Spacelike Surfaces with a Common Line of Curvature in Lorentz-Minkowski 3-Space

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Abstract: This paper aims to study spacelike surfaces from a given spacelike curve in Minkowski 3–space. Also, we investigate the necessary and sufficient conditions for the given space-like curve to be the line of curvature on the space-like surface. Depending on the causal character of the curve, the necessary and sufficient conditions for the given space-like curve to satisfy the line of curvature and the geodesic (resp. asymptotic) requirements are also analyzed. Furthermore, we give with illustration some computational examplesinsupportofourmainresults.

Key–Words: Serret-Frenet frame, Line of curvature, Spacelike surface pencil, Marching-scale functions.

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1 Introduction

A curve in $\mathbb{E}_1^3$, the three-dimensional Minkowski space, is said to be time-like, space-like or null if all its tangent vectors are always time-like, space-like or null, respectively. Indeed, the distance function $\langle,\rangle$ in the Euclidean 3-space can only be positive while in $\mathbb{E}_1^3$ it can be positive, negative or zero. Based on the classification of distance function mentioned above, curves are classified as space-like, time-like or null curve, respectively, [1,2].

The problem of investigating surfaces with a given curve plays an important role in geometric design. It was Wang et.al. [3], who proposed and studied such type of problem for the first time. They provided a method for constructing a surface family from a given spatial geodesic. They expressed the required surface as a linear combination of the marching-scale functions $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$, $\mathcal{W}(s,t)$ and the Serret–Frenet frame $\{\mathbf{r}_1(s), \mathbf{r}_2(s), \mathbf{r}_3(s)\}$. Also, they derived necessary and sufficient conditions with correct parametric representation of the surface pencil for a given curve. Further, Kasap et al. [4] investigated the sufficient conditions for a given curve to be a geodesic on a surface by generalized the marching-scale functions. Li et.al. [5] reported the surface pencil with a common line of curvature by replacing the characteristic curve from geodesic to line of curvature. Bayram et.al. [6] tackled the problem of constructing surfaces passing through a given asymptotic curve. Important contributions to surface passing through a given curve have been studied in [5–8].

With the inspiration of work of Wang [3], we extend the work of Lie et al. [4] to deduce the necessary and sufficient conditions for a space-like surface pencil to contain $\beta=\beta(s)$ as a line of curvature. Subsequently, we analyze these conditions when the given curve is a geodesic (resp. asymptotic) and line of curvature. As an application, we verified the method by exact space-like surface pencil formulations for some simple surfaces, such as surfaces of revolution and ruled surfaces.

2 Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the three-dimensional real vector space $\mathbb{R}^3$ with standard flat metric given by

$$\langle d\mathbf{a}, d\mathbf{a}\rangle = da_1^2 + da_2^2 - da_3^2$$

where (a_1, a_2, a_3) stands for rectangular coordinate system of $\mathbb{E}_1^3$. An arbitrary vector $\mathbf{a}\neq 0$ of $\mathbb{E}_1^3$ is said to be space-like, time-like or null (light-like) provided $\langle\mathbf{a},\mathbf{a}\rangle>0$, $\langle\mathbf{a},\mathbf{a}\rangle<0$ or $\langle\mathbf{a},\mathbf{a}\rangle=0$, respectively. A time-like or light-like vector in $\mathbb{E}_1^3$ is also known as causal. The norm of a vector $\mathbf{a}\in\mathbb{E}_1^3$ is defined by $\|\mathbf{a}\|=\sqrt{|\langle\mathbf{a},\mathbf{a}\rangle|}$. The vector $\mathbf{a}\in\mathbb{E}_1^3$ is called a space-like or time-like unit vector if $\langle\mathbf{a},\mathbf{a}\rangle=1$ or $\langle\mathbf{a},\mathbf{a}\rangle=-1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be space-like, time-like or null (light-like), if all its velocity vectors are so, for more details, we refer, [1,2]. For any two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ of $\mathbb{E}_1^3$, the inner product and the vector product is defined as $\langle \mathbf{a}, \mathbf{b} \rangle = a_1b_1 + a_2b_2 - a_3b_3$ and $\mathbf{a} \times \mathbf{b} = ((a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), -(a_1b_2 - a_2b_1))$, respectively.

Let $\beta = \beta(s)$ be a unit speed space-like curve in $\mathbb{E}_1^3$; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\beta = \beta(s)$, respectively.

Consider the Serret–Frenet frame $\{\mathbf{r}_1(s), \mathbf{r}_2(s), \mathbf{r}_3(s)\}$ associated with $\beta = \beta(s)$ such that $\mathbf{r}_1(s)$, $\mathbf{r}_2(s)$ and $\mathbf{r}_3(s)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. Depending on the causal character of the curve $\beta = \beta(s)$, we have the following Serret–Frenet formulae:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{r}_1(s) \\ \mathbf{r}_2(s) \\ \mathbf{r}_3(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_1(s) \\ \mathbf{r}_2(s) \\ \mathbf{r}_3(s) \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned} \mathbf{r}_1(s) \times \mathbf{r}_2(s) &= \mathbf{r}_3(s), \\ \mathbf{r}_2(s) \times \mathbf{r}_3(s) &= \mathbf{r}_1(s), \\ \mathbf{r}_3(s) \times \mathbf{r}_1(s) &= -\mathbf{r}_2(s), \end{aligned} \tag{2}$$

if $\beta = \beta(s)$ is a space-like curve with a time-like principal normal $\mathbf{r}_2$, and

$$\frac{d}{ds}\begin{pmatrix} \mathbf{r}_1(s) \\ \mathbf{r}_2(s) \\ \mathbf{r}_3(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_1(s) \\ \mathbf{r}_2(s) \\ \mathbf{r}_3(s) \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} \mathbf{r}_1(s) \times \mathbf{r}_2(s) &= -\mathbf{r}_3(s), \\ \mathbf{r}_2(s) \times \mathbf{r}_3(s) &= \mathbf{r}_1(s), \\ \mathbf{r}_3(s) \times \mathbf{r}_1(s) &= \mathbf{r}_2(s), \end{aligned} \tag{4}$$

if $\beta = \beta(s)$ is a space-like curve with a space-like principal normal $\mathbf{r}_2$.

Let $\mathbf{P} = \mathbf{P}(s,t)$ be a parametric spacelike surface in $\mathbb{E}_1^3$ based on the given spacelike curve $\beta = \beta(s)$ as follows:

$$M: \ \mathbf{P}(s,t) = \beta(s) + \mathcal{U}(s,t)\mathbf{r}_1(s) + \mathcal{V}(s,t)\mathbf{r}_2(s)$$

$$+\mathcal{W}(s,t)\mathbf{r}_3(s); \quad 0 \le t \le T, \quad 0 \le s \le L, \tag{5}$$

where $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$ stand for C^1 functions. If t is the time parameter, then $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$ can be viewed as directed marching distances of a point unit in the time t in the direction $\mathbf{r}_1$; $\mathbf{r}_2$; and $\mathbf{r}_3$, respectively, where the position vector $\beta(s)$ is the initial location of this point. A space-like surface with the unit speed space-like curve $\beta = \beta(s)$ satisfying Eq. (1) or Eq. (3) is indicated by type M^- or type M^+, respectively.

