\pi(G) \times O_{\pi'}(G)$.

(ii) $|x^G|$ is a $\pi-$number for every primary $\pi'-$element x if and only if G has an abelian Hall $\pi'-$subgroup.

Lemma 1.6 ([2, Proposition 1.4]) *Suppose that $G/Z(G) \cong S_3$, the symmetric group of degree 3. Then, up to a central direct factor, $G = TD$, where T is a normal subgroup of order 3, D a cyclic 2−group.*

The following Lemma 1.7 is a consequence of [7, Theorem A].

Lemma 1.7 *If $ppcs(G) = \{1, m\}$, then $m = p^a$ for some prime p and some positive integer a, and $G = P \times A$, where P is a $p-$group and A is an abelian group.*

2 Results and Proofs

A positive integer m is called a Hall number of a group G if $m||G|$ and $(m, |G|/m) = 1$.

Theorem 2.1 *Suppose that G satisfies the following two conditions:*

(1) $pcs(G) = \{1, m_1, m_2\}$ with $(m_1, m_2) = 1$.

(2) For each $p \in \pi(G)$ and any noncentral $p-$element x of G, $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$ and $C_G(x)_{p'}$ is abelian.

Then, the following statements are true:

(I) G is a quasi-Frobenius group with abelian kernel and complement.

(II) Assume that G has no non-trivial abelian direct factor. Set $\pi := \pi(m_1)$. If m_1 is a Hall-number of G, then $Z(G)_\pi = 1$. If m_2 is a Hall-number of G, then $Z(G)_{\pi'} = 1$. If both m_1 and m_2 are Hall-number of G, then $Z(G) = 1$.

Proof We write $\pi := \pi(m_1)$. By hypothesis,we have $\pi(m_1) \cap \pi(m_2) = \emptyset$. If a prime divisor r of $|G|$ does not divide m_i for $i = 1, 2$, then by Lemma 1.2 we know that G has a central Sylow $r-$subgroup, and hence G has a non-trivial abelian direct factor.It is well-known that abelian direct factors are immaterial in this context, and so we can assume that G has no non-trivial abelian direct factor. Therefore, we have $\pi(G) = \pi(m_1) \cup \pi(m_2)$. Let $x, y \in G$ be primary elements such that $|x^G| = m_1$ and $|y^G| = m_2$. Since $(m_1, m_2) = 1$, by Lemma 1.3 we have $G = C_G(x)C_G(y)$. By hypothesis(condition (2)), $C_G(x)$ and $C_G(y)$ are nilpotent, and so by Lemma 1.4 we conclude that G is solvable. Hence, G has Hall $\pi-$subgroups and Hall $\pi'-$subgroups. Notice that any two Hall $\pi(\pi')-$subgroups of G are conjugate, and any $\pi(\pi')-$subgroup of G is contained a Hall $\pi(\pi')-$subgroup of G.

Let $z \in G$ be a primary $\pi-$element, and suppose that $|z^G| = m_1$. Then, by hypothesis(condition (2)) we have

$$(*) \quad C_G(z) = C_G(z)_\pi \times H,$$

where H is a Hall $\pi'-$subgroup of G and H is abelian. Let K be a Hall $\pi-$subgroup of G such that $C_G(z)_\pi \subseteq K$. Clearly, we have $C_G(z)_\pi < K$. For every noncentral primary element $x \in C_G(z)_\pi$, by equality $(*)$ it is obvious that $|x^G|$ is a $\pi-$number.There exists a noncentral primary element $w \in K - C_G(z)_\pi$ such that $|w^G|$ is a $\pi'-$number; otherwise,for every noncentral primary element y of K we have that $|y^G|$ is a $\pi-$number, and thus by Lemma 1.5(i) we have that $G = K \times H$ and the abelian Hall $\pi'-$subgroup H of G is a central direct factor of G, forcing $pcs(G) = \{1, m_1\}$, in contradiction to the hypothesis of the theorem. Then, there exists some element g such that $K \le C_G(w)^g = C_G(w^g)$. It follows that $w^g \le Z(K)$. Then, since $C_G(z)_\pi \subseteq K$, $w^g \in C_G(z)_\pi$, and thus by equality $(*)$ we have $H \le C_G(w^g)$. It follows that $w \in Z(G)$, contrary to the choice of w. Consequently, we conclude that every primary $\pi-$element of G has conjugacy class size 1 or m_2. Then, by Lemma 1.5(ii) we conclude that Hall $\pi-$subgroups of G are abelian. By the same arguments we conclude that Hall $\pi'-$subgroup of G are abelian. Let K be a Hall $\pi-$subgroup of G, and let H be a Hall $\pi'-$subgroup of G . Then, K and H are abelian.

