Set-valued contractions in b-Metric Spaces with Application

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Abstract: -In this article, we present a (α, F)-set-valued mapping in setting b-metric space by characterizing the weak contraction condition with the $\mathcal{C}$ function and the α-set-valued function of type S. There are examples and implementations accessible that illustrate the validity of our findings.

Key-Words: Set-valued mapping, Fixed point, b-metric space, α-admissible mapping of type S, Nadler's fixed point, Multi-valued, Hausdorff-Pompieu metric

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1 Introduction

Banach's contraction principle is one of the most useful tools in fixed point theory. Edelstein [1] proved the following version of the Banach contraction principle.

Let (X, d) be a compact metric space and let $f : X \to X$ be a self-mapping. Assume that $d(fx, fy) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. Then f has a unique fixed point in X.

Later, Suzuki [2] proved generalized versions of Edelstein's results in compact metric space as follows.

Let (X, d) be a compact metric space and let $f : X \to X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2}d(x, fx) < d(x, y) \Rightarrow d(fx, fy) < d(x, y)$$

Then f has a unique fixed point in X.

The fixed point theory for set-value mapping was developed after Nadler's famous paper [3].

Let (X, d) be a complete metric space and f be a multi-valued map on X such that fx is a nonempty closed bounded subset of X for any $x \in X$. If there exists $c \in (0, 1)$ such that

$$H(fx, fy) \leq cd(x, y), \quad \forall\ x, y \in X,$$

then f has a fixed point in X.

Several authors have defined multiple value mapping terms using the concept of the Hausdorff-Pompieu metric. $i.e.$,

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Then H is a metric on $CB(X)$, is the class of all nonempty closed and bounded subsets of X.

Aydi et al. [4] introduced a multi-valued mapping in b-metric spaces as follow:

Let (X, d_b) be a complete b-metric space and let $f : X \to CB(X)$ be a multi-valued mapping such that for all $x, y \in X$,

$$H(fx, fy) \leq cd_b(x, y),$$

where $0 \leq c < \dfrac{1}{s^2 + s} < 1$ and

$$M(x, y) = \max\{d_b(x, y), d_b(x, fx), d_b(y, fy), d_b(x, fy), d_b(y, fx)\}.$$

Then f has a fixed point in X, that is, there exists $u^* \in X$ such that $u^* \in fx$.

It is generally expanded or explained in different directions and many (general) fixed point theorems have been identified (see [6, 7, 8, 9, 11, 12, 13, 14, 15]).

In this work, we present a fixed point for (α, F)-set-valued mapping in setting b-metric space.

In Section 2, we present definition lemma in b-metric space.

In Section 3, we prove the fixed point theorem for (α, F)-set-valued mapping in setting b-metric space and give an example for support our theorem.

In Section 4, we show application are available to demonstrate the reliability of the our result.

2 Preliminaries

Definition 2.1. [17, 18] *Let X be a nonempty set and*

$s \geq 1$ be a given real number. Take $d_b : X \times X \to [0, \infty)$. Suppose that for all $x, y, z \in X$, we have:

(C1) $d_b(x, y) = 0$ *if and only if* $x = y$;

(C2) $d_b(x, y) = d_b(x, y)$;

(C3) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$.

Then, d_b is said a b-metric and the triplet (X, d_b, s) is called a b-metric space.

Let (X, d_b, s) be a b-metric space. Let $\{x_n\}$ be a sequence in X.

(C1) *$\{x_n\} \subseteq X$ converges to a point $u^* \in X$ if* $\lim_{n\to\infty} d_b(x_n, u^*) = 0$;

(C2) *$\{x_n\} \subseteq X$ is Cauchy if, for each $\epsilon > 0$, there is some $n(\epsilon) \in \mathbb{N}$ such that $d_b(x_n, x_m) < \epsilon$ for all $m, n \geq n(\epsilon)$;*

(C3) *(X, d_b, s) is said complete if any Cauchy sequence is convergent in X.*

Lemma 2.2. [19] *Let (X, d_b, s) be a b-metric space with $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\epsilon > 0$ and $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ two sequences of positive integers such that

(C1) $\epsilon \leq \liminf_{k\to\infty} d_b(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to\infty} d_b(x_{m(k)}, x_{n(k)}) \leq s\epsilon$;

(C2) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) \leq \limsup_{k\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon$;

(C3) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \leq \limsup_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon$;