Definition 1 *A surface in the Minkowski 3-space $\mathbb{E}_1^3$ is called a time-like surface or a space-like surface if the induced metric on the surface is a Lorentz metric or a positive definite Riemannian metric, which amounts to say the normal vector on space-like (time-like) surface is a time-like (space-like) vector.*

According to this definition, we obtain two time-like normal vector fields:

1- For the type M^-, we can write

$$\mathbf{r}_2(s,t) := \frac{\partial \mathbf{P}(s,t)}{\partial s} \times \frac{\partial \mathbf{P}(s,t)}{\partial t}$$

$$= \eta_1(s,t)\mathbf{r}_1(s) + \eta_2(s,t)\mathbf{r}_2(s) + \eta_3(s,t)\mathbf{r}_3(s), \tag{6}$$

where

$$\left.\begin{aligned} \eta_1(s,t) &= \left(\tfrac{\partial \mathcal{V}(s,t)}{\partial s} + \mathcal{U}(s,t)\kappa(s) + \mathcal{W}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{W}(s,t)}{\partial t} \\ &\quad - \left(\tfrac{\partial \mathcal{W}(s,t)}{\partial s} + \mathcal{V}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{V}(s,t)}{\partial t} \\ \eta_2(s,t) &= \left(1 + \tfrac{\partial \mathcal{U}(s,t)}{\partial s} + \mathcal{V}(s,t)\kappa(s)\right)\tfrac{\partial \mathcal{W}(s,t)}{\partial t} \\ &\quad - \left(\tfrac{\partial \mathcal{W}(s,t)}{\partial s} + \mathcal{V}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{U}(s,t)}{\partial t}, \\ \eta_3(s,t) &= (1 + \tfrac{\partial \mathcal{U}(s,t)}{\partial s} + \mathcal{V}(s,t)\kappa(s))\tfrac{\partial \mathcal{V}(s,t)}{\partial t} \\ &\quad -(\tfrac{\partial \mathcal{V}(s,t)}{\partial s} + \mathcal{U}(s,t)\kappa(s) + \mathcal{W}(s,t)\tau(s))\tfrac{\partial \mathcal{U}(s,t)}{\partial t}. \end{aligned}\right\}$$

2- For the type M^+, we can write

$$\mathbf{r}_2(s,t) = \eta_1(s,t)\mathbf{r}_1(s) + \eta_2(s,t)\mathbf{r}_2(s) + \eta_3(s,t)\mathbf{r}_3(s), \tag{7}$$

where

$$\left.\begin{aligned} \eta_1(s,t) &= \left(\tfrac{\partial \mathcal{V}(s,t)}{\partial s} + \mathcal{U}(s,t)\kappa(s) + \mathcal{W}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{W}(s,t)}{\partial t} \\ &\quad - \left(\tfrac{\partial \mathcal{W}(s,t)}{\partial s} + \mathcal{V}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{V}(s,t)}{\partial t} \\ \eta_2(s,t) &= -\left(1 + \tfrac{\partial \mathcal{U}(s,t)}{\partial s} - \mathcal{V}(s,t)\kappa(s)\right)\tfrac{\partial \mathcal{W}(s,t)}{\partial t} \\ &\quad + \left(\tfrac{\partial \mathcal{W}(s,t)}{\partial s} + \mathcal{V}(s,t)\tau(s)\right)\tfrac{\partial \mathcal{U}(s,t)}{\partial t}, \\ \eta_3(s,t) &= -(1 + \tfrac{\partial \mathcal{U}(s,t)}{\partial s} - \mathcal{V}(s,t)\kappa(s))\tfrac{\partial \mathcal{V}(s,t)}{\partial t} \\ &\quad +(\tfrac{\partial \mathcal{V}(s,t)}{\partial s} + \mathcal{U}(s,t)\kappa(s) + \mathcal{W}(s,t)\tau(s))\tfrac{\partial \mathcal{U}(s,t)}{\partial t}. \end{aligned}\right\}$$

3 $\mathbf{M}^-$ with a common space-like line of curvature

In this section, our objective is to derive the necessary and sufficient conditions for which the given space-like curve $\beta(s)$ is an isoparametric line of curvature on the type M^-.
Firstly, since the directrix $\beta(s)$ is an isoparametric curve on the surface then there exists a parameter $t = t_0$ such that $\beta(s) = \mathbf{P}(s, t_0)$, that is, we have

$$\mathcal{U}(s,t_0) = \mathcal{V}(s,t_0) = \mathcal{W}(s,t_0) = 0. \tag{8}$$

Thus the time-like normal vector field becomes

$$\mathbf{r}_2(s,t_0) = \eta_1(s,t_0)\mathbf{r}_1(s) + \eta_2(s,t_0)\mathbf{r}_2(s) + \eta_3(s,t_0)\mathbf{r}_3(s), \tag{9}$$

where

$$\left.\begin{array}{r}\eta_1(s,t_0) = \frac{\partial\mathcal{W}(s,t_0)}{\partial t}\frac{\partial\mathcal{V}(s,t_0)}{\partial s} - \frac{\partial\mathcal{W}(s,t_0)}{\partial s}\frac{\partial\mathcal{V}(s,t_0)}{\partial t},\\ \eta_2(s,t_0) = (1+\frac{\partial\mathcal{U}(s,t_0)}{\partial s})\frac{\partial\mathcal{W}(s,t_0)}{\partial t} - \frac{\partial\mathcal{U}(s,t_0)}{\partial t}\frac{\partial\mathcal{W}(s,t_0)}{\partial s},\\ \eta_3(s,t_0) = (1+\frac{\partial\mathcal{U}(s,t_0)}{\partial s})\frac{\partial\mathcal{V}(s,t_0)}{\partial t} - \frac{\partial\mathcal{U}(s,t_0)}{\partial t}\frac{\partial\mathcal{V}(s,t_0)}{\partial s}).\end{array}\right\} \tag{10}$$

Secondly, let us choose a time-like unit vector

$$\mathbf{e}(s) = \cosh\theta\mathbf{r}_2(s) + \sinh\theta\mathbf{r}_3(s). \tag{11}$$

Hence, from Eqs. (9) and (11), we have that $\mathbf{e}(s) \| \mathbf{r}_2(s,t_0)$ iff there exists a function $\lambda(s)$ such that

$$\begin{array}{rcl}\eta_1(s,t_0) &=& 0, \quad \eta_2(s,t_0) = \lambda(s)\cosh\theta, \\ \eta_3(s,t_0) &=& \lambda(s)\sinh\theta.\end{array} \tag{12}$$

Differentiating Eq. (11) and using the corresponding Serret-Frenet formulae (1), we find

$$\frac{d\mathbf{e}}{ds} = (\frac{d\theta}{ds} + \tau)\mathbf{e}^{\perp} + \kappa\cosh\theta\mathbf{r}_1.$$

Hence, $\beta = \beta(s)$ is a line of curvature on M^- if and only if $\frac{d\mathbf{e}}{ds} \parallel \mathbf{r}_1$, i.e. $\frac{d\theta}{ds} + \tau = 0$. This means that

$$\theta(s) = \theta_0 - \int_{s_0}^{s} \tau(s)ds, \tag{13}$$

where s_0 is the initial value of arc length and $\theta_0 = \theta(s_0)$. It is worthy to note that the technique we use is fundamentally different than in [9–13]. Now, we draw an important conclusion as follows:

Theorem 2 *The given space-like curve $\beta(s)$ is a line of curvature on the type M^- if and only if*

$$\left.\begin{array}{l}\mathcal{U}(s,t_0) = \mathcal{V}(s,t_0) = \mathcal{W}(s,t_0) = 0,\\ \quad 0 \le t_0 \le T, \quad 0 \le s \le L, \quad \lambda(s) \neq 0,\\ \eta_1(s,t_0) = 0, \quad \eta_2(s,t_0) = \lambda(s)\cosh\theta,\\ \eta_3(s,t_0) = \lambda(s)\sinh\theta,\end{array}\right\} \tag{14}$$

where $\lambda(s)$ and $\theta(s)$ are called controlling functions.