Let x be a noncentral primary $\pi-$element of G. We may assume that $x \in K$. Since K is abelian, by hypothesis we have $C_G(x) = K \times C_G(x)_{\pi'}$. We may assume that $C_G(x)_{\pi'} \le H$. Then, since H is abelian, we conclude that $C_G(x)_{\pi'} = Z(G)_{\pi'}$. Let $y \in G$ be a noncentral primary $\pi'-$element of G. By the same arguments as for x we conclude that $C_G(y)_\pi = Z(G)_\pi$.

Since K and H are abelian, we have $F(G) = O_\pi(G) \times O_{\pi'}(G)$. Since G is nonabelian, by [5, 4.2, p.277] we have $F(G) \not\le Z(G)$. Then, either $O_\pi(G) \not\le Z(G)$ or $O_{\pi'}(G) \not\le Z(G)$. Without loss of generality, we may assume that $O_\pi(G) \not\le Z(G)$. Suppose that $O_{\pi'}(G) \not\le Z(G)$, and let $y \in O_{\pi'}(G) - Z(G)$. By the above paragraph we have $C_G(y)_\pi = Z(G)_\pi$. Then, since $O_\pi(G) \le C_G(O_{\pi'}(G)) \le C_G(y)$, we have $O_\pi(G) \le Z(G)$, contradicting our assumption. Therefore, we conclude that $O_{\pi'}(G) = Z(G)_{\pi'}$. It follows that $K \le C_G(O_\pi(G)) = C_G(O_\pi(G)) \times Z(G)_{\pi'}) = C_G(O_\pi(G) \times O_{\pi'}(G)) = C_G(F(G)) \le F(G) = O_\pi(G) \times O_{\pi'}(G)$(see [5, 4.2, p.277]), and thus $K = O_\pi(G)$, that is, $K \lhd G$. Then, we have $G = K > \lhd H$.(The notation $> \lhd$ denotes a semidirect product.)

Consider the factor group $G/Z(G) = KH/Z(G)$, and we use the bar convention. Let $\overline{h}$ be a primary non-identity element of $\overline{H} = HZ(G)/Z(G)$, and let $\overline{k}$ be a primary non-identity element of $\overline{K} = KZ(G)/Z(G)$. By Lemma 1.1 we may assume that h and k are primary.Suppose that $[\overline{k}, \overline{h}] = \overline{1}$. Then we have that $[k, h] \leq Z(G)$. It follows that $[k, h] \leq [K, H] \cap C_K(H)$. On the other hand, we have that $K = [K, H] \times C_K(H)$(see [8, Theorem 2.3, p.177]). Hence, we get that $[k, h] = 1$.It follows that $h \in C_G(k)_{\pi'} = Z(G)_{\pi'}$, and thus $\overline{h} = \overline{1}$, a contradiction. So, we have that $[\overline{k}, \overline{h}] \neq \overline{1}$. Then, by Lemma 1.1 and [9, Problems 7.1(a), p.121] we conclude that $G/Z(G) = KH/Z(G)$ is a Frobenius group with the kernel $KZ(G)/Z(G)$ and a complement $HZ(G)/Z(G)$. Thus, noting that K and H are abelian, G is a quasi-Frobenius group with abelian kernel and complement. So, statement (I) holds.

We have that $m_1 = |K/Z(G)_\pi|$ and $m_2 = |H/Z(G)_{\pi'}|$, and so statement (II)) is obvious. This completes the proof of the theorem.

Theorem 2.2 [10, THEOREM 1.1] *Suppose that* $ppcs(G) = \{1, m_1, m_2\}$ *with* $(m_1, m_2) = 1$. *Then, G is a quasi-Frobenius group with abelian kernel and complement.*

Proof By hypothesis and Lemma 1.2 we have that $pcs(G) = \{1, m_1, m_2\}$ with $(m_1, m_2) = 1$.

For any $p \in \pi(G)$, let x be a $p-$element of G such that $|x^G| = m_i$ for $i = 1$ or 2. Let z be any primary $p'-$element of $C_G(x)$. We have that $C_G(xz) = C_G(x) \cap C_G(z) \leq C_G(x)$. Then, since $(m_1, m_2) = 1$, we conclude that $C_G(x) = C_G(xz) \leq C_G(z)$, and thus $z \in Z(C_G(x))$. Hence, $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$ and $C_G(x)_{p'}$ is abelian.

