MCMC Method for Exponentiated Lomax Distribution based on Accelerated Life Testing with Type I Censoring

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Abstract: - Accelerated Life Testing (ALT) is an effective technique which has been used in different fields to obtain more failures in a shorter period of time. It is more economical than traditional reliability testing. In this article, we propose Bayesian inference approach for planning optimal constant stress ALT with Type I censoring. The lifetime of a test unit follows an exponentiated Lomax distribution. Bayes point estimates of the model parameters and credible intervals under uniform and log-normal priors are obtained. Besides, optimum test plan based on constant stress ALT under Type I censoring is developed by minimizing the pre-posterior variance of a specified low percentile of the lifetime distribution at use condition. Gibbs sampling method is used to find the optimal stress with changing time. The performance of the estimation methods is demonstrated for both simulated and real data sets. Results indicate that both the priors and the sample size affect the optimal Bayesian plans. Further, informative priors provide better results than non-informative priors.

Key-words:- Accelerated life testing, constant stress, exponentiated Lomax distribution, Bayes estimates, MCMC method, non informative priors, credible intervals, Monte Carlo simulation.

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1 Introduction

Market competitiveness makes it necessary for companies to produce highly reliable products having longer life time, particularly, electronic devices, computer equipment, missiles, etc. However, for such reliable products/items, it is not easy to obtain enough failure information under normal operating conditions within a specific time period. To overcome such problems, ALT is the most common approach used to ensure rapid failure of products in order to obtain enough failure data or information about life characteristics economically in a short period of time. For greater details, see [1] and [2]. However, in practice, in CSALT the test needs a longer time at low stress levels to yield sufficient failure data.

[1] pointed out that the stress can be applied in different ways such as constant stress ALT (CSALT), step-stress, progressive stress, etc and each stress level has its own advantages and disadvantages. In CSALT, each unit is tested/run at a pre-specified stress level until failure or the test is terminated for any reason. Usually, electronic items like semiconductors, capacitors, etc., run at a constant stress. Due to simplicity in test design and data analysis, CSALTs are frequently adopted in appliance manufacture industries. Conventionally, engineering experience is needed to determine the stress levels of CSALT plans ([3]; [4]). [1] provided excellent review on past developments of CSALT. [5] observed that insufficient failures may cause difficulty in estimating reliability of the product design. In the recent past, CSALT has been studied by many authorsin varied contexts. Readers may refer to the works of [6]; [7]; [8] ; [9]; [10]; [11] ; [12] ; [13]; [14] and many others. If the parameters of the model are known and precise, then one can apply maximum likelihood approach for optimal test plan by using the optimization criterion which

defines as minimization of the asymptotic variance (AV) of the MLE of t_p (100pth percentile of the lifetime distribution at the normal stress condition). Using the Delta method and the Fisher information matrix ([15], [16]), AV of the MLE of t_p can be obtained. Since the true values parameters are unknown, planning information is usually dependent on uncertainty [17]. Bayesian methods [18] is an alternative for dealing with the uncertainty. The criterion that minimizes the preposterior variance of a quantity of interest determines the optimal test plan which is generally used in Bayesian ALT designs (see, [5]; [19] and [17]). The pre-posterior variance can be computed using Monte Carlo simulation.

This paper presents CSALT model for exponentiated Lomax (ELomax) distribution based on Type I censored failure data. It is assumed that at constant stress level the shape parameter of the distribution follows log linear model. Our aim is to obtain Bayes point estimates and credible intervals under uniform and log-normal priors of the model parameters and to perform sensitivity analysis to investigate how prior distribution and sample size affects the optimal stress changing point. As far as our knowledge goes, no work was carried out to study how prior distribution and sample size affects the optimal stress changing point. We aim to fill up this gap through this work.

The rest of article is organized as follows. In Section 2, we review the ELomax model. In Section 3, we describe the Bayesian approach and Optimization criterion. Simulation study is conductedand findings are listed in Section 4. The importance of ELomax distribution is illustrated by means of aircraft windshield failure time's data set in Section 5. Finally, Section 6 offers some concluding remarks.

2 The Model

[20] introduced the ELomax distribution. The ELomax is more flexible than Lomax distribution, for more details see [21]. The cdf, pdf, reliability function and hazard rate function of the ELomax distribution are defined as follows:

$$F(t|\varphi,\beta,\theta) = \left[1-(1+\varphi t)^{-\beta}\right]^{\theta}, \quad t,\beta,\theta,\varphi > 0, \qquad (1)$$

$$f(t|\varphi,\beta,\theta) = \beta\theta\varphi(1+\varphi t)^{-(\beta+1)}\left[1-(1+\varphi t)^{-\beta}\right]^{\theta-1}, t,\beta,\theta,\varphi > 0, \qquad (2)$$

$$R(t|\varphi,\beta,\theta) = 1-\left[1-(1+\varphi t)^{-\beta}\right]^{\theta}, \quad t > 0, \qquad (3)$$

and

$$h(t|\varphi,\beta,\theta) = \frac{\beta\theta\varphi(1+\varphi t)^{-(\beta+1)}\left[1-(1+\varphi t)^{-\beta}\right]^{\theta-1}}{1-\left[1-(1+\varphi t)^{-\beta}\right]^{\theta}}, \quad t > 0. \qquad (4)$$

where θ and β are the shape parameters and φ is a scale parameter of this distribution. When φ= 1, ELomax reduces to the Exponentiated Pareto distribution. Also, when φ=θ=1, ELomax reduces to Lomax distribution. The shape of the hazard rate function could be decreasing and inverted bathtub.