With M^- we denote the space-like surfaces with common space-like line of curvature described by Eq. (5) and Eq. (14). For the sake of simplicity, the marching-scale functions $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$ were decomposed into two factors [14,15]:

$$\begin{array}{c}\mathcal{U}(s,t) = l(s)\mathcal{U}(t),\\ \mathcal{V}(s,t) = m(s)\mathcal{V}(t),\\ \mathcal{W}(s,t) = n(s)\mathcal{W}(t).\end{array}$$

Here $l(s), m(s), n(s)$, $\mathcal{U}(t)$, $\mathcal{V}(t)$ and $\mathcal{W}(t)$ are C^1 functions and $l(s), m(s)$ and $n(s)$ are not identically zero. Thus, from the Theorem 1, we can get the following corollary:

Corollary 3 *A necessary and sufficient condition of the space-like curve $\beta(s)$ being a line of curvature on M^- is*

$$\left.\begin{array}{c}\mathcal{U}(t_0) = \mathcal{V}(t_0) = \mathcal{W}(t_0) = 0,\\ \frac{d\mathcal{W}(t_0)}{dt} = \frac{\lambda(s)\cosh\theta}{n(s)}, \ \frac{d\mathcal{V}(t_0)}{dt} = \frac{\lambda(s)\sinh\theta}{m(s)}.\end{array}\right\} \tag{15}$$

However, we can assume that the marching-scale functions depend only on the parameter t; that is, $l(s) = m(s) = n(s) = 1$. Then condition (15) can be analyzed according to the different expressions of $\theta(s)$:
(i) If the curve $\beta(s)$ is a twisted curve, i.e., $\tau(s) \neq 0$, then $\theta(s)$ is a non-constant function of variable s and the condition (15) can be represented as

$$\left.\begin{array}{c}\mathcal{U}(t_0) = \mathcal{V}(t_0) = \mathcal{W}(t_0) = 0,\\ \frac{d\mathcal{W}(t_0)}{dt} = \lambda(s)\cosh\theta, \ \frac{d\mathcal{V}(t_0)}{dt} = \lambda(s)\sinh\theta.\end{array}\right\} \tag{16}$$

(ii) If the curve $\beta(s)$ is a planar curve, i.e., $\tau(s) = 0$, then $\theta(s) = \theta_0$, i.e., is a constant. However, if $\theta_0 \neq 0$, for convenience, we can assume that $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$ depend only on the parameter t, that is $l(s) = m(s) = n(s) = 1$, then

$\lambda(s)$ is also a constant. After simplification, the condition (15) becomes

$$\left.\begin{array}{c}\mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0,\\ \frac{d\mathcal{W}(t_0)}{dt}=\lambda(s)\cosh\theta_0,\ \frac{d\mathcal{V}(t_0)}{dt}=\lambda(s)\sinh\theta_0.\end{array}\right\} \tag{17}$$

(iii) In Case (ii), if $\theta_0 = 0$, substituting it to Eq. (11) we have $\mathbf{r}_2(s)\|\mathbf{r}_2(s,t_0)$. According to the geodesic theory [4], the curve $\beta(s)$ is a geodesic on M^- if and only if at any point on the curve $\beta(s)$ the principal normal $\mathbf{r}_2(s)$ to the curve and the normal $\mathbf{r}_2(s,t_0)$ to M^- are parallel to each other. Hence, the curve $\beta(s)$ is not only a line of curvature but also a geodesic on M^-. In this case, the condition (15) has the simple form

$$\left.\begin{array}{c}\mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0,\\ \frac{d\mathcal{W}(t_0)}{dt}=\lambda(s),\ \frac{d\mathcal{V}(t_0)}{dt}=0.\end{array}\right\}$$

From above analysis, one can easily notice that there are no constraints for the curves given by Eqs. (14), (15) or (16). Thus, by choosing suitable marching-scale functions M^- can always be determined.

3.1 Examples

In this section, our adopted methods are verified by some illustrative examples.

Example 3.1 *In this example, we construct M^- in which all the surfaces share a space-like helix as common space-like line of curvature.*
Given the space-like circular helix:

$$\begin{array}{rcl}\beta(s) &=& (a\sinh\frac{s}{c}, b\frac{s}{c}, a\cosh\frac{s}{c}),\\ a &>& 0,\ \ b\neq 0,\ a^2-b^2=c^2,\ -4\leq s\leq 4.\end{array}$$

One can easy to show that

$$\left.\begin{array}{c}\mathbf{r}_1(s)=(\frac{a}{c}\cosh\frac{s}{c},\frac{b}{c},\frac{a}{c}\sinh\frac{s}{c}),\\ \mathbf{r}_2(s)=(\sinh\frac{s}{c},0,\cosh\frac{s}{c}),\\ \mathbf{r}_3(s)=(\frac{b}{c}\cosh\frac{s}{c},-\frac{a}{c},\frac{b}{c}\sinh\frac{s}{c}),\end{array}\right\}$$

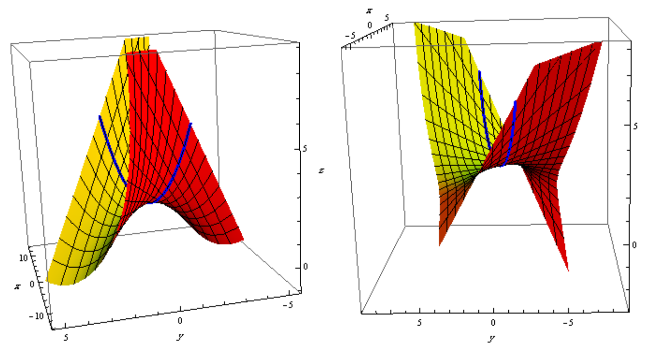
and $\tau=\frac{b}{c^2}$, then $\theta(s)=-\frac{b}{c^2}s+\theta_0$. If $\theta_0=0$, we have $\theta(s)=-\frac{b}{c^2}s$.
By choosing

$$\begin{array}{rcl}l(s) &=& m(s)=n(s)=1,\\ \mathcal{U}(t) &=& \alpha t,\ \mathcal{V}(t)=t\lambda(s)\sinh\theta,\\ \mathcal{W}(t) &=& t\lambda(s)\cosh\theta,\ \lambda\neq 0,\end{array}$$

and from formula (4), the equation of M^- is

$$\mathbf{P}(s,t;\alpha,\lambda)=(a\sinh\frac{s}{c}, b\frac{s}{c}, a\cosh\frac{s}{c})$$
$$+t(\alpha,\lambda\sinh\theta,\lambda\cosh\theta)\begin{pmatrix}\frac{a}{c}\cosh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\sinh\frac{s}{c}\\ \sinh\frac{s}{c} & 0 & \cosh\frac{s}{c}\\ \frac{b}{c}\cosh\frac{s}{c} & -\frac{a}{c} & \frac{b}{c}\sinh\frac{s}{c}\end{pmatrix}.$$

So, if we choose $t\in[-4,4]$, $a=2$, $b=1$, then for $\alpha=1$, $\lambda=1$ and $\alpha=-\frac{\sqrt{5}}{4}$, $\lambda=\frac{\sqrt{5}}{2}$, the corresponding space-like surfaces are shown in Fig. 1(a) and Fig. 1(b), respectively.