To sum up, G satisfies two conditions of Theorem 2.1. Hence, by Theorem 2.1 we conclude that the theorem holds. This completes the proof.

Note In the original proof of [10, THEOREM 1.1], the authors of [10] dealt separately with the following three cases: (i) Both m_1 and m_2 are Hall numbers of G;(ii) Only one of m_1 and m_2 is a Hall number of G; (iii) Neither m_1 nor m_2 is a Hall number of G. From the proof of Theorem 2.1, we see that it is not necessary to deal separately with the above-mentioned three cases. So, we have given a simple and clearer proof of [10, THEOREM 1.1]. However, we point out that, in fact, THEOREM 1.1 of [10] is a consequence of Theorem 1 and Theorem 2 of [4] (see the following Theorem 2.5 and its proof). In addition, we will see that THEOREM 1.1 of [10] is also a consequence of Corollary 2.12 in the present paper.

Corollary 2.3 *Suppose that* $ppcs(G) = \{1, n, n+1\}$. *Then, G is a quasi-Frobenius group with abelian kernel and complement. Furthermore,* $n + 1$ *is a prime power.*

Proof Clearly, $(n, n+1) = 1$. Then, by Theorem 2.2, $G/Z(G) = K/Z(G) > \lhd H/Z(G)$ is a Frobenius group with the kernel $K/Z(G)$ and a complement $H/Z(G)$, and K and H are abelian. Since $\{1, |H/Z(G)|, |K/Z(G)|\} = cs(G/Z(G)) = cs(G)$, we conclude that$\{1, |H/Z(G)|, |K/Z(G)|\} = \{1, n, n{+}1\}$.Hence, $G/Z(G)$ is a sharply $2-$transitive group of degree $n + 1$, and so by [11, XII Theorem 9.1] we conclude that $n + 1$ is a prime power. This completes the proof.

Corollary 2.4 *Suppose that* $ppcs(G) = \{1, 2, 3\}$. *Then,* $G/Z(G) \cong S_3$. *Furthermore, up to a central direct factor,* $G = DT$, *where D is a normal subgroup of order 3 and T is a cyclic $2-$group.*

Proof By Corollary 2.3 and its proof we conclude that $G/Z(G)$ is a nonabelian group of order 6, and so $G/Z(G) \cong S_3$. Thus, by Lemma 1.6 we get that, up to an abelian direct factor, $G = DT$, where D is a normal subgroup of order 3 and T is a cyclic $2-$group. This completes the proof.

Theorem 2.5 *Suppose that*

$$ppcs(G) = \{1, m_1, \cdots, m_t\}$$

with $t \geq 2$.*In addition, for* $i, j = 1, \cdots, t$, *if* $i \neq j$, *then* $(m_i, m_j) = 1$.*Then,* $t = 2$ *and G is a quasi-Frobenius group with abelian kernel and complement.*

Proof Let x be a noncentral element of G, and assume that $o(x) = p_1^{r_1}...p_n^{r_n}$, where $p_1, ..., p_n$ are distinct primes and $r_1, ..., r_n$ are positive integers. By Lemma 1.1 we have that $x = x_1...x_n$ with $o(x_i) = p_i^{r_i}$ and $x_r x_s = x_s x_r$ for $s, r = 1, ..., n$. Since x is noncentral, some x_i is noncentral for $1 \leq i \leq n$. Without loss of generality, we may assume that x_1 is noncentral. By hypothesis, we have $|x_1^G| = m_j$ for some j.Clearly, we may assume that $j = 1$, that is, $|x_1^G| = m_1$. By the same arguments as in the proof of Theorem 2.2 we conclude that $x_i \in Z(C_G(x_1))$ for $i = 1, \cdots, n$. It follows that $C_G(x_1) \subseteq C_G(x_i)$ for $i = 2, \cdots, n$.Thus,

$$C_G(x) = C_G(x_1 \cdots x_n) =$$
$$C_G(x_1) \cap C_G(x_2) \cap \cdots \cap C_G(x_n) = C_G(x_1).$$

It follows that $|x^G| = |x_1^G| = m_1$. So, we have proved that $cs(G) = ppcs(G) = \{1, m_1, \cdots, m_t\}$. Then, since $(m_i, m_j) = 1$ for $i \neq j$, by THEOREM 1 of [4] we conclude that $t = 2$, and so by THEOREM 2 of [4] (or by Theorem 2.2 in the precent paper) we conclude that G is a quasi-Frobenus group with abelian kernel and complement. This completes the proof.

