Toeplitz like operators on 2-nuclear tensor product of hardy spaces

AMANI SHATARAH
Department of Mathematics
Taibah University
Al Madina, SAUDI ARABIA

Abstract: in this paper, we investigated Toeplitz like operators on vector valued Hardy spaces. Toeplitz like operators on 2-nuclear tensor product of Hardy spaces are then constructed and described using the theory of p-nuclear tensor product of Banach spaces, and their basic algebraic properties and spectrum are analyzed.

Key-Words: Toeplitz like operators , 2-nuclear tensor product of hardy spaces.

Received: January 26, 2021. Revised: May 5, 2021. Accepted: May 21, 2021. Published: June 2, 2021.

1 Introduction

Toeplitz (1911), introduce Toeplitz (1911) introduced Toeplitz operators, and Douglas (1972) provided the sense in which Toeplitz operators appeared as a matrix operating on space $ell2(N)$, see [1] and [6]. Brown and Halmos (1964) studied Toeplitz operators as a composition of a multiplier of $L2$ and a projection on $H2$ (Hardy space) in a systematic way, see [5].

Toeplitz operators in multiple variables were studied by Davie, Jewell, and Mc Donald (1977).Douglas and Pearcy(1965) investigated generalized Toeplitz operators (see [4]).

According to Brown, Halmos, and Douglas, the main focus of my research is on a 2-nuclear tensor product of Hardy space, as seen in [2] and [3].On the tensor product space, a new operator is created that is not a Toeplitz operator but has a matrix representation that is close to that of a Toeplitz operator, hence the name Toeplitz like operators.

Some important concepts and properties of Toeplitz-like operators are discussed in this paper, as well as the spectrum and invariability of the new operator (see [4]).

Finally, Possible applications of this study can be found in problems of [7]and [8].

2 Preliminaries

Assume $H^2(T^2)\otimes_{n(2)}H^2(T^2)$ be the 2-nuclear tensor product of Hardy spaces on torus. Then $H^2(T^2)\otimes_{n(2)}H^2(T^2)$ is defining as the space which contains all functions with the following representation

$$\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} d_{n_1,n_2}\otimes b_{n_1,n_2},$$

with

$$(\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} \|d_{n_1,n_2}\|_2^2)^{\frac{1}{2}}\ .$$

$$\sup_{\|b^*\|\leq 1}(\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} |< b_{n_1,n_2}, b^* >|^2)^{\frac{1}{2}} < \infty,$$

where $d_{n_1,n_2},\ b_{n_1,n_2}\in H^2(T^2)$.

It is clear to see that $H^2(T^2)\otimes_{n(2)}H^2(T^2)$ is a Hilbert space with norm

$$\|\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} d_{n_1,n_2}\otimes b_{n_1,n_2}\|_{n(2)} =$$

$$\inf\{(\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} \|d_{n_1,n_2}\|_2^2)^{\frac{1}{2}}.$$

$$\sup_{\|b^*\|\leq 1}(\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} |< b_{n_1,n_2}, b^* >|^2)^{\frac{1}{2}}\},$$

where the infimum is taken over all representations of

$$\sum_{n_1\in Z_{n_1},n_2\in Z_{n_2}} d_{n_1,n_2}\otimes b_{n_1,n_2}.$$

Lemma 1:

Let $H^2(T^2)\otimes_{n(2)}H^2(T^2)$ be 2-nuclear tensor product of Hardy spaces on torus. Then $H^2(T^2)\otimes_{n(2)}H^2(T^2)$ is a closed subspace of $L^2(T^2)\otimes_{n(2)}L^2(T^2)$.

Remark 1:

1. We will consider $P_1 \otimes P_1 : L^2(T^2)\otimes_{n(2)} L^2(T^2) \rightarrow H^2(T^2)\otimes_{n(2)} H^2(T^2)$ is a unique

orthogonal projection, where P_1 is the orthogonal projection from $L^2(T^2)$ onto $H^2(T^2)$.

2. Let $\psi = \psi_1 \otimes \psi_2$, where $\psi_1, \psi_2 \in L^\infty(T^2)$. Then $\psi.A \in L^2(T^2) \otimes_{n(2)} L^2(T^2)$, for all $A \in L^2(T^2) \otimes_{n(2)} L^2(T^2)$.

3. $L^2(T^2, L^2(T^2))$ denotes the vector space of all 2-Bochner integrable functions (equivalence classes) from (T^2, σ) into $L^2(T^2)$, where is σ is a Haar measure.
   For $\omega \in L^2(T^2, L^2(T^2))$, define $\|\omega\|_{B(2)} = (\int_{T^2} \|\omega(t_1,\ t_2)\|_2^2\, d\sigma)^{\frac{1}{2}}$.

4. $\{e^{in_1\theta_1} e^{in_2\theta_2} e^{in_3\theta_3}\ e^{in_4\theta_4} : n_1,\ n_2\ , n_3\ , n_4\ \in Z\}$ is an orthonormal basis of $L^2(T^2, L^2(T^2))$, Then we can define the function $\omega\ in\ L^2(T^2, L^2(T^2))$ as:
   $\omega(t_1, t_2)(\theta_1, \theta_2) = \sum_{n_1,n_2 \in Z} \omega_{n_1,n_2}(t_1, t_2) e^{in_1\theta_1} e^{in_2\theta_2}$,
   where
   $\omega_{n_1,n_2} \in L^2(T^2), and \sum_{n_1,n_2 \in Z} \|\omega_{n_1,\ n_2}\|_2^2 < \infty.$

5. The vector valued Hardy space on torus $H^2(T^2, H^2(T^2))$ is the closed subspace of $L^2(T^2, L^2(T^2))$ consisting of all functions ω such that
   $\omega(t_1, t_2)(\theta_1, \theta_2) = \sum_{n_1,n_2 \in Z} \omega_{n_1,n_2}(t_1, t_2)\, e^{in_1\theta_1}\ e^{in_2\theta_2}$,
   with $\omega_{n_1,n_2}(t_1, t_2) = 0$ for all $n_1, n_2 > 0$ and $\omega_{n_1,n_2} \in H^2(T^2)$.

6. Let X_1 and X_2 be Hilbert spaces. Then a pseudo inner product on $X_1 \otimes X_2$ is defined by

$$\langle x_1 \otimes x_2, x_3 \otimes x_4\rangle = \langle x_1, x_3\rangle_X\ \langle x_2, x_4\rangle_Y.$$

We refer the reader to [6], for more about tensor product of Banach spaces.

Now, we will present some important results mentioned for Toeplitz operators which we will use in our study of Toeplitz like operators.