(C4) $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \leq \limsup_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \leq s^3\epsilon$;

Lemma 2.3. [19] *Let (X, d_b, s) be a b-metric space with $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent sequences to u^* and v^*, respectively. Then,*

$$\frac{1}{s} d_b(u^*, v^*) \leq \liminf_{n\to\infty} d_b(x_n, y_n) \leq \limsup_{n\to\infty} d_b(x_n, y_n) \leq s^2 d_b(u^*, v^*).$$

Lemma 2.4. [20] *Let (X, d_b, s) be a b-metric space. For $A \in CL(X)$ and $x \in X$, we have*

$$d_b(x, A) = 0 \Leftrightarrow x \in \bar{A} = A, \tag{1}$$

where the closure of the set A is denoted by $\bar{A}$.

Denote by 2^X (resp. $CL(X)$) the family of subsets (resp. of closed subsets) of X. Let $CB(X)$ be the class of all nonempty closed bounded subsets of X.

Lemma 2.5. [21] *Assume that $fx \in CL(X)$ for each $x \in X$. If f is upper semi-continuous then $Gr(f)$ is closed in X^2.*

Definition 2.6. [22] *Let X be a nonempty set. Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. Let $s \geq 1$ be a given real number. We say that f is weak α-admissible of type S if for $x \in X$ and $\alpha(x, fx) \geq s$, then $\alpha(fx, ffx) \geq s$.*

Definition 2.7. [23] *Let X be a nonempty set. Given $f : X \to CL(X)$ and $\alpha : X \times X \to [0, \infty)$. Let $s \geq 1$ be a given real number. Such that f is said to be α-admissible of type S if for each $x \in X$ and $y \in fx$ with $\alpha(x, y) \geq s$, we have $\alpha(y, z) \geq s$ for each $z \in fy$.*

Definition 2.8. [24] *A mapping $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a C-class function if it is continuous and for $s, t \in [0, \infty)$, F satisfies the following two conditions:*

(C1) $F(r, t) \leq r$;

(C2) $F(r, t) = r$ *implies that either* $r = 0$ *or* $t = 0$.

The family of C-class functions is denoted by $\mathcal{C}$.

3 Main Results

Definition 3.1. *Let (X, d_b, s) be a b-metric space with constant $s \geq 1$ and $f : X \to CL(X)$ such that f is an (α, F)-set-valued mapping if there exist $\alpha : X \times X \to [0, \infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$, $\theta \in \Theta$ and $\phi \in \Phi$ such that*

$$\begin{aligned} & x, y \in X \text{ with } \alpha(x, y) \geq s \\ & \Rightarrow H(fx, fy) \\ & \leq F(\psi(M(x, y)) + \theta(N(x, y)), \phi(M(x, y))), \end{aligned} \tag{2}$$

where

$$\begin{aligned} M(x, y) = \max \Big\{ & d_b(x, y), \frac{d_b(y, fy)[1 + d_b(x, fx)]}{1 + d_b(x, y)}, \\ & \frac{d_b(y, fx)[1 + d_b(x, fy)]}{1 + d_b(x, y)}, \\ & \frac{d_b(x, fy) + d_b(y, fx)}{2s} \Big\} \end{aligned} \tag{3}$$

and

$$\begin{aligned} N(x, y) = \min\{ & d_b(x, y), d_b(x, fx), d_b(y, fy), \\ & d_b(x, fy), d_b(y, fx)\}. \end{aligned} \tag{4}$$

Theorem 3.2. *Let (X, d_b, s) be a b-metric space with constant $s \geq 1$ and $f : X \to CL(X)$ be an (α, F)-admissible set-valued mapping. Assume that:*

(C1) *f is α-admissible of type S;*

(C2) *there exist $x_0 \in X$ and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq s$;*

(C3) *$Gr(f)$ is a closed subset of X^2.*

Then, f has a fixed point.