(a) (b)

Figure 1: Surfaces with a spacelike helix as a common spacelike line of curvature.

Example 3.2 *Suppose that a parametric spacelike curve is given by*

$$\beta(s)=(0,\sinh s,\cosh s),\quad -2\leq s\leq 2.$$

After simple computation, we have

$$\begin{array}{rcl}\mathbf{r}_1(s) &=& (0,\cosh s,\sinh s),\\ \mathbf{r}_2(s) &=& (0,\sinh s,\cosh s),\ \mathbf{r}_3(s)=(1,0,0),\end{array}$$

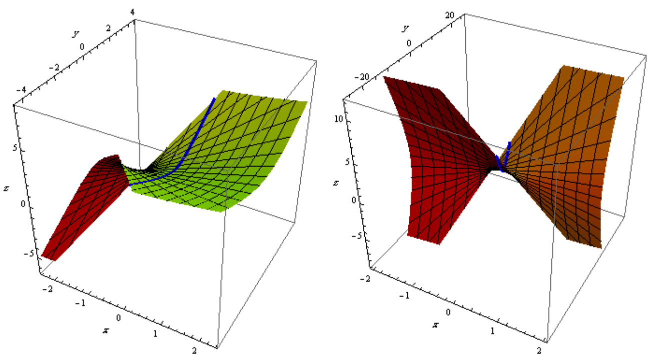
and $\tau=0$ which follows $\theta(s)=\theta_0$ is a constant. By choosing

$$\begin{array}{rcl}l(s) &=& m(s)=n(s)=1,\\ \mathcal{U}(t) &=& \alpha t,\ \mathcal{V}(t)=t\lambda(s)\sinh\theta_0,\\ \mathcal{W}(t) &=& t\lambda(s)\cosh\theta_0,\ \lambda\neq 0,\end{array}$$

and from formula (2.4), the equation of M^- is

$$\mathbf{P}(s,t;\alpha,\lambda)=(0,\sinh s,\cosh s)$$
$$+t(\alpha,\lambda\sinh\theta_0,\lambda\cosh\theta_0)\begin{pmatrix}0 & \cosh s & \sinh s\\ 0 & \sinh s & \cosh s\\ 1 & 0 & 0\end{pmatrix}. \tag{18}$$

So, if we choose $t \in [-2, 2]$ and $\theta_0 = 1.5$ (resp. $\theta_0 = 0$), then for $\alpha = 0.3$, $\lambda = 0.5$ and $\alpha = 3$, $\lambda = 1$, the corresponding spacelike surfaces are shown in Fig. 2(a), and Fig. 2(b), respectively.



(a) (b)

Figure 2: M^- spacelike surfaces with $\theta_0 = 1.5$ and $\theta_0 = 0$.

3.2 M$^-$ revolutions

In the following examples, we construct M^- revolutions in which all the surfaces passing through a given curve as a line of curvature.

Example 3.3 *Let*

$$\mathbf{h}(t) = (h_x(t), 0, h_z(t)), 0 \le t \le T,$$

be a parametric space-like curve in the oxz-plane with the following condition:

$$\begin{aligned} h(t_0) &= (0, 0, 1), \\ h'(t_0) &= (\lambda \cosh \theta_0, 0, \lambda \sinh \theta_0), \ \lambda \neq 0, \ 0 \le t_0 \le T. \end{aligned}$$

By revolving generating curve $\mathbf{h}(t)$ about the $x-$axis of our coordinate system. A parametrization of the surface of revolution is expressed as

$$\mathbf{R}(s, t) = (h_x(t), h_z(t) \sinh s, h_z(t) \cosh s), \ h_x(t) \neq 0. \tag{19}$$

If Eqs. (18) and (19) represent the same family, we have

$$h_z(t) = 1 + \lambda t \sinh \theta_0, \ h_x(t) = \lambda t \cosh \theta_0.$$

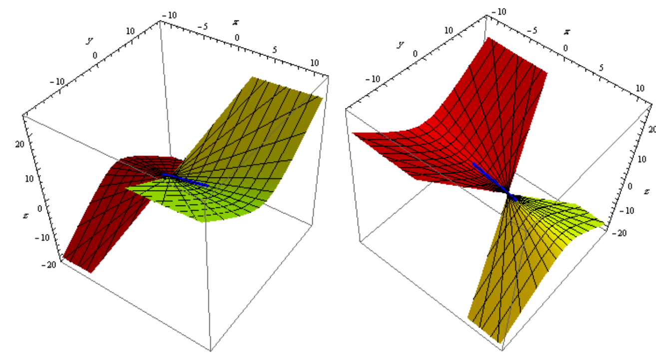
Hence, the equation of $M^-$of revolutions can be written as

$$\mathbf{P}(s, t, \lambda, \theta_0) = \begin{pmatrix} \lambda t \cosh \theta_0, \\ (1 + \lambda t \sinh \theta_0) \sinh s, \\ (1 + \lambda t \sinh \theta_0) \cosh s \end{pmatrix}.$$

The parametric curve $\mathbf{h}(t)$ has the representation of the form

$$\mathbf{h}(t) = (\lambda t \cosh \theta_0, 0, 1 + \lambda t \sinh \theta_0).$$

This means that M^- of revolutions is formed by a uniparametric family of vertical hyperbolas. We chose $t \in [-10, 10]$ and $s \in [-3, 3]$. For $\lambda = 1$, $\theta_0 = 0.5$ and $\lambda = -1$, $\theta_0 = -0.5$, the corresponding spacelike surfaces are shown in Fig. 3(a) and Fig. 3(b), respectively. In most of the practical applications, the parameter of a given curve is usually not its arc length. So, in the following example, we present an algorithm for constructing M^- of revolutions from an arbitrarily parameterized line of curvature being the generating curve.



(a) (b)

Figure 3: M^- space-like surfaces of revolutions.