3 Inference based on Type I Censored Samples

Suppose, in an experiment,there are r levels of high stress xj, j=1, 2, ..., r and the stress underuse condition is denoted by xuwhere xu< x1< x2< ...< xr. Using Type I censoring, at each stress level, the experiment terminates once all the items fail or when a fixed censoring time tcj is reached. Assuming that the lifetime at stress level xj , tij , i = 1, 2, . . . , n, j = 1, 2, . . . , r, follows ELomax distribution with pdf as given in (2). It is assumed that the stress xj affects only on the shape parameter θj of the ELomax distribution through a log linear model as follows:

$$\theta_j = exp\left(a+bx_j\right), \;\; j=1,2,\dots,r \qquad (5)$$

where a and b are two unknown parameters depending on the nature of the product.

Based on (1) and (2), the likelihood function is given by:

$$L\left(\beta,\varphi,\underline{\theta}|\underline{t}\right) = \prod_{j=1}^{r}\prod_{i=1}^{n_j}\left[\beta\varphi\theta_j(1+\varphi t_{ij})^{-(\beta+1)}\left[1-(1+\varphi t_{ij})^{-\beta}\right]^{\theta-1}\right]^{\delta_{ij}}\left[1-\left[1-(1+\varphi t)^{-\beta}\right]^{\theta}\right]^{1-\delta_{ij}}, \qquad (6)$$

where δ_{ij} is an indicator variable such that:

$$\delta_{ij}=\begin{cases}1 & for \;\; t_{ij} \le t_{cj} \\ 0 & for \;\; t_{ij} > t_{cj}\end{cases}, \qquad (7)$$

In our Bayesian analysis, the two sets of prior distributions are assumed and are reported in Table 1. We assume continuous uniform and log-normal prior distributions for the parameters a, b, β and φ to express our uncertainty in the values of the parameters. Since we do not have prior information, for illustrative purposes we use the following priors:

Table 1. Priors distribution for Bayesian analysis

Types	a	b	β	φ
P I	U(0,6)	U(0,8)	U(0,5)	U(0,7)
P II	lnor (0.001,1000)	lnor (0.001,1000)	lnor (0.001,1000)	lnor (0.001,1000)

where prior PI stands for non-informative prior and considered as, uniform distribution (U). For simplicity, let $\Omega = (a, b, \beta, \varphi)$ and the joint prior for the parameters are considered

$$\varepsilon(a, b, \beta, \varphi) \propto 1/ab\beta\varphi, \sigma_1 \le a \le \sigma_2, \sigma_3 \le b \le \sigma_4, \sigma_5 \le \beta \le \sigma_6, \sigma_7 \le \varphi \le \sigma_8, \qquad (8)$$

where $\sigma_1, \sigma_2, \ldots, \sigma_8$ are constants. Assume that the prior PII is informative prior. The informative prior pdf of Ω follows lognormal (lnor) distribution which are totally uncorrelated with the location and scale parameters $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8$, respectively and its pdf has the following form

$$\varepsilon_1(a) = \frac{1}{a\mu_2\sqrt{2\pi}} e^{-\left(\frac{ln(a)-\mu_1}{2\mu_2^2}\right)^2}, a > 0, \quad (9)$$

$$\varepsilon_2(b) = \frac{1}{b\mu_4\sqrt{2\pi}} e^{-\left(\frac{ln(b)-\mu_3}{2\mu_4^2}\right)^2}, b > 0, \quad (10)$$

$$\varepsilon_3(\beta) = \frac{1}{\beta\mu_6\sqrt{2\pi}} e^{-\left(\frac{ln(\beta)-\mu_5}{2\mu_6^2}\right)^2}, \beta > 0, \quad (11)$$

$$\varepsilon_4(\varphi) = \frac{1}{\varphi\mu_8\sqrt{2\pi}} e^{-\left(\frac{ln(\varphi)-\mu_7}{2\mu_8^2}\right)^2}, \varphi > 0, \quad (12)$$

The joint informative prior pdf for the parameter a, b, β and φ is provided as

$$\Psi(a, b, \beta, \varphi) = \frac{e^{-\left(\frac{ln(a)-\mu_1}{2\mu_2^2}+\frac{ln(b)-\mu_3}{2\mu_4^2}+\frac{ln(\beta)-\mu_5}{2\mu_6^2}+\frac{ln(\varphi)-\mu_7}{2\mu_8^2}\right)^2}}{ab\beta\varphi\mu_2\mu_4\mu_6\mu_8(2\pi)^2},$$

$$a, b, \beta, \varphi > 0, \qquad (13)$$

As a result, the posterior pdf with non-informative prior becomes:

$$\Psi 1(a, b, \beta, \varphi) \propto \Omega(a, b, \beta, \varphi) L\left(\mathrm{a}, \beta, \varphi, \underline{\theta} \middle| \underline{t}\right), \quad (14)$$

Then

$$\Psi 1(a, b, \beta, \varphi) \propto (1/ab\beta\varphi) \prod_{j=1}^{r} \prod_{i=1}^{n_j} \left[\beta\varphi\theta_j(1+\varphi t_{ij})^{-(\beta+1)}\left[1-(1+\varphi t_{ij})^{-\beta}\right]^{\theta-1}\right]^{\delta_{ij}} \left[1-\left[1-(1+\varphi t)^{-\beta}\right]^{\theta}\right]^{1-\delta_{ij}}, \qquad (15)$$

Similarly we can write the posterior distribution using informative priors and can obtained Bayes estimators.