Proposition 1:

Let $\omega \in L^\infty(T^2),\ and\ \omega_1\ and\ \overline{\omega_2}$ be functions in $H^\infty(T^2)$ $(L^\infty(T^2) \cap H^2(T^2))$. Then $T_\omega T_{\omega_1} = T_{\omega\omega_1}$ and $T_{\omega_2} T_\omega = T_{\omega_2\omega}$.

Theorem 1:

Let $\omega_1,\ \omega_2 \in L^\infty(T^2)$. Then $T_{\omega_1}\ T_{\omega_2} = T_{\omega_2}\ T_{\omega_1}$ if and only if one of the following conditions are satisfied:
i. $\omega_1\ and\ \omega_2$ are analytic.
ii. $\omega_1\ and\ \omega_2$ are co-analytic .
iii. $\omega_2 = \alpha\omega_1 + c,\ where\ \alpha \in C\ and\ c\ is\ constant.$

Corollary 1:

Assume that T_ω is an invertible Toeplitz operator. Then T_ω^{-1} is a Toeplitz operator if and only if φ is analytic or co-analytic.

Corollary 2:

T_ω is compact operator if and only if $\omega = 0$.

Corollary 3:

Suppose $\omega \in L^\infty(T^2)$. Then $\sigma(T_\omega)$ is connected.

We refer the reader to Douglas [1], and Brown and Halmos , for more about Toeplitz operators on Hardy spaces.

In the end of the preliminaries we will mention the key theorem of this paper.

3 Definition and first properties

Theorem 2:

The complex valued vector space $H^2(T^2, H^2(T^2))$ is isometrically isomorphic to the 2-nuclear Tensor product of Hardy spaces $H^2(T^2) \otimes_{n(2)} H^2(T^2)$.

Proof:

Assume that
$$W : H^2(T^2) \otimes_{n(2)} H^2(T^2) \to H^2(T^2, H^2(T^2))$$
, is defined as
$$W(\sum_{n_1 \in Z_{n_1}, n_2 \in Z_{n_2}} d_{n_1,n_2} \otimes b_{n_1,n_2})(t_1,\ t_2) = \sum_{n_1 \in Z_{n_1}, n_2 \in Z_{n_2}} d_{n_1,n_2}(t_1, t_2)\ b_{n_1,n_2}.$$

It is obvious W is a linear operator.

Now, we will show that W is a contraction. So, if we take $\omega = W(\sum_{n_1 \in Z_{n_1}, n_2 \in Z_{n_2}} d_{n_1,n_2} \otimes b_{n_1,n_2})$, then $\|\omega\|_{B(2)} = (\int_{T^2} \|\omega(t_1, t_2)\|_2^2\, d\sigma(t_1, t_2))^{\frac{1}{2}}$, Hence, by the Hahn-Banach theorem, we get
$\|\omega(t_1, t_2)\|_2 = \sup_{\|t\|=1} |\langle\omega(t_1, t_2), t\rangle|,\ where\ t \in H^2(T^2).$

Thus; we have

$$\|\omega\|_{B(2)} = (\int_{T^2} \sup_{\|t\|=1} | \sum_{n_1,n_2 \in Z} d_{n_1,n_2}(t_1,t_2) \langle b_{n_1,n_2}, t\rangle|^2 d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$\leq (\int_{T^2} \sup_{\|t\|=1} (\sum_{n_1,n_2 \in Z} |d_{n_1,n_2}(t_1,t_2)||\langle b_{n_1,n_2}, t\rangle|)^2 d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$\leq (\int_{T^2} \sup_{\|t\|=1} ((\sum_{n_1,n_2 \in Z} |d_{n_1,n_2}(t_1,t_2)|^2)^{\frac{1}{2}} (\sum_{n_1,n_2 \in Z} |\langle b_{n_1,n_2}, t\rangle|^2)^{\frac{1}{2}} . d\sigma(t_1,t_2))^{\frac{1}{2}} \; by\, Schwartz\, inequality$$

$$\leq (\int_{T^2} \sup_{\|t\|=1} ((\sum_{n_1,\, n_2 \in Z} |d_{n_1,\, n_2}(t_1,t_2)|^2). (\sum_{n_1,\, n_2 \in Z} |\langle b_{n_1,n_2}, t\rangle|^2)\, d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$= (\int_{T^2} (\sum_{n_1,\, n_2 \in Z} |d_{n_1,\, n_2}(t_1,t_2)|^2)\, d\sigma(t_1,t_2))^{\frac{1}{2}} (\sum_{n_1,\, n_2 \in Z} (\sup_{\|t\|=1} |\langle b_{n_1,n_2}, t\rangle|^2))^{\frac{1}{2}}$$

$$\leq \| \sum_{n_1 \in Z_{n_1}, n_2 \in Z_{n_2}} d_{n_1,n_2} \otimes b_{n_1,n_2} \|_{n(2)}.$$

Notice that, the set of all elements of the form

$$\sum_{n_1 \in Z_{n_1}, n_2 \in Z_{n_2}} d_{n_1,n_2} \otimes b_{n_1,n_2}$$

are dense in $H^2(T^2) \otimes_{n(2)} H^2(T^2)$, So, we have $\|W(H)\|_{B(2)} \leq \|H\|_{n(2)}$, for all $H \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$.

Hence $\|W\|_{B(2)} \leq 1$.

Now, Assume that $S \in H^2(T^2, H^2(T^2))$. Then $S(t_1,t_2) \in H^2(T^2)$, and

$$S(t_1,t_2)(\theta_1,\theta_2) = \sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2}(t_1,t_2)\, e^{in_1\theta_1}\, e^{in_2\theta_2}$$

, with $\|S(t_1,t_2)\| = \sum_{n_1,\, n_2 \in Z} |d'_{n_1,\, n_2}|^2 < \infty$. So; we can write S in the form

$$S = W(\sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2} \otimes e_{n_1,\, n_2})$$

, where $e_{n_1,\, n_2}(\theta_1,\theta_2) = e^{in_1\theta_1}\, e^{in_2\theta_2}$. And also,

$$\|W(\sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2} \otimes e_{n_1,\, n_2})\| = \|S\| = (\int_{T^2} \|S(t_1,t_2)\|^2\, d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$= (\int_{T^2} \sum_{n_1,\, n_2 \in Z} |d'_{n_1,\, n_2}(t_1,t_2)|^2\, d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$= (\sum_{n_1,\, n_2 \in Z} \int_{T^2} |d'_{n_1,\, n_2}(t_1,t_2)|^2\, d\sigma(t_1,t_2))^{\frac{1}{2}}$$

$$= (\sum_{n_1,\, n_2 \in Z} \|d'_{n_1,\, n_2}\|^2)^{\frac{1}{2}}.$$

But, $\sup_{\|t\|=1} \sum_{n_1,\, n_2 \in Z} |\langle e_{n_1,\, n_2}, t\rangle|^2)^{\frac{1}{2}} = 1$, for all $t \in H^2(T^2)$. Thus;

$$\|W(\sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2} \otimes e_{n_1,\, n_2})\|_{B(2)} = \| \sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2} \otimes e_{n_1,\, n_2})\|_{n(2)}.$$

And we can write S as

$$S = \sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2}\, e_{n_1,\, n_2} = W(\sum_{n_1,\, n_2 \in Z} d'_{n_1,\, n_2} \otimes e_{n_1,\, n_2}).$$

Thus; W is an isometry operator and onto.