Proof. Using condition (C2), we have $x_0 \in X$ and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq s$. If $x_0 = x_1$ or $x_1 \in fx_1$, we deduce that x_1 is a fixed point of f and hence the proof is done. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin fx_1$. Using Lemma 2.4, $d_b(x_1, fx_1) > 0$. It following equation (2), we have

$$\begin{aligned}0 &< d_b(x_1, fx_1)\\ &\leq H(fx_0, fx_1)\\ &\leq F(\psi(M(x_0, x_1)) + \theta(N(x_0, x_1)), \phi(M(x_0, x_1))),\end{aligned} \tag{5}$$

where

$$\begin{aligned}&M(x_0, x_1)\\ &= \max\Big\{d_b(x_0, x_1), \frac{d_b(x_1, fx_1)[1 + d_b(x_0, fx_0)]}{1 + d_b(x_0, x_1)},\\ &\qquad \frac{d_b(x_1, fx_0)[1 + d_b(x_0, fx_1)]}{1 + d_b(x_0, x_1)},\\ &\qquad \frac{d_b(x_0, fx_1) + d_b(x_1, fx_0)}{2s}\Big\}\\ &\leq \max\Big\{d_b(x_0, x_1), d_b(x_1, fx_1), d_b(x_1, fx_0),\\ &\qquad \frac{d_b(x_0, fx_1)}{2s}\Big\}\\ &\leq \max\Big\{d_b(x_0, x_1), d_b(x_1, fx_1),\\ &\qquad \frac{d_b(x_0, x_1) + d_b(x_1, fx_1)}{2s}\Big\}\\ &\leq \max\big\{d_b(x_0, x_1), d_b(x_1, fx_1)\big\}.\end{aligned} \tag{6}$$

Suppose now $\max\{d_b(x_0, x_1), d_b(x_1, fx_1)\} = d_b(x_1, fx_1)$, then by equation (5) becomes

$$\begin{aligned}&d_b(x_1, fx_1)\\ &\leq F(\psi(d_b(x_1, fx_1)) + \theta(N(x_0, x_1)), \phi(d_b(x_1, fx_1))).\end{aligned} \tag{7}$$

But

$$\begin{aligned}&N(x_0, x_1)\\ &= \min\{d_b(x_0, x_1), d_b(x_0, fx_0), d_b(x_1, fx_1),\\ &\qquad d_b(x_0, fx_1), d_b(x_1, fx_0)\}\\ &= 0.\end{aligned} \tag{8}$$

Thus,

$$\begin{aligned}0 &< d_b(x_1, fx_1)\\ &\leq F(\psi(d_b(x_1, fx_1)), \phi(d_b(x_1, fx_1)))\end{aligned} \tag{9}$$

Using $\psi(t) < t$ for each $t > 0$, we obtain

$$\begin{aligned}0 &< d_b(x_1, fx_1)\\ &\leq F(\psi(d_b(x_1, fx_1)), \phi(d_b(x_1, fx_1)))\\ &\leq \psi(d_b(x_1, fx_1))\\ &< d_b(x_1, fx_1),\end{aligned} \tag{10}$$

which is a contradiction. Hence, $\max\{d_b(x_0, x_1), d_b(x_1, fx_1)\} = d_b(x_0, x_1)$. Using again equation (5) and the fact that ψ is nondecreasing and θ is a continuous function, we obtain that

$$0 < d_b(x_1, fx_1) \leq \psi(d_b(x_0, x_1)).$$

This implies that there exists $x_2 \in fx_1$ (of course, $x_2 = x_1$) such that

$$0 < d_b(x_1, x_2) < \psi(d_b(x_0, x_1)).$$

Because $\alpha(x_0, x_1) \geq s$, $x_1 \in fx_0$ and $x_2 \in fx_1$, by the fact that f is α-admissible, we have $\alpha(x_1, x_2) \geq s$. If $x_2 \in fx_2$, x_2 is a fixed point of f. Otherwise, $x_2 \notin fx_2$, so we have $d_b(x_2, fx_2) > 0$. It following equation (2), we have

$$\begin{aligned}0 &< d_b(x_2, fx_2)\\ &\leq H(fx_1, fx_2)\\ &\leq F(\psi(M(x_1, x_2)) + \theta(N(x_1, x_2)), \phi(M(x_1, x_2))),\end{aligned} \tag{11}$$

where

$$\begin{aligned}&M(x_1, x_2)\\ &= \max\Big\{d_b(x_1, x_2), \frac{d_b(x_2, fx_2)[1 + d_b(x_1, fx_1)]}{1 + d_b(x_1, x_2)},\\ &\qquad \frac{d_b(x_2, fx_1)[1 + d_b(x_1, fx_2)]}{1 + d_b(x_1, x_2)},\\ &\qquad \frac{d_b(x_1, fx_2) + d_b(x_2, fx_1)}{2s}\Big\}\\ &\leq \max\Big\{d_b(x_1, x_2), d_b(x_2, fx_2), d_b(x_2, fx_1),\\ &\qquad \frac{d_b(x_1, fx_2)}{2s}\Big\}\\ &\leq \max\Big\{d_b(x_1, x_2), d_b(x_2, fx_2),\\ &\qquad \frac{d_b(x_1, x_2) + d_b(x_2, fx_2)}{2s}\Big\}\\ &\leq \max\big\{d_b(x_1, x_2), d_b(x_2, fx_2)\big\}.\end{aligned} \tag{12}$$