Inference on each parameter is based on its marginal posterior density. Thus the marginal posterior density for the parameter a, b, β and φ,respectively,

$$f(a|t) = \iiint f\left(\varphi, \beta, \mathrm{b} \middle| \underline{t}\right) d\varphi d\beta db, \qquad (16)$$

$$f(b|t) = \iiint f\left(\varphi, \mathrm{a}, \beta \middle| \underline{t}\right) d\varphi da d\beta, \qquad (17)$$

$$f(\beta|t) = \iiint f\left(\varphi, \mathrm{a}, \mathrm{b} \middle| \underline{t}\right) d\varphi da db, \qquad (18)$$

$$f(\varphi|t) = \iiint f\left(\mathrm{b}, \mathrm{a}, \beta \middle| \underline{t}\right) db da d\beta. \qquad (19)$$

Gibbs sampling is used to draw a random sample of the parameters a, b, β and φ from their own marginal posterior distribution $f(a|t), f(b|t), f(\beta|t)$ and $f(\varphi|t)$, respectively, and then estimate the expected value using the sample mean. The squared error loss (SEL) function is used in order to obtain Bayes estimators from marginal posterior pdf.

3.1 Bayesian Estimators under the Squared Loss Function

The posterior mean is a well-known Bayesian estimate based on the SEL function. The SEL function is a symmetric loss function with the following formula:

$$SEL(a,\hat{a}) = c(\hat{a}-a)^2,$$

where $\hat{a}$ is an estimator of a and c is a constant. When the parameters are unknown, the Bayes estimators of a, b, β and φ under SEL can be obtained as follows:

$$\hat{a}_{SEL} = E_1(a|\underline{t}) = \iiint_0^\infty c(\hat{a} - a)^2\, \varepsilon_1(a,b,\beta,\varphi)\,db\,d\beta\,d\varphi,$$

$$\hat{b}_{SEL} = E_1(b|\underline{t}) = \iiint_0^\infty c(\hat{b} - b)^2\, \varepsilon_1(a,b,\beta,\varphi)\,da\,d\beta\,d\varphi,$$

$$\hat{\beta}_{SEL} = E_1(\beta|\underline{t}) = \iiint_0^\infty c(\hat{\beta} - \beta)^2\, \varepsilon_1(a,b,\beta,\varphi)\,db\,da\,d\varphi,$$

$$\hat{\varphi}_{SEL} = E_1(\varphi|\underline{t}) = \iiint_0^\infty c(\hat{\varphi} - \varphi)^2\, \varepsilon_1(a,b,\beta,\varphi)\,db\,d\beta\,da, \qquad (20).$$

The credible intervals for a, b, β and φ under SEL can be obtained easily, so the $100(1-\alpha)\%$ CIs for the parameters at $(L_a, U_a), (L_b, U_b), (L_\beta, U_\beta)$ and (L_φ, U_φ),are respectively satisfying

$$p(L_a \le a \le U_a) = 1-\alpha = \int_{L_a}^{U_a} f(a|t)\,da, \qquad (21)$$

$$p(L_b \le b \le U_b) = 1-\alpha = \int_{L_b}^{U_b} f(b|t)\,db \qquad (22)$$

$$p(L_\beta \le \beta \le U_\beta) = 1-\alpha = \int_{L_\beta}^{U_\beta} f(\beta|t)\,d\beta, \qquad (23)$$

$$p(L_\varphi \le \varphi \le U_\varphi) = 1-\alpha = \int_{L_\varphi}^{U_\varphi} f(\varphi|t)\,d\varphi, \qquad (24)$$

where the *L* and *U* are the lower and upper limit of the interval, respectively.

3.2 Optimization Criterion

The posterior mean and posterior variance of t_p are given by

$$E(t_p(x_u)|\underline{t}) = \int_0^\infty t_p\, f\big(t_p(x_u)\big|t\big)\,dt_p(x_u), \qquad (25)$$

and

$$V(t_p(x_u)|\underline{t}) = \int_0^\infty \Big[t_p(x_u) - E\Big(t_p(x_u)\Big)\Big]^2 f\big(t_p(x_u)\big|t\big)\,dt_p(x_u) \qquad (26)$$

respectively. The variance given by (21) cannot be used as an optimization criterion because it depends on the data. However, the preposterior variance of t_p, which is defined as:

$$E(V\Big(t_p(x_u)\Big|\underline{t}\Big) = \int_0^\infty V t_p(x_u)\, f\big(t_p(x_u)\big|t\big)\,dt_p(x_u). \qquad (27)$$

does not depend on $\underline{t}$ and therefore it will be used as the objective function. The optimal changing time will be the one to minimize$E(V\Big(t_p(x_u)\Big|\underline{t}\Big)$ where $t_p = \frac{1}{\varphi}\left[1-\left(1-p^{\frac{1}{\theta}}\right)^{\frac{-1}{\beta}}\right]$, and p is the p^{th} percentile.