Which this implies $H^2(T^2, H^2(T^2))$ is isometrically isomorphic to $H^2(T^2) \otimes_{n(2)} H^2(T^2)$.

Definition 1:

Assume that $\psi = \psi_1 \otimes \psi_2$, where $\psi_1, \psi_2 \in L^\infty(T^2)$ and T_ψ is an operator defined as

$$T_\psi : H^2(T^2)n(2)\otimes H^2(T^2) \to H^2(T^2)n(2)\otimes H^2(T^2)$$

such that

$$\begin{aligned} T_{\psi_1\otimes\psi_2}(d \otimes b) &= P_1 \otimes P_1((\psi_1 \otimes \psi_2)(d \otimes b)) \\ &= P_1(\psi_1 d) \otimes P_1(\psi_2 b). \end{aligned}$$

Then the operator T_ψ is said to be Toeplitz like operator with symbol ψ.

Lemma 2:

$T_{\phi_1\otimes\phi_2}$ is linear.

Proof:

It is easy to see the proof.

Now, we will go to write the form of the matrix of a Toeplitz like operator.

The orthonormal basis of $X \otimes Y$ and the order of these basis have been studied by Holub [3]. Hence an orthonormal basis of $H^2(T^2) \otimes_{n(2)} H^2(T^2)$ is $\{e^{in_1\theta_1}e^{in_2\theta_2} \otimes e^{in_3\theta_3}e^{in_4\theta_4} : n_1, n_2, n_3, n_4 \in Z^+\}$.

WLOG, we will order the sequence of tensors $(e^{in_1\theta_1}e^{in_2\theta_2} \otimes e^{in_3\theta_3}e^{in_4\theta_4})$ as the following

$\underline{1 \otimes 1}$ | $1 \otimes e^{i(\theta_3+\theta_4)}$ | $1 \otimes e^{i2(\theta_3+\theta_4)}$ | ...
$\underline{e^{i(\theta_1+\theta_2)} \otimes 1}$ | $\underline{e^{i(\theta_1+\theta_2)} \otimes e^{i(\theta_3+\theta_4)}}$ | ...
$\underline{e^{i2(\theta_1+\theta_2)} \otimes 1}$ | $\underline{e^{i2(\theta_1+\theta_2)} \otimes e^{i\theta_3+\theta_4)} \quad e^{i2(\theta_1+\theta_2)} \otimes e^{i2(\theta_3+\theta_4)}}$ |
...
$\underline{e^{i3(\theta_1+\theta_2)} \otimes 1}$ | $\underline{e^{i3(\theta_1+\theta_2)} \otimes e^{i(\theta_3+\theta_4)} \quad e^{i3(\theta_1+\theta_2)} \otimes e^{i2(\theta_3+\theta_4)}}$ |
...

And also, these basis is called the tensor product basis. Now, suppose that $q_0 = 1 \otimes 1$, $q_1 = 1 \otimes e^{i(\theta_3+\theta_4)}$, $q_2 = e^{i(\theta_1+\theta_2)} \otimes 1$, $q_3 = 1 \otimes e^{i2(\theta_3+\theta_4)}$, $q_4 = e^{i(\theta_1+\theta_2)} \otimes e^{i(\theta_3+\theta_4)}$, $q_5 = e^{i2(\theta_1+\theta_2)} \otimes 1$, ...

Now, we will construct the matrix of a Toeplitz like operator on 2-nuclear tensor product of Hardy spaces on torus with respect to the orthonormal basis $\{e^{in_1\theta_1}e^{in_2\theta_2} \otimes e^{in_3\theta_3}e^{in_4\theta_4} : n_1, n_2, n_3, n_4 \in Z^+\}$.

Let $\psi_1 \otimes \psi_2 \in L^\infty(T^2) \otimes_{n(\infty)} L^\infty(T^2)$. Then $\psi_1,\ \psi_2$ are in $L^2(T^2)$.

Thus; $\psi_1 = \sum_{n_1,\ n_2 \in N} d_{n_1,\ n_2}\, e^{in_1\theta_1}\, e^{in_2\theta_2}$, and $\psi_2 = \sum_{n_1,\ n_2 \in N} d'_{n_3,\ n_4}\, e^{in_3\theta_3}\, e^{in_4\theta_4}$.

Let $R = (r_{ij})$ be the matrix representation of $T_{\psi_1 \otimes \psi_2}$. Then $(r_{ij}) = \langle T_{\psi_1 \otimes \psi_2} q_i, q_j \rangle$.

Now, we will give an example to see how we compute :

$(r_{44}) = \langle T_{\psi_1 \otimes \psi_2} q_4, q_4 \rangle$

$= \langle P_1(\psi_1.e^{i(\theta_1+\theta_2)}) \otimes P_1(\psi_2.e^{i(\theta_3+\theta_4)}),$
$\mathrm{e}^{i(\theta_1+\theta_2)} \otimes e^{i(\theta_3+\theta_4)} \rangle$

$= \langle P(\psi_1.e^{i(\theta_1+\theta_2)}), e^{i(\theta_1+\theta_2)} \rangle.$
$\langle P(\psi_2.e^{i(\theta_3+\theta_4)}), e^{i(\theta_3+\theta_4)} \rangle$

$= \langle \psi_1.e^{i(\theta_1+\theta_2)}, e^{i(\theta_1+\theta_2)} \rangle \langle \psi_2.e^{i(\theta_3+\theta_4)}, e^{i(\theta_3+\theta_4)} \rangle$

$= \langle \sum_{n_1,\ n_2 \in N} d_{n_1,\ n_2}\, e^{i(n_1+1)\theta_1}\, e^{i(n_2+1)\theta_2}, e^{i(\theta_1+\theta_2)} \rangle.$
$\langle \sum_{n_1,\ n_2 \in N} d'_{n_1,\ n_2}\, e^{i(n_3+1)\theta_3}\, e^{i(n_4+1)\theta_4}, e^{i(\theta_3+\theta_4)} \rangle$

$= \sum_{n_1,\ n_2 \in N} d_{n_1,\ n_2} \langle e^{i(n_1+1)\theta_1}\, e^{i(n_2+1)\theta_2}, e^{i(\theta_1+\theta_2)} \rangle.$
$\sum_{n_1,\ n_2 \in N} d'_{n_1,\ n_2} \langle e^{i(n_3+1)\theta_3}\, e^{i(n_4+1)\theta_4}, e^{i(\theta_3+\theta_4)} \rangle$

$= \mathrm{d}_{0,0}\ d'_{0,0}$

Then Continue in this procedure to get matrix representation of Toeplitz like operator.