Example 3.4 *Suppose we are given a parametric space-like curve in the oxz-plane:*

$$\beta(r) = (f(r), 0, g(r)),$$

which when revolved about the x-axis produces the space-like surface of revolution. Simple computation leads to

$$\begin{aligned} \mathbf{r}_1(r) &= \left(f', 0, g'\right) \left\|\beta'\right\|^{-1}, \\ \mathbf{r}_2(r) &= \left(g', 0, f'\right) \left\|\beta'\right\|^{-1}, \\ \mathbf{r}_3(r) &= (0, -1, 0), (' = \frac{d}{dr}), \end{aligned}$$

where $\left\|\beta'\right\|^2 = f'^2 - g'^2 > 0$. Then the equation of type M^- is expressed as

$$\mathbf{P}(r, t) = \begin{pmatrix} f + \frac{\mathcal{U}(r,t) f' + \mathcal{V}(s,t) g'}{\left\|\beta'\right\|}, \\ -\mathcal{W}(r, t), \\ g + \frac{\mathcal{U}(r,t) g' + \mathcal{V}(r,t) f'}{\left\|\beta'\right\|} \end{pmatrix}, \tag{20}$$

where $0 \le t \le T$. Now we consider a member of the above family. If the surface is generated by rotation and parameter t as an angle of rotation with the initial angle $t_0 = 0$; then it took the following form

$$\mathbf{P}(r,t) = (f(r), g(r)\sinh t, g(r)\cosh t). \qquad (21)$$

Comparing to Eqs. (20) and (21), we have

$$\mathcal{W}(r,t) = -g(r)\sinh t, \ \frac{\mathcal{U}(r,t)f' + \mathcal{V}(s,t)g'}{\left\|\beta'\right\|} = 0,$$

$$\frac{\mathcal{U}(r,t)g' + \mathcal{V}(r,t)f'}{\left\|\beta'\right\|} = g(\cosh t - 1).$$

Then, the marching-scale functions are given by

$$\begin{aligned}
\mathcal{U}(r,t) &= \frac{gg'(1-\cosh t)}{\left\|\beta'\right\|}, \\
\mathcal{V}(r,t) &= -\frac{gf'(1-\cosh t)}{\left\|\beta'\right\|}, \\
\mathcal{W}(r,t) &= -g\sinh t. \qquad (22)
\end{aligned}$$

Hence

$$\mathcal{U}(t) = 1-\cosh t, \ \mathcal{V}(t) = 1-\cosh t, \ \mathcal{W}(t) = -\sinh t. \qquad (23)$$

According to Eqs. (15) and (23), we have

$$\begin{aligned}
\frac{d\mathcal{W}(t_0)}{dt} &= \frac{\lambda(r)\cosh\theta}{n(r)\left\|\beta'\right\|} = -1, \\
\frac{d\mathcal{V}(t_0)}{dt} &= \frac{\lambda(r)\sinh\theta}{m(r)\left\|\beta'\right\|} = 0, \ \lambda \neq 0. \qquad (24)
\end{aligned}$$

Therefore, we have $\sinh\theta = 0$, that is $\beta(r)$ is a space-like line of curvature and also a geodesic on M^-. Thus, we set the marching-scale functions as

$$\begin{aligned}
\mathcal{U}(r,t) &= \frac{\beta(r)gg'(1-\cosh t)}{\left\|\beta'\right\|}, \\
\mathcal{V}(r,t) &= -\frac{\gamma(r)gf'(1-\cosh t)}{\left\|\beta'\right\|}, \\
\mathcal{W}(r,t) &= -g\sinh t, \qquad (25)
\end{aligned}$$

where $\beta(r)$, $\gamma(r)$ are all C^1 functions and $\gamma(r) \neq 0$, $0 \le t \le L$, to obtain M^-,

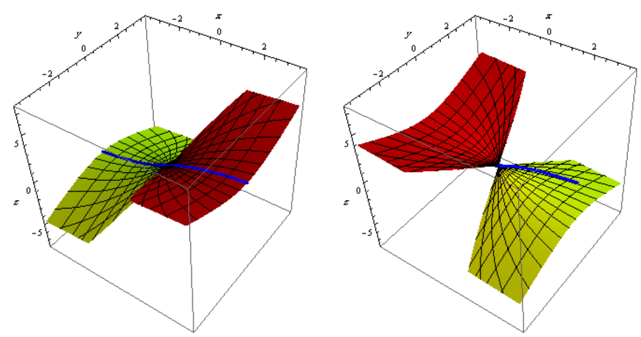
$$\{\mathbf{P}(r,t,\beta(r),\gamma(r)) \mid \beta(r),\gamma(r) \in C^1, \ \gamma(r) \neq 0\}.$$

When $\beta(r) = \gamma(r) = 1$, the corresponding,

$$\mathbf{P}(r,t,1,1) = \mathbf{P}(r,t) = (f(r), g(r)\sinh t, g(r)\cosh t)$$

is thus also M^- of revolutions whose generating curve is a line of curvature and also a geodesic. It is easy to validate that the marching-scale functions satisfy the conditions $\mathbf{r}_2(r)\|\mathbf{r}_2(r,t_0)$, $0 \le t \le L$ given in Eq. (14) with $t_0 = 0$. Thus, Eqs. (20) and (25) are the required conditions. For $\beta = \gamma = 1$ and $\beta(r) = \gamma(r) = -1$, the two members are shown in Fig 4(a). Fig. 4(b); where the generating curve is given as

$$\beta(r) = (\sinh r, 0, r), \ -2 \le r \le 2,$$



(a) (b)

Figure 4: M^- of revolutions whose generating curve is a line of curvature and also a geodesic.

3.3 M^- ruled surfaces

In what follows, the M^- ruled surfaces in which all the surfaces share the same directrix are constructed. Let $\beta(s)$ be a 3D space-like curve with arc-length parameter s. Consider ruled surface $\mathbf{P}(s,t)$ with the space-like directrix $\beta(s)$, where $\beta(s)$ is the parametric curve of $\mathbf{P}(s,t)$, then there exists t_0 such that $\mathbf{P}(s,t_0) = \beta(s)$. It follows that

$$\begin{aligned}
\mathbf{P}(s,t) &= \mathbf{P}(s,t_0) + (t-t_0)\mathbf{d}(s), \\
0 &\le s \le L, \ 0 \le t \le T, \ 0 \le t_0 \le T,
\end{aligned}$$

where $\mathbf{d}(s)$ represents the direction of the rulings. According to the formula (3), we have

$$\begin{aligned}
(t-t_0)\mathbf{d}(s) &= \begin{pmatrix} \mathcal{U}(s,t)\mathbf{r}_1(s) \\ +\mathcal{V}(s,t)\mathbf{r}_2(s) \\ +\mathcal{W}(s,t)\mathbf{r}_3(s), \end{pmatrix} \qquad (26) \\
0 &\le s \le L, \ 0 \le t \le T, \ 0 \le t_0 \le T,
\end{aligned}$$

which is a system of three equations with three unknown functions $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$. The solutions of the above system can be deduced as

$$\begin{aligned} \mathcal{U}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_1(s)\rangle, \\ -\mathcal{V}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_2(s)\rangle, \\ \mathcal{W}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_3(s)\rangle, \end{aligned} \tag{27}$$

where $0 \le s \le L,\ 0 \le t \le T,\ 0 \le t_0 \le T$. The above equations are the necessary and sufficient conditions for which $\mathbf{P}(s,t)$ is a ruled surface with a directrix $\beta(s)$.