The following Theorem 2.6 is an extension of [4, COROLLARY 3].

Theorem 2.6 *Let G be a nonabelian group, and suppose that G satisfies the following condition:*

$(*)$ *Let $x, y \in G - Z(G)$ be primary or biprimary. If $x^G \neq y^G$, then $(|x^G|, |y^G|) = 1$.*

Then, $G \cong S_3$.

Proof In view of condition $(*)$, by Theorem 2.5 and Lemma 1.7, one of the following two cases may occur:

(1) $G = P \times A$, where P is a $p-$group for some prime p and A is an abelian group.Furthermore,if $x, y \in G - Z(G)$ are primary or biprimary, then $x^G = y^G$.

In this case, it is clear that $A = 1$ and G is a $p-$group. Furthermore,G has only one conjugacy class of noncentral elements. It follows that $|G| = |Z(G)| + |G|/n$, where n is a positive integer and $n \geq 2$. Then, $|Z(G)| \geq |G|/2$, and this implies that G is abelian, a contradiction because G is nonabelian by hypothesis. So, this case can not occur.

(2) G is a quasi-Frobenius group. Furthermore, G has exactly two classes of noncentral primary and biprimary elements.

In this case, it is easy to see that $|\pi(G)| = 2$ and $Z(G) = 1$. Then, $G = P > \lhd Q$ is a Frobenius group with the kernel P and a complement Q, where P is an abelian $p-$group and Q is an abelian $q-$group, p and q are two distinct primes. Suppose that $|Q| > 2$. Then, there exist non-identity elements $x, y \in Q$ such that $x \neq y$. We have $x^G = y^G$, and thus there exists a non-identity element $w \in P$ such that $x^w = y$. It follows that $y \in Q^w \cap Q = 1$(see [5, 8.5, p.497]), and so $y = 1$, a contradiction. Hence, we have $|Q| = 2$. Then, we have $|P| - 1 = 2$, and thus $|P| = 3$ and $G \cong S_3$. The proof is complete.

Theorem 2.7 *Suppose that*

$$ppcs(G) = \{1, n, n+1, \cdots, n+r\}$$

with $r \geq 1$. If $ppcs(G)$ does not contain any prime, then $r = 1$ and G is a quasi-Frobenius group with abelian kernel and complement. Furthermore, $n+1$ is a prime power.

Proof Let x be a noncentral element of G. We have that that $o(x) = p_1^{m_1}...p_k^{m_k}$, where $p_1, ..., p_k$ are distinct primes and $m_1, ..., m_k$ are positive integers. By Lemma 1.1 we have that $x = x_1...x_k$ with $o(x_i) = p_i^{m_i}$ and $x_t x_s = x_s x_t$ for $s, t = 1, ..., k$. Since x is noncentral, some x_i is noncentral for $1 \leq i \leq k$. Without loss of generality, we may assume that x_1 is noncentral. For $i \neq 1$, we have $C_G(x_1 x_i) = C_G(x_1) \cap C_G(x_i)$, and so $C_G(x_1 x_i) \leq C_G(x_1)$. If $C_G(x_1 x_i) < C_G(x_1)$, then $|x_1^G|$ is a proper divisor of $|(x_1 x_i)^G|$, so that $2|x_1^G| \leq |(x_1 x_i)^G|$. By the Bertrand's postulate, there exists a prime p such that $|x_1^G| < p < 2|x_1^G| \leq |(x_1 x_i)^G|$. Since $\{|x_1^G|, |(x_1 x_i)^G|\} \subseteq ppcs(G)$, we get that $p \in ppcs(G)$, contradicting the assumption of the theorem. Hence, $C_G(x_1) = C_G(x_1 x_i) = C_G(x_1) \cap C_G(x_i)$. It follows that $C_G(x_1) \leq C_G(x_i)$ for $i = 2, \cdots, k$. Hence, we have

$$C_G(x) = C_G(x_1 \cdots x_k) = \\ C_G(x_1) \cap C_G(x_2) \cap \cdots \cap C_G(x_k) = C_G(x_1).$$

It follows that $|x^G| = |x_1^G| \in ppcs(G)$. So, we have proved that

$$cs(G) = ppcs(G) = \{1, n, n+1, \cdots, n+r\}.$$

Then, by [2, Theorem 2] we conclude that the theorem holds. This completes the proof.