Now, we will go to study the important properties of Toeplitz like operators.

Theorem 3:

Assume that ξ_1, ξ_2, ξ_3, and $\xi_4 \in L^\infty(T^2)$. Then the Toeplitz like operator on 2-nuclear tensor product of Hardy spaces on torus have the following properties:

1. $T_{\xi_1 \otimes \xi_2}$ is bounded, and $\|T_{\xi_1 \otimes \xi_2}\| \le \|\xi_1 \otimes \xi_2\| = \|\xi_1\|\|\xi_2\|$.
2. $T_{u(\xi_1 \otimes \xi_2)+v(\xi_3 \otimes \xi_4)} = uT_{\xi_1 \otimes \xi_2} + vT_{\xi_3 \otimes \xi_4},\ where\ u,\ v\ \in C.$
3. $T_{\xi_1 \otimes \xi_2} = 0$ if and only if $\xi_1 \otimes \xi_2 = 0$
4. $T^*_{\xi_1 \otimes \xi_2} = T_{\overline{\xi_1 \otimes \xi_2}}$

Proof

1. Let $d \otimes b \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$. Then

$$\begin{aligned} \|T_{\xi_1 \otimes \xi_2}(d \otimes b)\| &= \|P_1(\xi_1 d) \otimes P_1(\xi_2 g)\| \\ &= \|P_1(\xi_1 d)\|\|P_1(\xi_2 b)\| \\ &\le \|P_1\|^2\|\xi_1 d\|\|\xi_2 b\| \\ &\le \|\xi_1\|\|d\|\|\xi_2\|\|b\| \\ &= \|d \otimes b\|\|\xi_1 \otimes \xi_2\| \end{aligned}$$

Thus; $T_{\xi_1 \otimes \xi_2}$ is bounded and $\|T_{\xi_1 \otimes \xi_2}\| \le \|\xi_1 \otimes \xi_2\|$.

2. Let $d \otimes b \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$. Then

$$\begin{aligned} T_{u(\xi_1 \otimes \xi_2)+v(\xi_3 \otimes \xi_4)}(d \otimes b) &= P_1 \otimes P_1(u(\xi_1 \otimes \xi_2) \\ &+ v(\xi_3 \otimes \xi_4))(d \otimes b) \\ &= P_1 \otimes P_1(a(\xi_1 \otimes \xi_2)(d \otimes b) \\ &+ v(\xi_3 \otimes \xi_3))(d \otimes b) \end{aligned}$$

$= P_1(u(\xi_1.d) \otimes P_1(\xi_2.b)) + vP_1(\xi_3.d) \otimes P_1(\xi_4.b)$
$= uP_1(\xi_1.d) \otimes P_1(\xi_2.b) + bP_1(\xi_3.d) \otimes P_1(\xi_4.b)$
$= uT_{\xi_1 \otimes \xi_2}(d \otimes b) + vT_{\xi_1 \otimes \xi_2}(d \otimes b)$

3. Let $d_1 \otimes b_1, d_2 \otimes b_2 \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$ be non zero functions. Then

$$\begin{aligned} 0 &= \langle T_{\xi_1 \otimes \xi_2}(d_1 \otimes b_1), d_2 \otimes b_2 \rangle \\ &= \langle P_1(\xi_1.d_1) \otimes P_1(\xi_2.b_1), d_2 \otimes b_2 \rangle \\ &= \langle P_1(\xi_1.d_1), d_2 \rangle \langle P_1(\xi_2.b_1), b_2 \rangle \\ &= \langle \xi_1.d_1, d_2 \rangle \langle \xi_2.b_1, b_2 \rangle \\ &= \langle \xi_1.d_1 \otimes \xi_2.b_1, d_2 \otimes b_2 \rangle \end{aligned}$$

Thus; $\xi_1 = \xi_2 = 0$, since $d_1, d_2, b_1, b_2 \neq 0$.

4. Let $d_1 \otimes b_1, d_2 \otimes b_2 \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$. Then

$\langle T^*_{\xi_1 \otimes \xi_2}(d_1 \otimes b_1), d_2 \otimes b_2 \rangle$
$= \langle d_1 \otimes b_1, T_{\xi_1 \otimes \xi_2}(d_2 \otimes b_2) \rangle$
$= \langle d_1 \otimes b_1), P_1(\xi_1.d_2) \otimes P_1(\xi_2.b_2) \rangle$
$= \langle d_1, P_1(\xi_1.d_2) \rangle \langle b_1, P_1(\xi_2.b_2) \rangle$
$= \langle d_1, \xi_1.d_2 \rangle \langle b_1, \xi_2.b_2 \rangle$
$= \langle \overline{\xi_1}.d_1, P_1(d_2) \rangle \langle \overline{\xi_2}.b_1, P_1(b_2) \rangle$
$= \langle P_1(\overline{\xi_1}.d_1) \otimes P_1(\overline{\xi_2}.b_1), d_2 \otimes b_2 \rangle$
$= \langle T_{\overline{\xi_1 \otimes \xi_2}}(d_1 \otimes b_1), d_2 \otimes b_2 \rangle$

Thus; $T^*_{\xi_1 \otimes \xi_2} = T_{\overline{\xi_1 \otimes \xi_2}}$.

Now, we will go to study the commutativity of Toeplitz like operators on 2-nuclear tensor product of Hardy spaces on torus.

In the following theorem, showing when the product of two Toeplitz like operators on $H^2(T^2) \otimes_{n(2)} H^2(T^2)$ will be a Toeplitz like operator.