Next, it remains to verify that the curve $\beta(s)$ is also line of curvature on the surface $\mathbf{P}(s,t)$ by using the conditions given in Eq. (15). Evidently, these conditions become:

$$\begin{aligned} \lambda(s)\cosh\theta &= \langle \mathbf{d}(s), \mathbf{r}_3(s)\rangle, \\ \lambda(s)\sinh\theta &= -\langle \mathbf{d}(s), \mathbf{r}_2(s)\rangle. \end{aligned} \tag{28}$$

It follows that at any point on the curve $\beta(s)$; the ruling direction $\mathbf{d}(s)$ must be in the plane formed by $\mathbf{r}_3(s)$ and $\mathbf{r}_2(s)$. Also, the ruling direction $\mathbf{d}(s)$ can be expressed as:

$$\mathbf{d}(s) = a_1(s)\mathbf{r}_1(s) + a_2(s)\mathbf{r}_2(s) + a_3(s)\mathbf{r}_3(s),\ 0 \le s \le L, \tag{29}$$

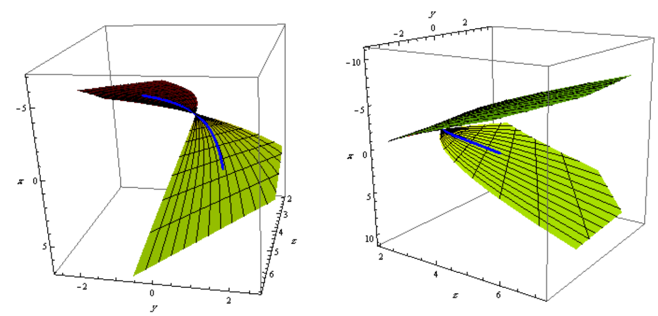
for some real functions $a_1(s)$, $a_2(s)$ and $a_3(s)$. Substituting it into the expressions in Eq. (27), we get

$$a_2(s)t = -\lambda(s)\sinh\theta, a_3(s)t = \lambda(s)\cosh\theta.$$

Hence, the $M^-$of ruled surfaces with the common directrix $\beta(s)$ can be expressed as

$$\mathbf{P}(s,t;\beta,\gamma) = \left[\begin{array}{c} \beta(s) + a_1(s)\mathbf{r}_1(s) \\ t\lambda(s)(-\sinh\theta\mathbf{r}_2(s) + \cosh\theta\mathbf{r}_3(s)), \\ 0 \le s \le L,\ 0 \le t \le T, \end{array}\right] \tag{30}$$

where the functions $a_1(s)$ and $\lambda(s)$ can control $M^-$of ruled surfaces. Every member of this family is decided by two pencil parameters $a_1(s)$ and $\lambda(s) \neq 0$; i.e., by the direction vector function $\mathbf{d}(s)$. Fig. 5(a) shows the member of $M^-$of ruled surfaces whose line of curvature is the space-like circular helix in Example 3.1, with $t \in [-2,2]$, $a = 2$, $b = 1$, and the controlling functions $a_1(s) = \lambda(s) = s$. For the suitable choices of controlling functions $a_1(s) = \lambda(s) = \frac{s}{2}$, the corresponding member is shown in Fig. 5(b). From these examples, the controlling functions can be chosen variant expressions and we can find that, they evidently control the shape of the members in M^-.



(a) (b)

Figure 5: M^- of ruled surfaces with the common directrix $\alpha(s)$.

4 $\mathbf{M^+}$ with a common space-like line of curvature

In this section, we investigate the case of the family of type M^+ with common space-like line of curvature. Firstly, as it was mentioned earlier, the normal vector field becomes

$$\mathbf{r}_2(s,t_0) = \eta_1(s,t_0)\mathbf{r}_1(s) + \eta_2(s,t_0)\mathbf{r}_2(s) + \eta_3(s,t_0)\mathbf{r}_3(s), \tag{31}$$

where

$$\left.\begin{aligned} \eta_1(s,t_0) &= \tfrac{\partial\mathcal{W}(s,t_0)}{\partial t}\tfrac{\partial\mathcal{V}(s,t_0)}{\partial s} - \tfrac{\partial\mathcal{W}(s,t_0)}{\partial s}\tfrac{\partial\mathcal{V}(s,t_0)}{\partial t}, \\ \eta_2(s,t_0) &= -(1+\tfrac{\partial\mathcal{U}(s,t_0)}{\partial s})\tfrac{\partial\mathcal{W}(s,t_0)}{\partial t} + \tfrac{\partial\mathcal{U}(s,t_0)}{\partial t}\tfrac{\partial\mathcal{W}(s,t_0)}{\partial s}, \\ \eta_3(s,t_0) &= -(1+\tfrac{\partial\mathcal{U}(s,t_0)}{\partial s})\tfrac{\partial\mathcal{V}(s,t_0)}{\partial t} + \tfrac{\partial\mathcal{U}(s,t_0)}{\partial t}\tfrac{\partial\mathcal{V}(s,t_0)}{\partial s}). \end{aligned}\right\} \tag{32}$$

Secondly, let us choose a timelike unit vector

$$\mathbf{e}(s) = \sinh\theta\mathbf{r}_2(s) + \cosh\theta\mathbf{r}_3(s). \tag{33}$$

Hence, from Eqs. (31) and (33), we have that $\mathbf{e}(s)\|\mathbf{r}_2(s,t_0)$ if and only if there exists a function $\lambda(s)$ such that

$$\begin{aligned} \eta_1(s,t_0) &= 0,\quad \eta_2(s,t_0) = \lambda(s)\sinh\theta, \\ \eta_3(s,t_0) &= \lambda(s)\cosh\theta. \end{aligned} \tag{34}$$

Hence, we get the corresponding conditions and we omit the details here.

Theorem 4 *A space-like curve $\beta(s)$ is a line of curvature on M^+ if and only if*

$$\left.\begin{array}{c} \mathcal{U}(s,t_0) = \mathcal{V}(s,t_0) = \mathcal{W}(s,t_0) = 0, \\ 0 \le t_0 \le T,\quad 0 \le s \le L,\quad \lambda(s) \neq 0, \\ \eta_1(s,t_0) = 0,\quad \eta_2(s,t_0) = \lambda(s)\sinh\theta, \\ \eta_3(s,t_0) = \lambda(s)\cosh\theta, \end{array}\right\} \tag{35}$$

where $\lambda(s)$ and $\theta(s)$ are the controlling functions.

By a similar procedure we have:

Corollary 5 *A necessary and sufficient condition of the space-like curve $\beta(s)$ being a line of curvature on M^+ is*

$$\left.\begin{array}{c} \mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0, \\ \frac{d\mathcal{W}(t_0)}{dt}=-\frac{\lambda(s)\sinh\theta}{n(s)},\ \frac{d\mathcal{V}(t_0)}{dt}=-\frac{\lambda(s)\cosh\theta}{m(s)}. \end{array}\right\} \tag{36}$$

By similar argument, we can also assume that the marching-scale functions varies with parameter t; that is, $l(s)=m(s)=n(s)=1$. Then, we analyze the condition (36) for different expressions of $\theta(s)$:

(i) If $\beta(s)$ is a twisted curve, that is, $\tau(s)\neq 0$, then $\theta(s)$ is a non-constant function of variable s and the condition (36) can be expressed as

$$\left.\begin{array}{c} \mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0, \\ \frac{d\mathcal{W}(t_0)}{dt}=-\lambda(s)\sinh\theta,\ \frac{d\mathcal{V}(t_0)}{dt}=-\lambda(s)\cosh\theta, \end{array}\right\} \tag{37}$$