The following Theorem 2.8 is an extension of [12, Theorem 1].

Theorem 2.8 *Suppose that G satisfies the following two conditions:*

(1) For $r, t \in ppcs(G) - \{1\}$, if $r \neq t$, then $r \nmid t$ and $t \nmid r$;

(2) $|ppcs(G) - \{1\}| \geq 3$.

Then, for any three distinct numbers r_1, r_2 and r_3 in $ppcs(G) - \{1\}$, if $(r_1, r_2) = 1$,then $(r_1, r_3) \neq 1$ and $(r_2, r_3) \neq 1$.

Proof It is easy to show that $pcs(G) = ppcs(G)$. Let $x \in G$ be a noncentral $p-$element for some prime p, and let $y \in C_G(x)$ be a primary $p'-$element. We have that $C_G(xy) = C_G(x) \cap C_G(y) \leq C_G(x)$. By condition (1) we conclude that $C_G(x) = C_G(xy) \leq C_G(y)$, and so $y \in Z(C_G(x))$. It follows that $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$ and $C_G(x)_{p'}$ is abelian. Suppose that $C_G(x)_{p'} \nleq Z(G)$. Let $u \in C_G(x)_{p'} - Z(G)$ be a $q-$element for some prime q. We have $q \neq p$.Reapeating the previous arguments, we conclude that $C_G(u) = C_G(u)_q \times C_G(u)_{q'}$ and $C_G(u)_{q'}$ is abelian. Then, since $C_G(x)_p \leq C_G(u)_{q'}$, $C_G(x)_p$ is abelian. It follows that $C_G(x)$ is abelian. So, we have proved that if x is a noncentral primary element of G,then (i)$C_G(x)$ is abelian, or (ii) $C_G(x) = C_G(x)_p \times Z(G)_{p'}$ for some prime p.

Suppose that $(r_1, r_3) = 1$. Let x_1, x_2 and x_3 be three noncentral primary elements of G such that $|x_i^G| = r_i$ for $i = 1, 2, 3$. We claim that $C_G(x_1)$ is abelian. Suppose on the contrary that $C_G(x_1)$ is not abelian. By the above paragraph we have that $C_G(x_1) = C_G(x_1)_p \times Z(G)_{p'}$ for some prime p. Then, we have

$$|G| = |x_1^G||C_G(x_1)| = r_1|C_G(x_1)_p|Z(G)_{p'}| = \\ r_1 p^a |Z(G)_{p'}|,$$

where a is some positive integer. It follows that $|G : Z(G)||r_1p^a$. Then, noting that $r_2||G/Z(G)|$ and $r_3||G/Z(G)|$, we get that $r_2|r_1p^a$ and $r_3|r_1p^a$. Then, since $(r_1, r_2) = (r_1, r_3) = 1$, we conclude that $r_2|p^a$ and $r_3|p^a$, and thus either $r_2|r_3$ or $r_3|r_2$, against condition (1).Hence, we have proved that $C_G(x_1)$ is abelian. Suppose that $C_G(x_2)$ is not abelian. By using the arguments used for $C_G(x_1)$ we conclude that $|G : Z(G)||r_2q^b$, where q is some prime and b is some positive integer.Then, since $r_1||G : Z(G)||r_2q^b$ and $(r_1, r_2) = 1$, we conclude that r_1 is a power of q. Since $(r_1, r_3) = 1$, we have $q \not| r_3$. Then, since $r_3||G : Z(G)||r_2q^b$ we get that $r_3|r_2$, against condition (1). Hence, $C_G(x_2)$ is also abelian.

Since $(r_1, r_2) = 1$, by Lemma 1.3 we have $G = C_G(x_1)C_G(x_2)$. Then, since $C_G(x_1)$ and $C_G(x_2)$ are abelian, we have that $C_G(x_1) \cap C_G(x_2) = Z(G)$. It follows that

$$|G : Z(G)| = |G : C_G(x_1) \cap C_G(x_2)| = |G||C_G(x_1)C_G(x_2)|/|C_G(x_1)||C_G(x_2)|,$$

$$|G : Z(G)| = |G|^2/|C_G(x_1)||C_G(x_2)| = |x_1^G||x_2^G| = r_1r_2.$$

Then, $r_3|r_1r_2$. Thus, since $(r_1, r_3) = 1$, we get that $r_3|r_2$, against condition (1). Hence, we have $(r_1, r_3) \neq 1$.