Similarly as above, we obtain that $\max\{d_b(x_1, x_2), d_b(x_2, fx_2)\} = d_b(x_1, x_2)$. Using (11) and (12),

$$0 < d_b(x_2, fx_2) \leq \psi(d_b(x_1, x_2)) < \psi^2(d_b(x_0, x_1)).$$

This implies again that there exists $x_3 \in fx_2$ (of course, $x_3 = x_2$) such that

$$0 < d_b(x_2, x_3) < \psi^2(d_b(x_0, x_1)).$$

Because $\alpha(x_1, x_2) \geq s$, $x_2 \in fx_1$ and $x_3 \in fx_2$, by the fact that f is α-admissible, we have $\alpha(x_2, x_3) \geq s$. If $x_3 \in fx_3$, x_3 is a fixed point of f. Otherwise, $x_3 \notin fx_3$, so we have $d_b(x_3, fx_3) > 0$. In the same way, we get

$$0 < d_b(x_3, fx_3) \leq \psi(d_b(x_2, x_3)) < \psi^3(d_b(x_0, x_1)).$$

By continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_n \notin fx_n$, $x_{n+1} \in fx_n$, $\alpha(x_n, x_{n+1}) \geq s$ and

$$0 < d_b(x_n, fx_n) \leq d_b(x_n, x_{n+1}) \leq \psi^n(d_b(x_0, x_1))$$

for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ be such that $m > n$. Then,

$$\begin{aligned} d_b(x_n, x_m) &\leq \sum_{i=n}^{m-1} s^{i-n+1} d_b(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} s^i \psi^i(d_b(x_0, x_1)) \end{aligned}$$

Because $\psi \in \Psi$, $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (X, d_b). Thus, there exists $u^* \in X$ such that $u_n \to u^*$. Because $x_{n+1} \in fx_n$, we have $(x_n, x_{n+1}) \in Gr(f)$. The graph is closed, so as $n \to \infty$, we obtain that $(x_n, x_{n+1}) \to (u^*, u^*)$, with $(u^*, u^*) \in Gr(f)$. We deduce that $u^* \in fu^*$, that is, u^* is a fixed point of f. □

Theorem 3.3. *Let (X, d_b, s) be a b-metric space with constant $s \geq 1$ and $f : X \to CL(X)$ be an (α, F)-set-valued mapping. Assume that:*

(C1) *f is α-admissible of type S and ψ, θ are continuous*;

(C2) *there exist $x_0 \in X$ and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq s$;*

(C3) *if $\{x_n\}$ is a sequence in X with $x_n \to u^* \in X$ and $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, u^*) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then, f has a fixed point.

Proof. From the proof of Theorem 3.2, there exists a sequence $\{x_n\}$ such that

$$x_{n+1} \in fx_n,\ x_n \notin fx_n \text{ and } \alpha(x_n, x_{n+1}) \geq s \quad (13)$$

for all $n \in \mathbb{N} \cup \{0\}$. Next, we will show $\{x_n\}$ is a Cauchy sequence in X, which converges to some u^* as $n \to \infty$. Using condition (C3), we obtain

$$\alpha(x_n, u^*) \geq s \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (14)$$