(ii) If $\beta(s)$ is a planar curve, that is, $\tau(s)=0$, then $\theta(s)=\theta_0$, i.e. is a constant. However, if $\theta_0\neq 0$, for the sake of simplicity, assume that $\mathcal{U}(s,t)$, $\mathcal{V}(s,t)$ and $\mathcal{W}(s,t)$ are functions of the parameter t only, that is, $l(s)=m(s)=n(s)=1$, then $\lambda(s)$ is also a constant. By simplifying, the condition (36) becomes

$$\left.\begin{array}{c} \mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0, \\ \frac{d\mathcal{W}(t_0)}{dt}=-\lambda(s)\sinh\theta_0,\ \frac{d\mathcal{V}(t_0)}{dt}=-\lambda(s)\cosh\theta_0. \end{array}\right\} \tag{38}$$

(iii) In Case (ii), if $\theta_0=0$, substituting it to (4.3) we have $\mathbf{r}_3(s)\|\mathbf{r}_2(s,t_0)$. According to the asymptotic theory [4], the curve $\beta(s)$ is asymptotic on M^+ if and only if at any point on the curve $\beta(s)$ the binormal $\mathbf{r}_3(s)$ to the curve and the normal $\mathbf{r}_2(s,t_0)$ to M^+ are parallel to each other. Hence, the curve $\beta(s)$ is a line of curvature as well as asymptotic of M^+. In this case, the condition (38) has the simple form

$$\left.\begin{array}{c} \mathcal{U}(t_0)=\mathcal{V}(t_0)=\mathcal{W}(t_0)=0, \\ \frac{d\mathcal{W}(t_0)}{dt}=0,\ \frac{d\mathcal{V}(t_0)}{dt}=-\lambda(s). \end{array}\right\} \tag{39}$$

4.1 Examples

Next, we verify the validity of the formulae derived above from the following examples.

Example 4.1 *Given the space-like circular helix:*

$$\begin{aligned} \beta(s) &= (b\frac{s}{c}, a\cosh\frac{s}{c}, a\sinh\frac{s}{c}), \\ a &> 0,\ \ b\neq 0,\ b^2-a^2=c^2,\ -1\leq s\leq 1. \end{aligned}$$

It is easy to show that

$$\left.\begin{array}{c} \mathbf{r}_1(s)=(\frac{b}{c},\frac{a}{c}\sinh\frac{s}{c},\frac{a}{c}\cosh\frac{s}{c}), \\ \mathbf{r}_2(s)=(0,\cosh\frac{s}{c},\sinh\frac{s}{c}), \\ \mathbf{r}_3(s)=(\frac{a}{c},\frac{b}{c}\sinh\frac{s}{c},\frac{b}{c}\cosh\frac{s}{c}), \end{array}\right\}$$

and $\tau=\frac{b}{c^2}$, then $\theta(s)=-\frac{b}{c^2}s+\theta_0$. If $\theta_0=0$, we have $\theta(s)=-\frac{b}{c^2}s$. By choosing

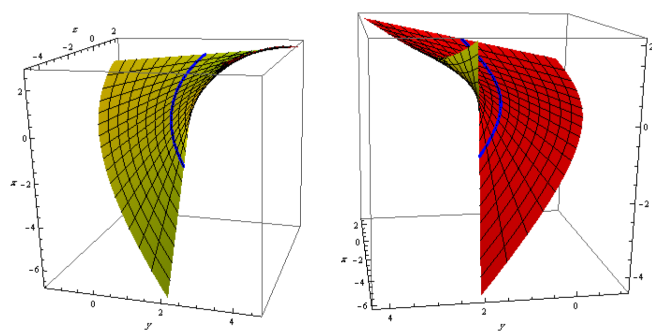
$$\begin{aligned} l(s) &= m(s)=n(s)=1, \\ \mathcal{U}(t) &= \alpha t,\ \mathcal{V}(t)=-t\lambda(s)\cosh\theta, \\ \mathcal{W}(t) &= -t\lambda(s)\sinh\theta,\ \lambda\neq 0, \end{aligned}$$

and from formula (4), the equation of family M^+ is

$$\mathbf{P}(s,t;\alpha,\lambda)=(b\frac{s}{c},a\cosh\frac{s}{c},a\sinh\frac{s}{c})$$

$$+t(\alpha,-\lambda\cosh\theta,-\lambda\sinh\theta)\begin{pmatrix} \frac{b}{c} & \frac{a}{c}\sinh\frac{s}{c} & \frac{a}{c}\cosh\frac{s}{c} \\ 0 & \cosh\frac{s}{c} & \sinh\frac{s}{c} \\ \frac{a}{c} & \frac{b}{c}\sinh\frac{s}{c} & \frac{b}{c}\cosh\frac{s}{c} \end{pmatrix}.$$

So, if we choose $t\in[-1,1]$, $a=1,\ b=2$, then for $\alpha=1,\ \lambda=-1$ and $\alpha=\frac{\sqrt{3}}{2},\lambda=-\frac{\sqrt{3}}{2}$, the corresponding space-like surfaces are shown in Fig. 6(a) and Fig. 6(b), respectively.



(a) (b)

Figure 6: M^+ space-like surfaces with a common space-like line of curvature.

Example 4.2 *Consider a parametric space-like curve*

$$\beta(s)=(\cos s,\sin s,0),\quad 0\leq s\leq 2\pi.$$

After straightforward computation, we get

$$\begin{aligned} \mathbf{r}_1(s) &= (-\sin s,\cos s,0), \\ \mathbf{r}_2(s) &= (-\cos s,-\sin s,0),\ \mathbf{r}_3(s)=(0,0,-1), \end{aligned}$$

and $\tau = 0$ which follows $\theta(s) = \theta_0$ is a constant. By choosing

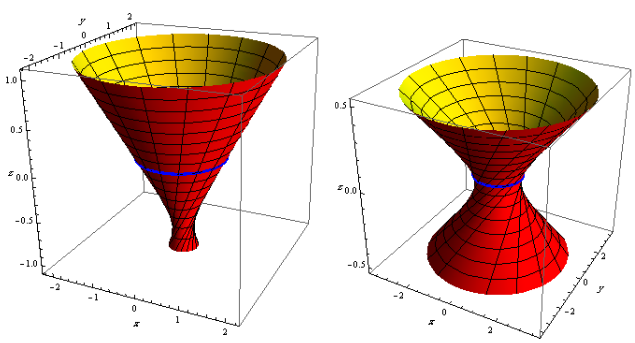
$$\begin{aligned} l(s) &= m(s) = n(s) = 1, \\ \mathcal{U}(t) &= \alpha t,\ \mathcal{V}(t) = -t\lambda(s)\cosh\theta_0, \\ \mathcal{W}(t) &= -t\lambda(s)\sinh\theta_0,\ \lambda \neq 0, \end{aligned}$$

and from formula (4), the equation of M^+ is

$$\mathbf{P}(s,t;\alpha,\lambda) = (\cos s, \sin s, 0)$$

$$+t(\alpha, -\lambda\cosh\theta_0, -\lambda\sinh\theta_0)\begin{pmatrix} -\sin s & \cos s & 0 \\ -\cos s & -\sin s & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So, if we choose $t \in [-1,1]$ and $\theta_0 = 1.5$ (resp. $\theta_0 = 0$), then for $\alpha = 0.3,\ \lambda = 0.5$ and $\alpha = 3,\ \lambda = 1$, the corresponding space-like surfaces are shown in Fig. 7(a), and Fig. 7(b), respectively.



(a) (b)

Figure 7: M^+ space-like surfaces corresponding to space-like plane curves.

4.2 M$^+$ revolutions

By the following examples, we showed that there is no M^+ revolutions.