By symmetry, we have $(r_2, r_3) \neq 1$. This completes the proof of the theorem.

Corollary 2.9 *Suppose that ppcs$(G) = \{1, 2k+1, 2k+3, \cdots, 2l+1\}$ (continuous odd numbers) with $l \leq 3k$. Then, ppcs$(G) = \{1, 2k+1\}$, $2k+1 = p^a$ for some prime p, and $G = P \times A$, where P is a $p-$group and A is an abelian group.*

Proof Since $l \leq 3k$ by hypothesis,G satisfies condition (1) of Theorem 2.8. Since $(2k + 1, 2k + 3) = 1$ and $(2k + 1, 2k + 5) = 1$, by Theorem 2.8 we conclude that $|ppcs(G) - \{1\}| \leq 2$.

(i) Assume that $|ppcs(G) - \{1\}| = 2$.

In this case, $ppcs(G) = \{1, 2k + 1, 2k + 3\}$. By Theorem 2.2, $G/Z(G) = KH/Z(G)$ is a Frobenius group with the kernel $K/Z(G)$ and a complement $H/Z(G)$, and K and H are abelian. Then,we have $cs(G) = \{1, |H/Z(G)|, |K/Z(G)|\}$. It follows that $2k + 3 = |K/Z(G)|$ and $2k + 1 = |H/Z(G)|$. Then, $2k + 1 = |H/Z(G)|||K/Z(G)| - 1 = 2k + 2$(see [5, 8.3, p.497]). But, this is impossible.

(ii) Assume that $|ppcs(G) - \{1\}| = 1$.

In this case, $ppcs(G) = \{1, 2k + 1\}$. Then, by Lemma 1.7 we conclude that $2k + 1 = p^a$ for some prime p, and $G = P \times A$, where P is a $p-$group and A is an abelian group. This completes the proof of theorem.

We recall that a nonabelian group G is an $\mathbf{F}$-group if the centralizers of its noncentral elements are pairwise incomparable with respect to inclusion, that is, for $x, y \in G - Z(G)$, we have that $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$.

We have the following

Theorem A [13, Rebmann] *A nonabelian solvable group G is an* $\mathbf{F}$*-group if and only if it is one of the following types:*

(i) $G = P \times A$, where P is an $\mathbf{F}$*-group of prime power order and A is abelian;*

(ii) G has an abelian normal subgroup of prime index;

(iii) G is a quasi-Frobenius group with abelian kernel and complement;

(iv) $G/Z(G)$ is a Frobenius group with the kernel $K/Z(G)$ and a complement $L/Z(G)$, where L is abelian, $Z(K) = Z(G), K/Z(G)$ has prime power order and K is an $\mathbf{F}$*-group.*

(v) $G/Z(G) \cong S_4$, and V is not abelian if $V/Z(G)$ is the Klein four group in $G/Z(G)$.

Theorem 2.10 *Let G be a nonabelian group.Suppose that for noncentral primary or biprimary elements x, y of G,$C_G(x) \leq C_G(y)$ implies $C_G(x) = C_G(y)$. Then, $cs(G) = pcs(G) = ppcs(G)$ and G is an* $\mathbf{F}$*-group.*

Proof Let z be any noncentral element of G.We have that $o(z) = p_1^{m_1}...p_n^{m_n}$, where $p_1, ..., p_n$ are distinct primes and $m_1, ..., m_n$ are positive integers. By Lemma 1.1 we have that $z = z_1...z_n$ with $o(z_i) = p_i^{m_i}$ and $z_rz_s = z_sz_r$ for $s, r = 1, ..., n$. Since z is noncentral, some z_i is noncentral. Without loss of generality we may assume that z_1 is noncentral. For $i = 2, \cdots, n$, we have that $C_G(z_1z_i) = C_G(z_1) \cap C_G(z_i) \leq C_G(z_1)$. Then, by hypothesis we have $C_G(z_1z_i) = C_G(z_1)$. On the other hand, we have $C_G(z_1z_i) = C_G(z_1) \cap C_G(z_i) \leq C_G(z_i)$. It follows that $C_G(z_1) \leq C_G(z_i)$ for $i = 2, \cdots, n$. Hence, we have

$$C_G(z) = C_G(z_1 \cdots z_n) = C_G(z_1) \cap C_G(z_2 \cdots z_n) = C_G(z_1) \cap C_G(z_2) \cap \cdots \cap C_G(z_n) = C_G(z_1).$$

So, we have $C_G(z) = C_G(z_1)$ and we have proved that $cs(G) = pcs(G) = ppcs(G)$.