Theorem 4:

Let ξ_1, ξ_2, ξ_3, and $\xi_4 \in L^\infty(T^2)$. Then $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)}$ is a Toeplitz like operator if and only if one of the following conditions is satisfied:

i. ξ_3 and ξ_4 are analytic.
ii. ξ_1 and ξ_2 are co-analytic .
iii. ξ_2 is analytic and ξ_2 is co-analytic.
iv. ξ_4 is analytic and ξ_1 is co-analytic.

and if one of the above condition is satisfied, then $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} = T_{(\xi_1 \otimes \xi_2)(\xi_3 \otimes \xi_4)}$.

proof:

Suppose $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)}$ is a Toeplitz like operator. Then

$$\begin{aligned} T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} &= (T_{\xi_1} \otimes T_{\xi_2})(T_{\xi_3} \otimes T_{\xi_4}) \\ &= T_{\xi_1} T_{\xi_3} \otimes T_{\xi_2} T_{\xi_4} \end{aligned}$$

is a Toeplitz-like operator. Hence; $T_{\xi_1} T_{\xi_3}$ and $T_{\xi_2} T_{\xi_4}$ are Toeplitz operators. But by proposition 1, if $T_{\xi_1} T_{\xi_3}$ is Toeplitz operator, then ξ_3 is analytic or ξ_1 is co-analytic, and also if $T_{\xi_2} T_{\xi_4}$ is a Toeplitz operator, then ξ_4 is analytic or ξ_2 is co-analytic. So, we get i-iv above.
Conversely, Assume that

$$T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} = T_{\xi_1} T_{\xi_3} \otimes T_{\xi_2} T_{\xi_4}.$$

and one of the above condition is satisfied. Therefore ,by proposition 1, we have $T_{\xi_1} T_{\xi_3}$ and $T_{\xi_2} T_{\xi_4}$ are Toeplitz operators, Thus; $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)}$ is a Toeplitz-like operator.
Indeed, if one of the above condition is satisfied, then we obtain

$$\begin{aligned} T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} &= (T_{\xi_1} \otimes T_{\xi_2})(T_{\xi_3} \otimes T_{\xi_4}) \\ &= T_{\xi_1} T_{\xi_3} \otimes T_{\xi_2} T_{\xi_4} \\ &= T_{\xi_1 \xi_3} \otimes T_{\xi_2 \xi_4} \\ &= T_{(\xi_1 \otimes \xi_2)(\xi_3 \otimes \xi_4)} \end{aligned}$$

Corollary 4:

Assume that $T_{(\xi_1 \otimes \xi_3)} and T_{(\xi_2 \otimes \xi_4)}$ are Toeplitz like operators. Then the product of them is equal to zero if and only if at least one of them is to zero.

Proof:

If $T_{(\xi_1 \otimes \xi_3)} T_{(\xi_2 \otimes \xi_4)} = 0$, then since zero is a Toeplitz like operator.

$$T_{(\xi_1 \otimes \xi_3) T_{(\xi_2 \otimes \xi_4)}} = T_{(\xi_1 \otimes \xi_3)(\xi_2 \otimes \xi_4)} = 0.$$

Therefore $\xi_1 \xi_2 \otimes \xi_3 \xi_4 = 0$. Thus; $\xi_1 \xi_2 = 0$ or $\xi_3 \xi_4 = 0$.

Theorem 5:

Assume that ξ_1, ξ_2, ξ_3, and $\xi_4 \in L^\infty(T^2)$. Then

$$T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} = T_{(\xi_3 \otimes \xi_4)}\, T_{(\xi_1 \otimes \xi_2)}$$

if and only if one of the following equivalence conditions is satisfied:

1. $\xi_1,\ \xi_2,\ \xi_3,\ and\ \xi_4$ are analytic (or co-analytic).
2. $\xi_1,\ \xi_3$ are analytic (or co-analytic) and $\xi_2,\ \xi_4$ are co-analytic (or analytic).
3. $\xi_1,\ \xi_3$ are analytic (or co-analytic) and $a\xi_2 + b\xi_4$ is constant.
4. $c\xi_1 + h\xi_3$ is constant and $\xi_2,\ \xi_4$ are analytic (or co-analytic).
5. $c\xi_1 + h\xi_3$ and $a\xi_2 + b\xi_4$ are constants.

Proof:

Suppose that $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_3 \otimes \xi_4)} = T_{(\xi_3 \otimes \xi_4)}\, T_{(\xi_1 \otimes \xi_2)}$.
But $T_{(\xi_1 \otimes \xi_2)}\, T_{(\xi_2 \otimes \xi_4)} = T_{\xi_1} T_{\xi_3} \otimes T_{\xi_2} T_{\xi_4}$ and $T_{(\xi_3 \otimes \xi_4)}\, T_{(\xi_1 \otimes \xi_2)} = T_{\xi_3} T_{\xi_1} \otimes T_{\xi_4} T_{\xi_2}$, this implies

$$T_{\xi_1} T_{\xi_3} \otimes T_{\xi_2} T_{\xi_4} = T_{\xi_3} T_{\xi_1} \otimes T_{\xi_4} T_{\xi_2},$$

so we obtain
$T_{\xi_1} T_{\xi_3} = \eta T_{\xi_3} T_{\xi_1}\, and\, T_{\xi_2} T_{\xi_4} = \frac{1}{\eta} T_{\xi_4} T_{\xi_2},\, where\, \eta \neq 0.$

Now, without loss of generality, let $\eta = 1$. so,

$$T_{\xi_1} T_{\xi_3} = T_{\xi_3} T_{\xi_1} \quad (1)$$

$$T_{\xi_2} T_{\xi_4} = T_{\xi_4} T_{\xi_2} \quad (2).$$

But now, by (Theorem 1), equations (1) and (2) satisfied if and only if one of the conditions above is satisfied.

Theorem 6:

Let $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4$, and $\eta \in L^\infty(T^2)$. Then $T_{(\xi_1 \otimes \eta)+(\xi_2 \otimes \eta)}$ commutes with $T_{(\xi_3 \otimes \eta)+(\varphi_4 \otimes \psi)}$ if and only if one of the following are satisfied :
i. $\xi_1 + \xi_2$, and $\xi_3 + \xi_4$ are analytic.
ii. $\xi_1 + \xi_2$, and $\xi_3 + \xi_4$ are co-analytic.
iii. $\xi_1 + \xi_2 = \beta(\xi_3 + \xi_4) + r$, where $\beta \in C$ and r is a constant function.

Proof:

Note that
$T_{\xi_1} \otimes T_\eta + T_{\xi_2} \otimes T_\eta = T_{(\xi_1 \otimes \eta)+(\xi_2 \otimes \eta)}$ and $T_{\xi_3} \otimes T_\eta + T_{\xi_4} \otimes T_\eta = T_{(\xi_3 \otimes \eta)+(\xi_4 \otimes \eta)}$.