4 Simulation Algorithm

It's simple to simulate a data vector with $t_p = \frac{1}{\varphi}\left[1-(1-)p^{\frac{1}{\theta_u},}{}^{\frac{-1}{\beta}}\right]$ two-step process. To begin, the joint prior distribution is used to generate a set of values for the model parameters a, b, β and φ. Second, the conventional inverse transformation method is used to simulate n i.i.d. failure times from the distribution function provided by (2). The right-censored observations are simulated failure times that are longer than the censoring time t_c. The posterior variance $V(t_p|t)$ supplied by (21) must be evaluated for each simulated data vector. The Gibbs sampling method is utilized in this work to select a random sample of t_p from the marginal posterior distribution $f(t_p|t)$ and estimate $V(t_p|t)$ using the sample variance. The preposterior variance of t_p is estimated by averaging $V(t_p|t)$ over all simulated data vectors. The preposterior variance of t_p is evaluated at discrete points in time to determine the ideal stress changing time of the preposterior variance of t_p, in order to discover the ideal stress changing point.

4.1 Simulation

The following are the simulation steps: The R language is used to produce accelerated life data from the ELomax distribution at different sample sizes (20,40) at $\varphi = 1.2\, \beta = 1.1, \theta = 1.9$. To eliminate posterior dependency on a simulation's starting point, numerous chains with over-dispersed beginning points should be conducted in a single MCMC simulation.

- For three chains, the initial values of a, b, β *and* $\varphi$are as follows: The first chain has beginning values of a = 0.1, b = 0.1,$\beta = 0.1$ *and* $\varphi = 0.3$; the second chain has beginning values of a = 0.3, b = 0.7, $\beta = 0.2$ *and* $\varphi = 0.4$; and the third chain has beginning values of a =0.5, b=0.9,$\beta = 0.3$ *and* $\varphi = 0.5$.
- This is a non-formal method of determining convergence. The Monte Carlo standard error (MC error) of the mean is used to calculate the precision of a posterior estimate. According to [22], MC error for each estimate should be less than 5% of the sample standard deviation (SSD. This rule has been followed in this situation; see Appendix B for more information.
- Assume the experiment ends when all of the items fail or when a specific censoring time t is reached (Type I censoring). When n =n =10, t_{c1}=15,t_{c2}=7, n =n =20, t_{c1}=19,t_{c2}=7, and n =n =30, t_{c1}=19,t_{c2}=7.
- It is assumed that values of parameters are known and apply Bayesian method to determine the optimal stress changing point. The objective function is to minimize the asymptotic variance of the p^{th} percentile at normal stress, low stress and high stress. Tables 2-5 summarizes the optimal stress changing times $t_c^*$and the corresponding posterior variance of tp at p = 0.1. The summary of the samples are displayed in Tables 2-5. It is observed that as sample size increases, accuracy of the estimates improves. Although the stress level saves the experiment time, the results are better in usual conditions. As expected, we observe that the variance of the estimated parameters decreases when we assume informative prior. It is also reasonably to conclude that the interval length is narrower with informative priors. The interval of the parameters a, b, β, φ ,$\theta_u, t_p(x_u)$, $t_p(x_l)$ and $t_p(x_h)$, becomes narrower as the sample size increases. Tables 2-5 summarise the results of the samples. As the sample size grows larger, the accuracy of the estimates improves. The results are better in normal conditions, despite the fact that the stress level saves the experiment time. When we assume an informative prior, the variance of the calculated parameters appears to decrease, as expected. With informative priors, it is likewise reasonable to conclude that the interval length is shorter. As the sample size grows, the interval of the parameters a, b, β, φ ,$\theta_u, t_p(x_u)$, $t_p(x_l)$ and $t_p(x_h)$, becomes narrower.