If $u^* \in fu^*$, the proof is completed. We assume that $d_b(u^*, fu^*) > 0$. Then

$$\begin{aligned} 0 &< d_b(u^*, fu^*) \\ &\leq s[d_b(u^*, x_{n+1}) + d_b(x_{n+1}, fu^*)] \\ &\leq sd_b(u^*, x_{n+1}) + sH(fx_n, fu^*) \\ &\leq sd_b(u^*, x_{n+1}) \\ &\quad + sF(\psi(M(x_n, u^*)) + \theta(N(x_n, u^*), \phi(M(x_n, u^*)))) \\ &\leq sd_b(u^*, x_{n+1}) + s[\psi(M(x_n, u^*)) + \theta(N(x_n, u^*)] \end{aligned} \quad (15)$$

where

$$\begin{aligned} &M(x_n, u^*) \\ &= \max\Big\{d_b(x_n, u^*), \frac{d_b(u^*, fu^*)[1 + d_b(x_n, fx_n)]}{1 + d_b(x_n, u^*)}, \\ &\qquad \frac{d_b(u^*, fx_n)[1 + d_b(x_n, fu^*)]}{1 + d_b(x_n, u^*)}, \\ &\qquad \frac{d_b(x_n, fu^*) + d_b(u^*, fx_n)}{2s}\Big\} \\ &\leq \max\Big\{d_b(x_n, u^*), d_b(u^*, fu^*), d_b(u^*, fx_n), \\ &\qquad \frac{d_b(x_n, fu^*) + d_b(u^*, fx_n)}{2s}\Big\} \\ &\leq \max\Big\{d_b(x_n, u^*), d_b(u^*, fu^*), d_b(u^*, x_{n+1}), \\ &\qquad \frac{d_b(x_n, fu^*) + d_b(u^*, x_{n+1})}{2s}\Big\} \end{aligned} \quad (16)$$

and

$$\begin{aligned} &N(x_n, u^*) \\ &= \min\{d_b(x_n, u^*), d_b(x_n, fx_n), d_b(u^*, fu^*), \\ &\qquad d_b(x_n, fu^*), d_b(u^*, fx_n)\}. \end{aligned} \quad (17)$$

Taking $n \to \infty$, we have $\limsup_{n\to\infty} M(x_n, u^*) \leq d_b(u^*, fu^*)$ and $\limsup_{n\to\infty} N(x_n, u^*) \leq 0$. Using the continuity of ψ and θ, we have $\limsup_{n\to\infty} \psi(M(x_n, u^*)) \leq \psi(d_b(u^*, fu^*))$ and $\limsup_{n\to\infty} \psi(N(x_n, u^*)) \leq \psi(0) = 0$. Taking $n \to \infty$ in equation (15), we obtain

$$0 < d_b(u^*, fu^*) \leq s\psi(d_b(u^*, fu^*)) < d_b(u^*, fu^*),$$

which is a contradiction. Hence, $u^* \in fu^*$ and so f has a fixed point. □

Corollary 3.4. *Let (X, d_b, s) be a b-metric space with a constant $s \geq 1$ and $f : X \to CL(X)$ be an (α, F) set-valued mapping. Assume that there exist $\alpha : X \to [0, \infty)$, $\psi \in \Psi$, $\theta \in \Theta$, and $\phi \in \Phi$ such that*

$$\begin{aligned} &\alpha(x, y)H(fx, fy)) \\ &\leq F(\psi(M(x, y)) + \theta(N(x, y), \phi(M(x, y))) \end{aligned} \quad (18)$$

where $M(x, y)$ and $N(x, y)$ were defined by (4) *and* (5) *for all $x, y \in X$. Assume that:*

(C1) *f is α-admissible of type S and ψ, θ are continuous*;

(C2) *there exist $x_0 \in X$ and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq s$*;

(C3) *$\{x_n\}$ is a sequence in X with $x_n \to u^* \in X$ and $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, u^*) \geq s$) for all $n \in \mathbb{N} \cup \{0\}$.*

Then, f has a fixed point.

Proof. Using inequality (18) and the contraction (2) holds for all $x, y \in X$ such that $\alpha(x, y) \geq s$. Thus, f is an (α, F)-set-valued mapping. Using Theorem 3.3, f has a fixed point. □

Corollary 3.5. *Let (X, d_b, s) be a b-metric space with constant $s \geq 1$ and $f : X \to CL(X)$ be an (α, F)-set-valued mapping. Suppose there exist $\alpha : X \to [0, \infty)$, $\psi \in \Psi$, $\theta \in \Theta$, and $\phi \in \Phi$ such that*

$$\begin{aligned} &\alpha(x, y)H(fx, fy)) \\ &\leq F(\psi(M(x, y)) + \theta(N(x, y), \phi(M(x, y))) \end{aligned} \tag{19}$$

$M(x, y)$ and $N(x, y)$ were defined by (4) *and* (5) *for all $x, y \in X$. Assume that:*

(C1) *f is α-admissible of type S*;

(C2) *there exist $x_0 \in X$ and $x_1 \in fx_0$ such that $\alpha(x_0, x_1) \geq s$*;

(C3) *the graph of f is closed.*

Then, f has a fixed point.