But, the sum of two atoms is an atom if either the first components or the second ones are dependant, [2]. Thus

$$\begin{aligned} T_{\xi_1} \otimes T_\eta + T_{\xi_2} \otimes T_\eta &= (T_{\xi_1} + T_{\xi_2}) \otimes T_\eta \\ &= T_{\xi_1+\xi_2} \otimes T_\eta (1) \end{aligned}$$

Similarly,

$$T_{\xi_3} \otimes T_\eta + T_{\xi_4} \otimes T_\eta = T_{\xi_3+\xi_4} \otimes T_\eta \ (2)$$

But now, the problem is when two atomic Toeplitz operators commute? That is when
$(T_{\xi_1+\xi_2} \otimes T_\eta)(T_{\xi_3+\xi_4} \otimes T_\eta) = (T_{\xi_3+\xi_4} \otimes T_\eta)(T_{\xi_1+\xi_2} \otimes T_\eta)$.

Which is equivalent to :
$(T_{\xi_1+\xi_2} T_{\xi_3+\xi_4}) \otimes T_\eta T_\eta = (T_{\xi_3+\xi_4} T_{\xi_1+\xi_2}) \otimes T_\eta T_\eta$ (3).

Of course if $\xi_1 = -\xi_2$ or $\xi_3 = -\xi_4$ or $\eta = 0$, then trivially, we get the commutativity.

Hence, we assume that $\xi_1 + \xi_2 \neq 0,\ \xi_3 + \xi_4 \neq 0$, and $\eta \neq 0$. but (3) is valid if and only if

$$T_{\xi_1+\xi_2} T_{\xi_3+\xi_4} = T_{\xi_3+\xi_4} T_{\xi_1+\xi_2} \ (4)$$

However, by (Theorem 1), equation (4) is true if and only if one of the conditions (i), (ii) or (iii) is satisfied.

Theorem 7:

Let $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \eta_1$, and $\eta_2 \in L^\infty(T^2)$. Then $T_{(\xi_1 \otimes \eta_1)+(\xi_2 \otimes \eta_1)}$ commutes with $T_{(\xi_3 \otimes \eta_2)+(\xi_4 \otimes \eta_2)}$ if and only if one of the following conditions is satisfied:
i. $\xi_1 + \xi_2,\ \xi_3 + \xi_4,\ \eta_1$, and η_2 are analytic.
ii. $\xi_1 + \xi_2,\ \xi_3 + \xi_4$ are analytic and $\eta_1,\ \eta_2$ are co-analytic.
iii. $\xi_1 + \xi_2,\ \xi_3 + \xi_4$ are co-analytic and $\eta_1,\ \eta_2$ are analytic.
iv. $\xi_1 + \xi_2,\ \xi_3 + \xi_4$ are analytic and $\eta_1 = \beta\, \eta_2 + h$.
v. $\xi_1 + \xi_2,\ \xi_3 + \xi_4$ are co-analytic and $\eta_1 = \beta\, \eta_2 + h$.
vi. $\xi_1 + \xi_2 = \beta(\xi_3 + \xi_4) + h$ and $\eta_1,\ \eta_2$ are analytic.
vii. $\xi_1 + \xi_2 = \beta(\xi_3 + \xi_4) + h$ and $\eta_1,\ \eta_2$ are analytic.
viii. $\xi_1 + \xi_2 = \beta_1(\xi_3 + \xi_4) + h_1$ and $\eta_1 = \beta_2\, \eta_2 + h_2$.

Proof:

First, we have

$$T_{\xi_1} \otimes T_{\eta_1} + T_{\xi_2} \otimes T_{\eta_1} = T_{(\xi_1 \otimes \eta_1)+(\xi_2 \otimes \eta_1)}$$

and

$$T_{\xi_3} \otimes T_{\eta_2} + T_{\xi_4} \otimes T_{\eta_2} = T_{(\xi_3 \otimes \eta_2)+(\xi_4 \otimes \eta_2)}.$$

$$T_{\xi_1} \otimes T_{\eta_1} + T_{\xi_2} \otimes T_{\eta_1} = T_{\xi_1+\xi_2} \otimes T_{\eta_1},$$

and

$$T_{\xi_3} \otimes T_{\eta_2} + T_{\xi_4} \otimes T_{\eta_2} = T_{\xi_3+\xi_4} \otimes T_{\eta_2}.$$

So, we need to prove the Theorem, just to see the commutativity of the two atoms

$$(T_{\xi_1+\xi_2} \otimes T_{\eta_1})\ and\ (T_{\xi_3+\xi_4} \otimes T_{\eta_2}).$$

Since
$(T_{\xi_1+\xi_2} \otimes T_{\eta_1})\,(T_{\xi_3+\xi_4} \otimes T_{\eta_2}) = T_{\xi_1+\xi_2} T_{\xi_3+\xi_4} \otimes T_{\eta_1}\, T_{\eta_2}$ (1)
and
$(T_{\xi_3+\xi_4} \otimes T_{\eta_2})\,(T_{\xi_1+\xi_2} \otimes T_{\eta_1}) = T_{\xi_3+\xi_4} T_{\xi_1+\xi_2} \otimes T_{\eta_2}\, T_{\eta_1}$ (2).

Since (1) and (2) commutativity are satisfying if and only if
$T_{\xi_1+\xi_2}\, T_{\xi_3+\xi_4} = T_{\xi_1+\xi_2} T_{\xi_3+\xi_4}\, and\, T_{\eta_1}\, T_{\eta_2} = T_{\eta_2}\, T_{\eta_1}$.

Theorem 8:

A Toeplitz like operator $T_{\xi_1 \otimes \xi_2}$ is an isometry on $H^2(T^2) \otimes_{n(2)} H^2(T^2)$ if and only if $\xi_1 \otimes \xi_2$ is a constant and is satisfying $|\xi_1| = |\xi_2| = 1$.

Proof:

Assume that $T_{\xi_1\otimes\xi_2}$ is an isometry, then

$$\begin{aligned}\|T_{\xi_1\otimes\xi_2}(h\otimes d)\|^2 &= \langle T_{\xi_1\otimes\xi_2}(h\otimes d), T_{\xi_1\otimes\xi_2}(h\otimes d)\rangle\\ &= \langle T^*_{\xi_1\otimes\xi_2}T_{\xi_1\otimes\xi_2}(h\otimes d), h\otimes d\rangle\\ &= \langle h\otimes d, h\otimes d\rangle,\end{aligned}$$

$for\ all\ h\otimes d\in H^2(T^2)\otimes_{n(2)} H^2(T^2).$

Hence $T^*_{\xi_1\otimes\xi_2}T_{\xi_1\otimes\xi_2} = I_{1\otimes 1}$, so

$$T_{\overline{\xi_1\otimes\xi_2}}T_{\xi_1\otimes\xi_2} = I_{1\otimes 1}.$$

Similarly, one gets

$$T_{\xi_1\otimes\xi_2}T_{\overline{\xi_1\otimes\xi_2}} = I_{1\otimes 1} = T_{1\otimes 1}.$$

Therefore $T_{\xi_1\otimes\xi_2}T_{\overline{\xi_1\otimes\xi_2}} = T_{\overline{\xi_1\otimes\xi_2}}T_{\xi_1\otimes\xi_2} = I_{1\otimes 1} = T_{1\otimes 1}$.