Table 2. Posterior summaries of the model parameter at n = 20

P	Estimate	Mean	Sd	MC error	median	95% CI	Interval length
I	$\hat{a}$	0.3588	0.2067	0.00142	0.3443	(0.0265, 0.7994)	0.7729
	$\hat{b}$	0.12040	0.1013	5.011E-4	0.09409	(0.0037,0.3724)	0.3687
	$\hat{\beta}$	3.8210	0.827	0.00917	3.961	(2.025, 4.953)	2.928
	$\hat{\varphi}$	0.0511	0.03035	3.916E-4	0.043	(0.0258,0.1213)	0.0955
	$\hat{\theta}_u$	1.5490	0.3146	0.002336	1.495	(1.115,2.297)	1.1820
	$\hat{t}_p(x_u)$	1.6600	0.3365	0.001105	1.639	(1.065,2.3710)	1.3060
	$\hat{t}_p(x_l)$	1.8360	0.3782	0.00101	1.819	(1.149,2.6260)	1.4770
	$\hat{t}_p(x_h)$	1.5011	0.3508	0.001473	1.47	(0.9223,2.2590)	1.3367
II	$\hat{a}$	0.9607	0.02867	7.598E-5	0.9603	(0.9057,1.0180)	0.1123
	$\hat{b}$	0.9421	0.02744	7.369E-5	0.9417	(0.8893,0.9971)	0.1078
	$\hat{\beta}$	1.0660	0.03015	8.116E-5	1.065	(1.008 , 1.126)	0.1180
	$\hat{\varphi}$	1.0340	0.0319	8.778E-5	1.033	(0.9727, 1.097)	0.1243
	$\hat{\theta}_u$	4.1880	0.1289	3.345E-4	4.185	(3.945 , 4.4500)	0.505
	$\hat{t}_p(x_u)$	2.1240	0.1322	3.131E-4	2.119	(1.877,2.3950)	0.5180
	$\hat{t}_p(x_l)$	3.5770	0.2548	6.099E-4	3.566	(3.1050, 4.1050)	1.000
	$\hat{t}_p(x_h)$	1.2040	0.06838	1.623E-4	1.203	(1.076,1.345 0)	0.2690

Table 3. Optimum Bayesian design under Type I censoring at n=20

	P	stress	$E(var(t_p \mid \underline{t}))$	t^*_{c1}	t^*_{c2}
n	I	x_u	0.1925	3.6575	1.3475
		x_l	0.2324	4.4156	1.6268
		x_h	0.2846	5.4075	1.9922
	II	x_u	0.07556	1.4383	0.5293
		x_l	0.1573	2.9887	1.1011
		x_h	0.3158	6.0002	2.2106

Table 4. Posterior summaries of model parameters at n=40

P	Estimate	Mean	Sd	MC error	median	95% CI	Interval length
I	$\hat{a}$	0.1409	0.1117	7.427E-4	0.1155	)0.0047,0.4103)	0.4056
	$\hat{b}$	0.1786	0.1064	8.383E-4	0.1695	(0.0117,0.4076)	0.3959
	$\hat{\beta}$	3.691	0.8279	0.01215	3.764	(2.0360,4.9400)	2.9040
	$\hat{\varphi}$	0.03417	0.01657	2.636E-4	0.02919	(0.0181,0.0786)	0.0605
	$\hat{\theta}_u$	1.265	0.1313	0.001108	1.24	(1.0800,1.5870)	0.5070
	$\hat{t}_p(x_u)$	1.903	0.2525	8.067E-4	1.889	(1.4470,2.4330)	0.9860
	$\hat{t}_p(x_l)$	2.245	0.3598	0.00123	2.23	(1.5890,2.9870)	1.3980
	$\hat{t}_p(x_h)$	1.603	0.2675	0.001632	1.566	(1.1760,2.2190)	1.0430
	$\hat{a}$	0.9475	0.0278	7.621E-5	0.9472	(0.894,1.003)	0.1090
	$\hat{b}$	0.9317	0.02618	7.47E-5	0.9315	(0.8813,0.9836)	0.1023
	$\hat{\beta}$	1.073	0.02712	7.744E-5	1.072	(1.0200,1.1270)	0.1070

II	$\hat{\varphi}$	1.042	0.03185	9.015E-5	1.042	(0.9813,1.106)	0.1247
	$\hat{\theta}_u$	4.112	0.1204	3.399E-4	4.108	(3.884, 4.356)	0.4720
	$\hat{t}_p(x_u)$	2.025	0.104	2.179E-4	2.022	(1.8300, 2.2380)	0.4080
	$\hat{t}_p(x_l)$	3.391	0.1976	4.162E-4	3.385	(3.0220, 3.7960)	0.7740
	$\hat{t}_p(x_h)$	1.155	0.05573	1.215E-4	1.153	(1.0500,1.2680)	0.2180

Table 5. Optimum Bayesian design under Type I censoring at n=40

	P	stress	$E(var(t_p \mid \underset{\sim}{t}))$	t^*_{c1}	t^*_{c2}
n	I	x_u	0.1192	1.192	2.384
		x_l	0.1662	1.662	3.324
		x_h	0.2333	2.333	4.666
	II	x_u	0.0603	0.603	1.206
		x_l	0.1856	1.856	3.712
		x_h	0.5206	5.206	10.412

5 Application

In this section, for illustration purpose, we analyse one real data set which represents the failure times of 84 Aircraft windshield. This data set was first discussed by [23]. These data were recently studied by [24] and [25]. The data set is divided into two samples, n_1 and n_2 where ($n_1 + n_2 = 84$). n_1 units have failed during the interval (0, t_c) and $n_2(84 - n_1)$ units are still active, where a, b, β, φ are the population parameters. For achieving the convergence, we apply three chains of the Brooks-Gelman-Rubin statistics for a given parameter. The summary of the real data set with respect to the unknown parameters$a$, b, β, φ ,θ_u, $t_p(x_u)$, $t_p(x_l)$ and $t_p(x_h)$, where p = 0.1, are displayed in Table 6.