Let x, y be noncentral elements of G such that $C_G(x) \leq C_G(y)$. By the above paragraph, there exist noncentral primary elements x_1 and y_1 of G such that $C_G(x) = C_G(x_1)$ and $C_G(y) = C_G(y_1)$. Then, $C_G(x_1) \leq C_G(y_1)$. By hypothesis, we have $C_G(x_1) = C_G(y_1)$. It follows that $C_G(x) = C_G(y)$. So, we have proved that, for any noncentral elements x, y of G, $C_G(x) \leq C_G(y)$ implies $C_G(x) = C_G(y)$.

Hence, G is an **F**-group. This completes the proof of the theorem.

Theorem 2.11 *Let G be a nonabelian group, and suppose that $ppcs(G)$ satisfies the following condition: For $r, t \in ppcs(G)$, if $r \neq t$, then $r \not| t$ and $t \not| r$. Then, the following two propositions hold:*

(1) $cs(G) = pcs(G) = ppcs(G)$;

(2) If G is solvable, then G is one of the following types:

(i) $G = P \times A$, where P is a $p-$group for some prime p and A is an abelian group. Further, $ppcs(G) = cs(P) = \{1, p^a\}$, where a is a fixed positive integer;

(ii) G is a quasi-Frobenius group with abelian kernel and complement;

(iii) $G = PL$, where P is a normal Sylow $p-$subgroup of G for some prime p and L is an abelian $p-$complement of G, and $G/Z(G)$ is a Frobenius group. Furthermore, $Z(P) = Z(G) \cap P$ and $|cs(P)| = 2$.

Proof By hypothesis and Theorem 2.10 we conclude that $cs(G) = ppcs(G) = pcs(G)$ and G 1s an **F**-group. Thus, proposition (1) holds.

Now, we prove proposition (2). By the assumption of (2), G is solvable. Then, G is a solvable **F**-group. Therefore, by Theorem A, we have the following types of groups:

(a) $G = P \times A$,where P is an $\mathbf{F}-$group of prime power order and A is abelian.

In this case, P is a $p-$group for some prime p. We can assume that $A = 1$. Then $G = P$, and by hypothesis we have $ppcs(G) = cs(G) = cs(P) = \{1, p^a\}$, where a is a fixed positive integer. So, G is of type (i).

(b) G is nonabelian and has an abelian normal subgroup of prime index p.

If G is nilpotent, then it is easy to see that G is of type (i). So, we may assume that G is not nilpotent. Then, it is clear that $G = KP$, where P is a Sylow $p-$subgroup of G and K is an abelian normal $p-$complement of G, and $|P/O_p(G)| = p$.

Since K is abelian normal and $(|K|, |P|) = 1$, we have $K = [K, P] \times C_K(P)$. Clearly, $C_K(P)$ is an abelian direcy factor of G. We can assume that G has no nontrivial abelian direct factor, and so $C_K(P) = 1$ and $K = [K, P]$. Then, it is obvious that $G/O_p(G)$ is a Frobenius group with the kernel $KO_p(G)/O_p(G) \cong K$ and a complement $P/O_p(G)$ of order p. Suppose that P is nonabelian. Then, it is easy to see that $cs(G) = \{1, p, p^a|K|\}$ for some positive integer a (see also [14, Lemma 3.3]).Thus, since $ppcs(G) = cs(G)$(see (1)), $ppcs(G)$ does not satisfy the condition of the theorem, a contradiction. Therefore, P is abelian, $O_p(G) = Z(G)$ as $C_K(P) = 1$, and $G/Z(G) = KP/Z(G)$ is a Frobenius group. Then, G is of type (ii).

(c) G is a quasi-Frobenius group with abelian kernel and complement.

This is of type (ii).

(d) $G/Z(G)$ is a Frobenius group with the kernel $K_0/Z(G)$ and a complement $L_0/Z(G)$, where L_0 is abelian, $Z(K_0) = Z(G), K_0/Z(G)$ has prime power order and K_0 is an **F**-group.