By Theorem 1, we obtain $\xi_1,\ \xi_2,\ \overline{\xi_1},\ \overline{\xi_2}$ are analytic(or co-analytic) or there is a linear combination between ξ_1 and $\overline{\xi_1}$ or there is a linear combination between ξ_2 and $\overline{\xi_2}$, So ξ_1 and ξ_2 should be constants in all cases. Thus; $\xi_1\otimes\xi_2$ is constant and also $\xi_1\overline{\xi_1} = |\xi_1|^2 = 1$ and $\xi_2\overline{\xi_2} = |\xi_2|^2 = 1$ since

$$T_{\xi_1\otimes\xi_2}T_{\overline{\xi_1\otimes\xi_2}} = T_{\xi_1\overline{\xi_1}\otimes\xi_2\overline{\xi_2}} = T_{|\xi_1|^2\otimes|\xi_2|^2} = T_{1\otimes 1}.$$

4 Spectrum and invertibility of Toeplitz like operators

In this last section, we study the spectrum and the invertibility of Toeplitz like operators acting on 2-nuclear tensor Product of Hardy Spaces.

Definition 2:

Assume that $\xi_1,\ \xi_2\in L^\infty(T^2)$. Then $T_{\xi_1\otimes\xi_2}$ is invertible if T_{ξ_1} and T_{ξ_2} are invertible.

Lemma 3:

Let $\xi_1,\ \xi_2\in L^\infty(T^2)$ be invertible such that $\sigma(M_{\xi_1}\otimes M_{\xi_2})$ is contained in the open right - half plane. Then $T_{\xi_1\otimes\xi_2}$ is invertible.

Proof:

Let $\Delta = \{z\in C : |z-1| < 1\}$. Since $\sigma(M_{\xi_1}\otimes M_{\xi_2})$ is a compact set in C, then there exists $\epsilon > 0$ such that $\epsilon\sigma(M_{\xi_1}\otimes M_{\xi_2})\subset\Delta$, where

$$\epsilon\sigma(M_{\xi_1}\otimes M_{\xi_2}) = \{\epsilon\mu_1\mu_2 : \mu_1\mu_2\in\sigma(M_{\xi_1}\otimes M_{\xi_2})\}.$$

Therefore, $|\epsilon\mu_1\mu_2 - 1| < 1$, for all $\mu_1\mu_2\in\sigma(M_{\xi_1}\otimes M_{\xi_2})$. Consequently

$$\sup_{\mu_1\mu_2\in\sigma(M_{\xi_1}\otimes M_{\xi_2})}|\epsilon\mu_1\mu_2 - 1| < 1.$$

Now, by applying the Spectral Mapping Theorem, we get

$$\epsilon\mu_1\mu_2 - 1\in\sigma(\epsilon M_{\xi_1}\otimes M_{\xi_2} - I_{1\otimes 1}).$$

So

$$\|\epsilon\xi_1\otimes\xi_2 - I_{1\otimes 1}\| = \|\epsilon M_{\xi_1}\otimes M_{\xi_2} - I_{1\otimes 1}\| =$$

$$\sup_{\zeta\in\ \sigma(\epsilon M_{\xi_1}\otimes M_{\xi_2} - I_{1\otimes 1})}|\zeta| < 1.$$

However $\|T_{\xi_1\otimes\xi_2}\|_{n(2)} = \|\xi_1\otimes\xi_2\|$, Hence

$$\|I_{1\otimes 1} - \epsilon T_{\xi_1\otimes\xi_2}\|_{n(2)} = \|T_{1\otimes 1 - \epsilon\xi_1\otimes\xi_2}\|_{n(2)} =$$

$$\|1\otimes 1 - \epsilon\xi_1\otimes\xi_2\| < 1.$$

So, $\|I_{1\otimes 1} - \epsilon T_{\xi_1\otimes\xi_2}\|_{n(2)} < 1$, this; $\epsilon T_{\xi_1\otimes\xi_2}$ is invertible and then $T_{\xi_1\otimes\xi_2}$ is invertible.

Lemma 4:

Assume that $\xi_1,\ \xi_2\in L^\infty(T^2)$. Then

$$\sigma(T_{\xi_1\otimes\xi_2})\subset[\sigma(M_{\xi_1}\otimes M_{\xi_2})](convex\ hull\ of\ \sigma(\xi_1\otimes\xi_2)).$$

Proof:

From the definition of $[\sigma(M_{\xi_1}\otimes M_{\xi_2})]$, it is enough to prove that if H is an open half plane which contains the spectrum of $M_{\xi_1}\otimes M_{\xi_2}$, then $\sigma(T_{\xi_1\otimes\xi_2})\subset H$.

Let $\mu_1\mu_2\notin H$, so $\mu_1\mu_2\notin\sigma(M_{\xi_1}\otimes M_{\xi_2})$, and $\sigma(M_{\xi_1}\otimes M_{\xi_2} - \mu_1\mu_2 I_{1\otimes 1})\subset H - \mu_1\mu_2$. Since $H - \mu_1\mu_2$ does not contain zero (as $\mu_1\mu_2\notin H$), there exists a real number θ_1 such that $e^{i\theta_1}(H - \mu_1\mu_2)\subset H_e$, where H_e is the open right half plane. Further, $e^{i\theta_1}\sigma(H - \mu_1\mu_2)\subset H_e$. Since $(M_{\xi_1}\otimes M_{\xi_2} - \mu_1\mu_2 I_{1\otimes 1})$ is invertible, $e^{i\theta_1}(M_{\xi_1}\otimes M_{\xi_2} - \mu_1\mu_2 I_{1\otimes 1})$ is still invertible and by the spectral mapping theorem $\sigma(e^{i\theta_1}(M_{\xi_1}\otimes M_{\xi_2} - \mu_1\mu_2 I_{1\otimes 1}))\subset H_e$. which implies that, by lemma 3,($T_{\xi_1\otimes\xi_2 - \mu_1\mu_2}$) is invertible.