- Assume that the experiment is terminated once all the items fail or when a fixed censoring time t_{cj} is reached (Type I censoring) . t_{c1}=2.49 and t_{c2}=3.69 at n_1 and n_2, respectively.
- From Table 7 ,it is observed that the MC error for each estimate is less than 5% of the SSD, thus the rule of MC error has been achieved. Also, to check the convergence, Gelman-Rubin convergence statistic, R, is applied.
- For WinBUGS simulation convergence, R should be one, or close to one.
- The two-sided 95% credible intervals for the estimates of parameters θ_u, $t^p(x^u)$, $t^p(x^l)$ and $t^p(x^h)$ of ELomax distribution are reported in Table 6. As the sample size increases, length of the interval becomes narrower.
- It appears that in case of informative prior, the variance of estimated parameters decreases.

Table 6. Posterior statistics of model parameters under Type I censoring

P	Estimate	Mean	Sd	MC error	median	95% CI	Interval length
I	$\hat{a}$	0.4155	0.2634	0.004032	0.3840	(0.0245,0.9966)	0.9721
	$\hat{b}$	0.9960	0.1998	0.003059	1.0190	(0.5565,1.3100)	0.7535
	$\hat{\beta}$	0.9950	0.0049	2.102E-5	0.9966	(0.9819,0.9999)	0.0180
	$\hat{\varphi}$	1.9110	0.0828	3.600E-4	1.9350	(1.6900,1.9980)	0.3080
	$\hat{\theta}_u$	2.5300	0.4535	0.006768	2.4450	(1.9260,3.616)	1.6900
	$\hat{t}_p(x_u)$	0.3560	0.0781	0.001457	0.3370	(0.2300,0.5890)	0.3590
	$\hat{t}_p(x_l)$	0.7041	0.0963	9.33E-4	0.6960	(0.5754,0.8769)	0.3015
	$\hat{t}_p(x_h)$	1.3040	0.1165	8.598E-4	1.3010	(1.0830,1.5390)	0.4560
II	$\hat{a}$	0.9351	0.0276	7.934E-5	0.9347	(0.8824, 0.9904)	0.1080
	$\hat{b}$	0.9233	0.02633	8.242E-5	0.923	(0.8726, 0.9758)	0.1032
	$\hat{\beta}$	1.411	0.03529	1.212E-4	1.41	(1.342, 1.481)	0.1395
	$\hat{\varphi}$	1.126	0.03286	1.002E-4	1.126	(1.063, 1.192)	0.1290
	$\hat{\theta}_u$	4.044	0.1193	3.484E-4	4.041	(3.819, 4.287)	0.4680
	$\hat{t}_p(x_u)$	0.7171	0.02729	5.838E-5	0.7165	(1.7200 ,2.031)	1.3110
	$\hat{t}_p(x_l)$	1.191	0.0458	8.642E-5	1.19	(1.283, 1.1030)	0.1800
	$\hat{t}_p(x_h)$	1.87	0.07924	1.504E-4	1.869	(2.031, 1.72)	0.3110

Table7. Optimum Bayesian design under Type I censoring

P	stress	$E(var(t_p)\mid \underline{t})$	t^*_{c1}	t^*_{c2}
I	x_u	0.0026	0.0065	0.0182
	x_l	0.0096	0.0239	0.0672
	x_h	0.0329	0.0819	0.2303
II	x_u	0.0099	0.0247	0.0365
	x_l	0.0273	0.0679	0.1007
	x_h	0.0673	0.1676	0.2483

6 Conclusion

In this study, we present Bayesian analysis for the type I censored data under CSALT for the exponentiated Lomax distribution usinglog-linear life-stress function. We have investigated how prior distribution and sample size affects the optimal stress changing point. We observe that the variance of the estimated parameters decreases when we assume informative prior. We also observe that length of the credible intervals for the parameters of interest becomes narrower as the sample size increases. Similar results echoed in the real data analysis.It would be of great interest for statistician/ reliability engineers to study the different classical methods of estimation under CSALT for the exponentiated Lomax distribution and log linear acceleration model based on type II, progressive type II censoring data. The work in this direction is in progress and will be reported later. Although we have considered log-linear acceleration model, however, there are several other models where it can be applied.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Refah Alotaibi and H. Rezk carried out the methodology, simulation and the optimization.
Sanku Dey was responsible for editing and review.

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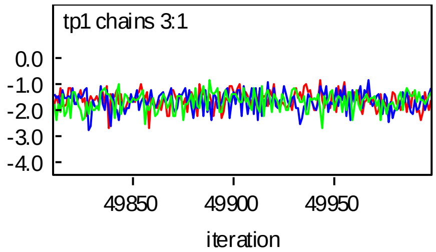
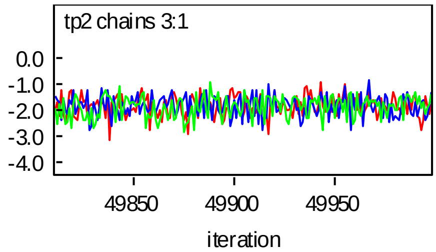
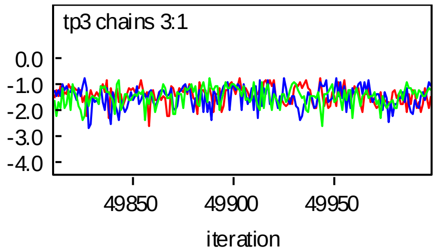
Appendix A