In this case, $K_0/Z(G)$ is a $p-$group for some prime p. Then, $K_0 = P \times A$, where P is a Sylow $p-$subgroup of K, and A is an abelian group.Clearly, $P \lhd G$ and $Z(G) = Z(K_0) = Z(P) \times A$. Further, since K_0 is an **F**-group and A is abelian, we have $|cs(P)| = 2$. We have

$$G/Z(G) = K_0/Z(G) > \lhd L_0/Z(G)$$
$$\cong P/Z(P) > \lhd L_0/Z(G).$$

Then, since $G/Z(G)$ is a Frobenius group with the kernel $K_0/Z(G)$ and a complement $L_0/Z(G)$, we have $p \not| |L_0/Z(G)|$. Let L be the $p-$complement of L_0. Since L_0 is abelian, L is abelian. So, we have that $G = PL$ and G is of type (iii).

(e) $G/Z(G) \cong S_4$, and V is not abelian if $V/Z(G)$ is the Klein four group in $G/Z(G)$.

In this case we have $cs(G) = \{1, 6, 8, 12\}$ (see [14, p.925]), and so by (1) we get that $ppcs(G) = cs(G) = \{1, 6, 8, 12\}$, against the condition of the theorem. This completes the proof of the theorem.

We know that if $|cs(G)| \leq 3$, then G is solvable (see [14, THEOREM 2.4(Ito)]). Hence, by Theorem 2.11 we obtain two corollaries. One corollary is Lemma 1.7, and another corollary is the following:

Corollary 2.12 *Suppose that $ppcs(G) = \{1, n, m\}$ with $n < m$ and $n \not| m$. Then, G is one of the following types:*

(1) G is a quasi-Frobenius group with abelian kernel and complement;

(2) $G = PL$, where P is a normal Sylow $p-$subgroup of G for some prime p and L is an abelian $p-$complement of G, and $G/Z(G)$ is a Frobenius group. Furthermore, $Z(P) = Z(G) \cap P$ and $|cs(P)| = 2$.

Note (i) Corollary 2.12 is an improvement of a part of [14, THEOREM A].

(ii) If G is of type (2) in Corollary 2.12, then $ppcs(G) = cs(G) = \{1, p^a, p^b|L/L \cap Z(G)|\}$ and $a > b$ (see [14, Lemma 3.3 and p.924]). So, Theorem 2.2, that is, THEOREM 1.1 of [10], is also a consequence of Corollary 2.12.

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References:

[1] B.Huppert, Character Theory Of Finite Groups, Berlin: Walter der Gruyter, 1998.

[2] M.Bianchi, D.Chilag, A.G.B.Mauri, M.Herzog and C.M.Scoppola, Applications of a graph related to conjugacy classes in finite groups, Arch. Math.58,1992, pp.126-132.

[3] Q.Kong, On an extension of Camina's theorem on conjugacy class sizes, WSEAS Transactions on Mathematics, 11, 2012, pp.898-907.

[4] E.A. Bertram, M.Herzog, A.Mann, On a graph related to conjugacy classes of groups, Bull.London. Math.Soc. 22, 1990, pp.569-575.

[5] B.Huppert, Endlich Gruppen I. Berlin, Springer-Verlag, 1967.

[6] C.G.Shao, Q.H.Jiang, On conjugacy class sizes of primary and biprimary elements of a finite group, Sci.China.Math. 57(3), 2014, pp.491-498.

[7] Q.Kong, Finite Groups with Two Class Sizes of Some Elements,Mathematical Journal of Interdisciplinary Sciences, 2, 2014, pp.191-193.

[8] D.Gorenstein, Finite Groups. New York/London: Harper & Row, 1968.

[9] I.M.Issacs, Character Theory of Finite Groups, New York, Acadamic Press, 1976.

[10] Q.H.Jiang and C.G.Shao, FINITE GROUPS WITH THREE CONJUGACY CLASS SIZES OF CERTAIN ELEMENTS, PUBLICATION DE L'INSTITUT MATHEMATIQUE, 98(112), 2015, pp.265-270.

[11] N.BLACKBURN and B.HUPPERT, Endlich Gruppen III. Berlin-Heidelberg-New York 1982.

[12] Bo.Chen, SOME RESULTS OF LENGTH OF THE CONJUGACY CLASSES, Journal of Sichuan University (Natural Science Edition), 38, 2001, pp.176-179.

[13] J.REBMANN, F-gruppen, Arch.Math. 22, 1971, pp.225-230.

[14] S.Dolfi and E.Jabara, The structure of finite groups of conjugate renk 2,Bull London. Math. Soc. 41, 2009, pp. 916-926.