That is $(T_{\xi_1\otimes\xi_2} - \mu_1\mu_2 I)^{-1}$ exists and therefore $\mu_1\mu_2\notin\sigma(T_{\xi_1\otimes\xi_2})$. Thus; $\sigma(T_{\xi_1\otimes\xi_2})\subset H$.

However, this is vaild for all open half planes H containing $\sigma(M_{\xi_1}\otimes M_{\xi_2})$. Hence $\sigma(T_{\xi_1\otimes\xi_2})\subset[\sigma(M_{\xi_1}\otimes M_{\xi_2})]$.

Theorem 9:

Let $T_{\xi_1}\otimes T_{\xi_2}$ be invertible. Then $(T_{\xi_1}\otimes T_{\xi_2})^{-1}$ is a Toeplitz like operator if and only if one of the following is satisfied:

i. $\xi_1\otimes\xi_2\in H^2(T^2)\otimes_{n(2)} H^2(T^2)$.
ii. $\overline{\xi_1\otimes\xi_2}\in H^2(T^2)\otimes_{n(2)} H^2(T^2)$.
iii. $\overline{\xi_1}\otimes\xi_2\in H^2(T^2)\otimes_{n(2)} H^2(T^2)$.
v. $\xi_1\otimes\overline{\xi_2}\in H^2(T^2)\otimes_{n(2)} H^2(T^2)$.

Theorem 9:

Let $T_{\xi_1} \otimes T_{\xi_2}$ be invertible. Then $(T_{\xi_1} \otimes T_{\xi_2})^{-1}$ is a Toeplitz like operator if and only if one of the following is satisfied:
i. $\xi_1 \otimes \xi_2 \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$.
ii. $\overline{\xi_1 \otimes \xi_2} \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$.
iii. $\overline{\xi_1} \otimes \xi_2 \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$.
v. $\xi_1 \otimes \overline{\xi_2} \in H^2(T^2) \otimes_{n(2)} H^2(T^2)$.

Proof:

Assume $(T_{\xi_1} \otimes T_{\xi_2})^{-1} = T_{\xi_1}^{-1} \otimes T_{\xi_2}^{-1}$ is a Toeplitz like operator, hence $T_{\xi_1}^{-1}$ and $T_{\xi_2}^{-1}$ are Toeplitz operators.

Now, $T_{\xi_1}^{-1}$ is a Toeplitz operator if and only if ξ_1 or $\overline{\xi_1} \in H^2(T^2)$.

Similarly, $T_{\xi_2}^{-1}$ is a Toeplitz operator if and only if ξ_2 or $\overline{\xi_2} \in H^2(T^2)$.

Therefore, $(T_{\xi_1} \otimes T_{\xi_2})^{-1}$ is a Toeplitz like operator if and only if one of the above conditions satisfies.

Corollary 5:

Assuming $\xi_1,\ \xi_2 \in L^\infty(T^2)$. Then $\sigma(T_{\xi_1 \otimes \xi_2})$ is connected.

Proof:

Since $\sigma(T_{\xi_1 \otimes \xi_2}) = \sigma(T_{\xi_1})\sigma(T_{\xi_2})$, and also $\sigma(T_{\xi_1})$ and $\sigma(T_{\xi_2})$ are connected sets, hence $\sigma(T_{\xi_1}) \times \sigma(T_{\xi_2})$ is a connected set.

Now, define a function

$$h : \sigma(T_{\xi_1}) \times \sigma(T_{\xi_2}) \longrightarrow C$$

$$h(a, b) \mapsto a.b$$

Clearly, h is a continuous function.

Thus; $h(\sigma(T_{\xi_1}) \times \sigma(T_{\xi_2})) = \sigma(T_{\xi_1})\sigma(T_{\xi_2})$ is a connected set. Which is implies $\sigma(T_{\xi_1 \otimes \xi_2})$ is a connected set.

Theorem 10:

Assuming $\xi_1,\ \xi_1 \in L^\infty(T^2)$. Then $T_{\xi_1 \otimes \xi_2}$ is a compact operator if and only if $\xi_1 \otimes \xi_2 = 0$.

Proof:

The proof directly will get it, from this theorem, Suppose $\xi \in L^\infty(T^2)$. Then T_ξ is a compact operator if and only if $\xi = 0$.

5 Outcome and questions

In this article, we discuss Toeplitz like operator on 2-nuclear tensor product of Hardy spaces, we conclude in the followings definitions and theorems:

Definition 1:

Assume that $\psi = \psi_1 \otimes \psi_2$, where $\psi_1, \psi_2 \in L^\infty(T^2)$ and T_ψ is an operator defined as

$$T_\psi : H^2(T^2)n(2)\otimes H^2(T^2) \to H^2(T^2)n(2)\otimes H^2(T^2)$$

such that

$$\begin{aligned} T_{\psi_1\otimes\psi_2}(d \otimes b) &= P_1 \otimes P_1((\psi_1 \otimes \psi_2)(d \otimes b)) \\ &= P_1(\psi_1 d) \otimes P_1(\psi_2 b). \end{aligned}$$

hen the operator T_ψ is said to be Toeplitz like operator with symbol ψ.

And

Theorem 2:

The complex valued vector space $H^2(T^2, H^2(T^2))$ is isometrically isomorphic to the 2-nuclear Tensor product of Hardy spaces $H^2(T^2) \otimes_{n(2)} H^2(T^2)$.

One can ask the following question:

What is the slant Toeplitz like opearator on ON THE Lebesgue space of unit circle and the torus.

References:

[1] Douglas, R, Banach Algebra Techniques in Operator Theory. Academic Press, New York, (1972).

[2] Light, W. Cheney, E, Approximation Theory in Tensor Product Spaces, New York: Springer-Verlag, (1985).

[3] Holub, J, Hilibertion Operators and Reflexive Tensor Product, J. Paci. Math, 36, (1971), $185 - 194$.

[4] Brown, A. Pearcy, C, Spectra of Tensor Products of Operators, Proc. Amer. Math. Soc, 17,(1966), $162 - 166$.

[5] Brown, A. Halmos, P, Algebraic Properties of Toeplitz Operators, J. Reine Angew Math, 213, (1964), $89 - 102$.

[6] Kreyzig, E, Introductory Functional Analysis with Applications, New York: John Wiley and Sons, (1978).

[7] Allabadi M, When Intersection Ideals of Graphs of Rings are a Divisor graph, WSEAS Transactions on Mathematics. 2020, 19: 430-433.

[8] Allabadi M, Al-muhur E, Planar of special idealization rings, WSEAS Transactions on Mathematics. 2020; 19: 606-609.