Trace of low stress ($t_p(x_l)$)

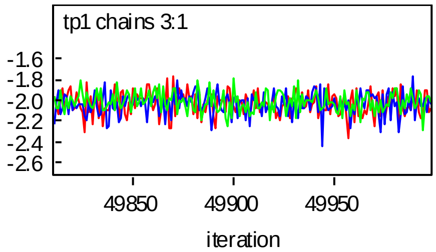


Trace of high stress $t_p(x_h)$

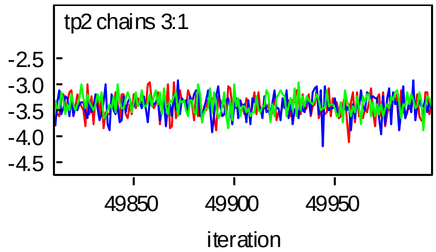


Trace of normal condition $t_p(x_u)$

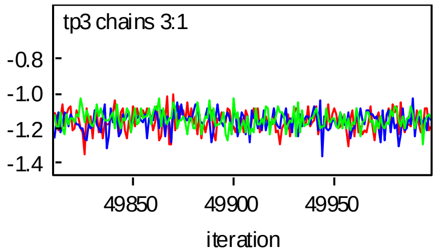
Fig. 1 The trace of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$ for informative prior at n=20



Trace of low stress ($t_p(x_l)$)

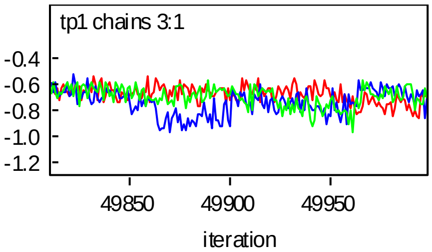


Trace of high stress $t_p(x_h)$

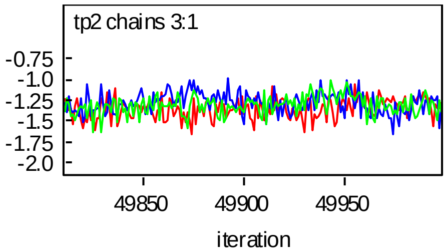


Trace of normal condition $t_p(x_u)$

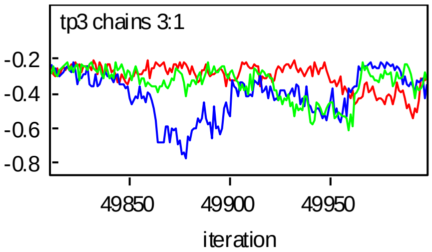
Fig. 2 The trace of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$ for informative prior at n=40



Trace of low stress ($t_p(x_l)$)

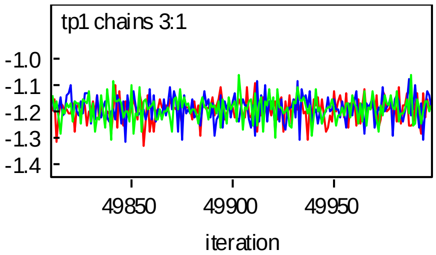


Trace of high stress ($t_p(x_h)$)

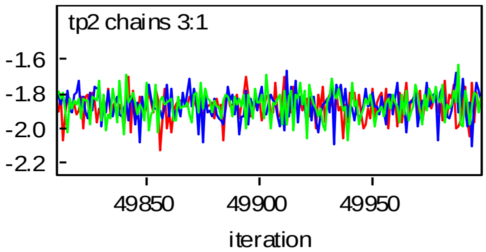


Trace of normal condition ($t_p(x_u)$)

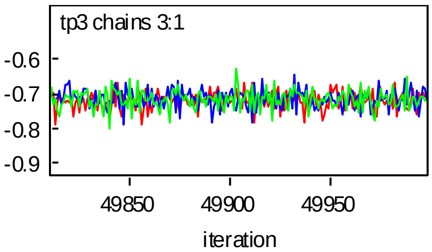
Fig. 3 Trace of non-informative prior for $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$ in real data



Trace of low stress ($t_p(x_l)$)



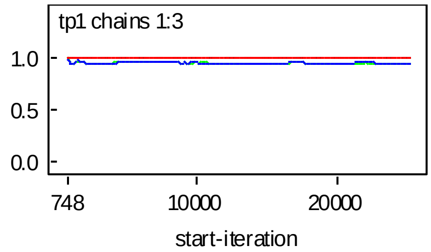
Trace of high stress ($t_p(x_h)$)



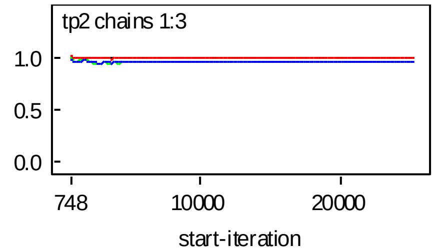
Trace of normal condition ($t_p(x_u)$)