Proof. Using inequality (19) and the contraction (2) holds for all $x, y \in X$ with $\alpha(x, y) \geq s$. Thus, f is an (α, F)-set-valued mapping. Using Theorem 3.3, the set-valued mapping f has a fixed point. □

Corollary 3.6. *Let (X, d_b, s) be a $b-$metric space with constant $s \geq 1$ and $f : X \to X$. Suppose there exist $\alpha : X \times X \to [0, \infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$, $\theta \in$, and $\phi \in$ such that*

$$\begin{aligned} &x, y \in X \text{ with } \alpha(x, y) \geq s \\ &\Rightarrow d_b(fx, fy)) \\ &\leq F(\psi(M(x, y)) + \theta(N(x, y)), \phi(M(x, y))), \end{aligned} \tag{20}$$

$M(x, y)$ and $N(x, y)$ were defined by (4) *and* (5) *for all $x, y \in X$. Assume that:*

(C1) *$\sigma, \eta \in X$, $\alpha(\sigma, \eta) \geq s$ implies $\alpha(f\sigma, f\eta) \geq s$. Also, ψ, θ are continuous*;

(C2) *there exists $\sigma_0 \in X$ such that $\alpha(\sigma_0, f\sigma_0) \geq s$*;

(C3) *if for $\mu_0 \in X$, the sequence $\{\mu_n = f^n\mu_0\}$ in X is such that $f^n\mu_0 \to \mu \in X$ and $\alpha(f^n\mu_0, f^{n+1}\mu_0) \geq s$ for each integer $n \geq 0$, then $\alpha(f^n\mu_0, \mu) \geq s$ for each $n \geq 0$.*

Then, f has a fixed point.

Example 3.7. *Let $X = [0, \infty)$ be endowed with the b-metric $d_b(x, y) = (x - y)^2$ with $s = 3$ for all $x, y \in X$. Define $f : X \to CL(X)$ and $\alpha : X \to [0, \infty)$ by*

$$fx = \begin{cases} [0, \frac{x}{9}] & \text{if } x \in [0, 1] \\ [x, x^2] & \text{if } x \in (1, \infty) \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 3 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Define the functions by $\psi(t) = t$, $\theta(t) = t$ and $\phi(t) = \frac{80}{81}t$. Take $F(r, t) = r - t$ for all $r, t \in [0, \infty)$.

Firstly, we show that f is α-admissible of type S. Let $x \in X$ and $y \in fx$ with $\alpha(x, y) \geq s = 3$. Then, $x, y \in [0, 1]$. Let $u \in fy$, then $u \in [0, \frac{y}{9}] \subset [0, \frac{x}{81}] \subset [0, 1]$. Then,

$$\alpha(y, u) = 3 = s.$$

Thus, f is α-admissible of type S. For $x_0 = \frac{1}{3}$ and $x_1 = \frac{1}{9} \in fx_0$, we have $\alpha(x_0, x_1) = 3 = s$. For any sequence $\{x_n = f^n x_0\} \subseteq X$ (where $x_0 \in X$ is arbitrary) such that $x_n \to u^*$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) = 3 = s$ for each $n \in \mathbb{N}$, we have x_n, $u^* \in [0, 1]$, and $\alpha(x_n, u^*) = 3 = s$ for each $n \in \mathbb{N}$.

Next, we will show that the conditions of Theorem 3.3 are fulfilled for all $x, y \in X$ with $\alpha(x, y) \geq s$, that is, $x, y \in [0, 1]$ with $x \neq y$. Then,

$$H(fx, fy) = \frac{(x - y)^2}{81}.$$

Hence,

$$\begin{aligned} &H(fx, fy) \\ &= \frac{(x - y)^2}{81} \\ &= \frac{1}{81} d_b(x, y) \\ &\leq M(x, y) \\ &= F(\psi(M(x, y)) + \theta(N(x, y)), \phi(M(x, y))). \end{aligned}$$

All hypotheses of Theorem 3 are satisfied and f has a fixed point.