Fig. 4 Trace of non-informative prior for $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$ in real data

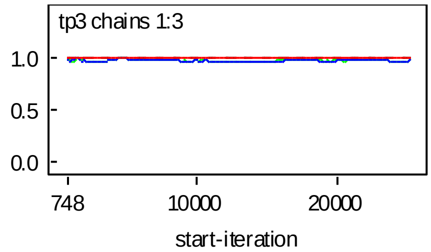
Appendix B



The Gelman Rubin of low stress ($t_p(x_l)$)

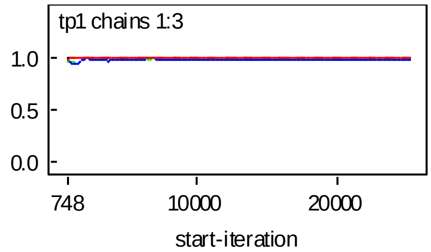


The Gelman Rubin of high stress $t_p(x_h)$

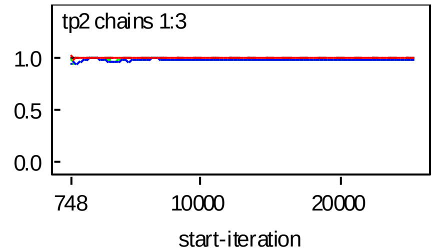


The Gelman Rubin of normal condition $t_p(x_u)$

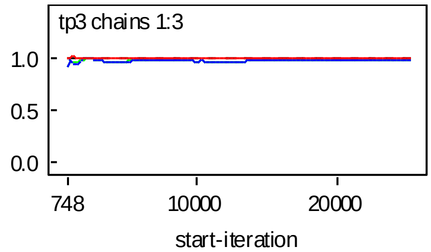
Fig. 5 The Gelman Rubin of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$, at n=20 with informative prior



The Gelman Rubin of low stress ($t_p(x_l)$)

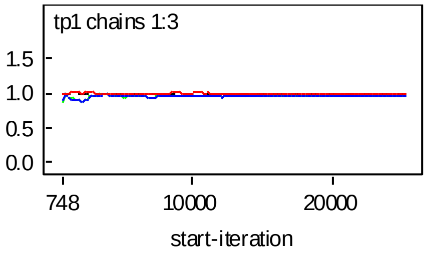


The Gelman Rubin of high stress $t_p(x_h)$

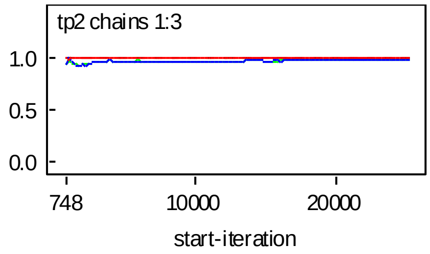


The Gelman Rubin of normal condition $t_p(x_u)$

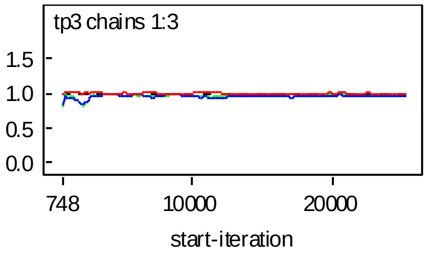
Fig.6 The Gelman Rubin of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$, at n=40 with informative prior



The Gelman Rubin of low stress ($t_p(x_l)$)

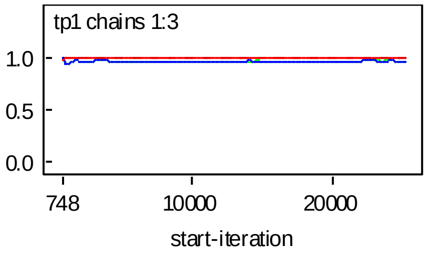


The Gelman Rubin of high stress $t_p(x_h)$

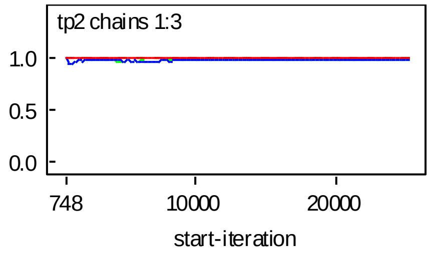


The Gelman Rubin of normal condition $t_p(x_u)$

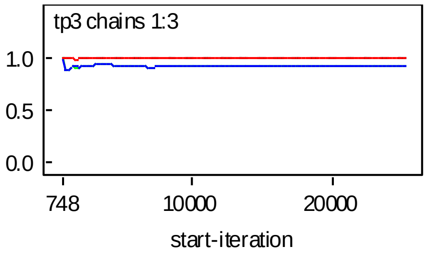
Fig. 7 The Gelman Rubin of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$, in real data with informative prior



The Gelman Rubin of low stress ($t_p(x_l)$)



The Gelman Rubin of high stress $t_p(x_h)$



The Gelman Rubin of normal condition $t_p(x_u)$

Fig. 8 The Gelman Rubin of $t_p(x_l)$, $t_p(x_h)$ and $t_p(x_u)$, in real data with non informative prior