4 Application

Let X be the set of continuous functions specified on the closed interval $[a, b]$. We endow X by the standard b-metric $d_b : X \times X \to [0, \infty)$:

$$d_b(x, y) = (\sup_{t \in [a, b]} |x(t) - y(t)|)^2,$$

for all $x, y \in X$. Then, (X, d_b) is a complete b-metric space with constant $s = 3$. We consider the following integral equation:

$$y(r) = y_0 + \int_a^b Q(r,t)R(t,y(t))\,dt \qquad (21)$$

where $y_0 \in \mathbb{R}$ and $Q : [a,b] \times [a,b] \to [0,\infty)$, $R : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Define $f : X \to X$ as

$$fy(r) = y_0 + \int_a^b Q(r,t)R(t,y(t))\,dt. \qquad (22)$$

Then, a solution of equation (21) is equivalent to stating that the map f has a fixed point.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(C1) *there exists* $\alpha : X \times X \to [0,\infty)$ *such that if* $\alpha(\sigma,\eta) \geq s = 3$ *for* $\sigma, \eta \in X$, *we have for each* $t \geq 0$,

$$|R(t,\sigma(t) - R(t,\eta(t)| \leq \sqrt{\ln(1 + (|\sigma(t) - \eta(t)|)^2)},$$

and

$$\sup_{r\in[a,b]} \int_a^b Q(r,t)\,dt \leq 1;$$

(C2) $\sigma, \eta \in X$, $\alpha(\sigma,\eta) \geq 1$ *implies* $\alpha(f\sigma, f\eta) \geq 1$;

(C3) *there exists* $\sigma_0 \in X$ *such that* $\alpha(\sigma_0, f\sigma_0) \geq s = 3$;

(C4) *if* $\{\mu_n =: f^n\mu_0\}$ *(where* μ_0 *is arbitrary in* X*) is a sequence in* X *with* $\mu_n \to \mu \in X$ *and* $\alpha(\mu_n, \mu_{n+1}) \geq s$ *for each integer* $n \geq 0$, *then* $\alpha(\mu_n, \mu) \geq s$ *for each* $n \geq 0$.

Then, the integral equation (21) *has a solution in* X.

Proof. For all $\sigma, \eta \in X$, we have

$$d_b(\sigma,\eta) = (\sup_{t\in[a,b]} |\sigma(t) - \eta(t)|)^2,$$

and for $r \in [a,b]$, we obtain

$$\begin{aligned}
&(|f\sigma(r) - f\eta(r)|)^2 \\
&= (|\int_a^b Q(r,t)|R(t,\sigma(t)) - R(t,\eta(t))|dt|)^2 \\
&\leq (\int_a^b Q(r,t)|R(t,\sigma(t)) - R(t,\eta(t))|dt)^2 \\
&\leq (\int_a^b Q(r,t)\sqrt{\ln(1 + (|\sigma(t) - \eta(t)|)^2)}dt)^2 \\
&\leq (\int_a^b Q(r,t)\sqrt{\ln(1 + d_b(\sigma,\eta))}dt)^2 \\
&\leq (\int_a^b Q(r,t)dt)^2 \ln(1 + d_b(\sigma,\eta)) \\
&\leq \ln(1 + d_b(\sigma,\eta)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&H(f\sigma, f\eta) \\
&\leq \ln(1 + d_b(\sigma,\eta)) \\
&\leq \ln(1 + M(\sigma,\eta) + N(\sigma,\eta)) \\
&= M(\sigma,\eta) + N(\sigma,\eta) \\
&-(M(\sigma,\eta) + N(\sigma,\eta) - \ln(1 + M(\sigma,\eta) + N(\sigma,\eta))) \\
&= F(\psi(M(\sigma,\eta)) + \theta(N(\sigma,\eta)), \phi(M(\sigma,\eta))),
\end{aligned}$$

where $\psi(t) = t$, $\theta(t) = t$ (it is continuous), $\phi(t) = t - \ln(1 + t)$, and $F(s,t) = s - t$. Using condition (C3) and (C4) hold, all hypotheses of Corollary 4 hold. Thus, f has a fixed point, that is, the integral equation (21) has a solution in X. □

5 Conclusion

We introduced the existence and uniqueness of fixed point results for (α, F)-admissible set-valued mappings in b-metric spaces using $\mathcal{C}$-functions and α-admissible set-valued mappings of type S in this paper. To illustrate the superiority of our results, we provided an example and an application of integral equations.

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