Example 4.3 *Consider a parametric space-like curve in the oxz-plane:*

$$\beta(t) = (h_x(t), 0, h_z(t)),\ \left\|\beta'\right\|^2 = h_x'^2(t) - h_z'^2(t) > 0. \tag{40}$$

which when revolved about the z-axis produces the space-like surface of revolution

$$\mathbf{P}(s,t) = (h_x(t)\cos s, h_x(t)\sin s, h_z(t)). \tag{41}$$

Since the surface is space-like, we must have that

$$\left\|\frac{\partial \mathbf{P}(s,t)}{\partial s} \times \frac{\partial \mathbf{P}(s,t)}{\partial t}\right\|^2 = h_x^2(h_x'^2(t) - h_z'^2(t)) < 0, \tag{42}$$

which is a contradiction, the result is clear.

Example 4.4 *Consider a parametric space-like curve in the oxy-plane:*

$$\beta(r) = (f(r), g(r), 0).$$

After straightforward calculations, we get

$$\begin{aligned} \mathbf{r}_1(r) &= \left(f', g', 0\right),\ \mathbf{r}_2(r) = \left(-g', f', 0\right), \\ \mathbf{r}_3(r) &= (0,0,-1), (' = \frac{d}{dr}), \end{aligned}$$

where $f'^2 + g'^2 = 1$. Then the equation of type M^+ is expressed as

$$\mathbf{P}(r,t) = \begin{pmatrix} f + \mathcal{U}(r,t)f' - \mathcal{V}(s,t)g', g(r) \\ +\mathcal{U}(r,t)g' + \mathcal{V}(r,t)f', -\mathcal{W}(r,t) \end{pmatrix}, 0 \leq t \leq T. \tag{43}$$

Since $\mathbf{P}(r,t)$ is a surface of revolution, it can also be expressed as

$$\mathbf{P}(r,t) = (f(r)\cosh t, g(r), f(r)\sinh t). \tag{44}$$

Comparing to Eqs. (43) and (44), we obtain

$$\begin{aligned} \mathcal{U}(r,t) &= f'(1-\cosh t), \\ \mathcal{V}(r,t) &= -g'(1-\cosh t), \\ \mathcal{W}(r,t) &= -f\sinh t. \end{aligned} \tag{45}$$

Hence

$$\mathcal{U}(t) = 1 - \cosh t,\ \mathcal{V}(t) = -1 + \cosh t,\ \mathcal{W}(t) = -\sinh t. \tag{46}$$

According to Eqs. (38) and (46), we have

$$\begin{aligned} \frac{d\mathcal{W}(t_0)}{dt} &= -\frac{\lambda(r)\sinh\theta}{n(r)\left\|\beta'\right\|} = -1, \\ \frac{d\mathcal{V}(t_0)}{dt} &= -\frac{\lambda(r)\cosh\theta}{m(r)\left\|\beta'\right\|} = 0,\ \lambda \neq 0. \end{aligned} \tag{47}$$

Therefore, we have $\cosh\theta = 0$ which is impossible.

4.3 M$^+$ ruled surfaces

In what follows, the M^+ ruled surfaces in which all the surfaces share the same directrix is constructed. Following the same procedures, we have

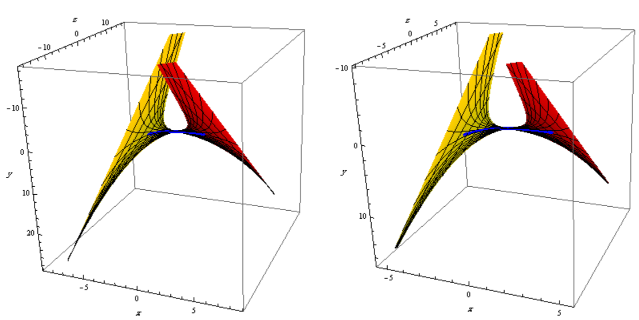
$$\begin{aligned} \mathcal{U}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_1(s)\rangle, \\ \mathcal{V}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_2(s)\rangle, \\ -\mathcal{W}(s,t) &:= (t-t_0)\langle \mathbf{d}(s), \mathbf{r}_3(s)\rangle, \end{aligned} \tag{48}$$

where $0 \le s \le L,\ 0 \le t \le T,\ 0 \le t_0 \le T$. The above equations are the necessary and sufficient conditions for which $\mathbf{P}(s,t)$ is a ruled surface with a directrix $\beta(s)$.

Likewise, a simple calculation shows that

$$\mathbf{P}(s,t;\beta,\gamma) = \begin{bmatrix} \beta(s) + a_1(s)\mathbf{r}_1(s) \\ +t\lambda(s)(-\cosh\theta\mathbf{r}_2(s) + \sinh\theta\mathbf{r}_3(s)), \\ 0 \le s \le L,\ 0 \le t \le T, \end{bmatrix} \tag{49}$$

where $a_1(s)$ and $\lambda(s)$ can control $M^+$of ruled surfaces. Every member of this family is decided by two pencil parameters $a_1(s)$ and $\lambda(s) \neq 0$; i.e., by the direction vector function $\mathbf{d}(s)$. Fig. 8(a) shows the member of $M^+$of ruled surfaces whose line of curvature is the space-like circular helix in Example 4.1, with $t \in [-2,2]$, $a=2$, $b=1$, and the controlling functions $a_1(s) = \lambda(s) = s$. For the choice of controlling functions $a_1(s) = \lambda(s) = \frac{s}{2}$, the corresponding member is shown in Fig. 8(b).



(a) (b)

Figure 8: M^+ ruled surfaces whose line of curvature is a space-like circular helix.

Concluding Remarks

In this article, we explored a method for finding a space-like surface family whose members share a given space-like line of curvature as an isoparametric line of curvature. We derive the necessary and sufficient conditions for the given space-like curve to be the line of curvature for the parametric space-like surface by combining of the given space-like curve and the three vectors decomposed along the directions of Serret–Frenet frame. In the process of derivation, we define two controlling functions $\theta(s)$ and $\lambda(s)$. Using these controlling functions, one can obtain variant forms of ruled and developable surfaces with unique features. Different form of surfaces are quite applicable in industry. Also, these parameters can improve the position of cutting tools which are used in the field of oil and energy. Some examples are constructed to illustrate that these controlling functions can control the shape of the space-like surface flexibly. The results, in addition to being of theoretical interest, have application in geometric modeling and the manufacturing of products. For example, designing agriculture machines' tools as development models of bulldozer's moldboard by geometric modeling method (for design engineering).

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References

[1] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, vol. 103 of Pure and Applied Mathematics, Academic Press, New York, NY, USA, 1983.

[2] J. Walrave, Curves and surfaces in Minkowski space, Ph.D. thesis, ProQuest LLC, Ann Arbor, Mich, USA, Katholieke Universiteit, Leuven, Belgium, 1995.

[3] G.J. Wang, K. Tang and C.L. Tai. Parametric representation of a surface pencil with a common spatial geodesic, Computer-Aided Design 36 (2004), 447-459.

[4] E. Kasap, F.T. Akyıldız and K. Orbay, A generalization of surfaces family with common spatial geodesic. Appl. Math. Comput. 201 (2008), 781-789.

[5] C. Y. Li, R. H. Wang and C. G. Zhu, Parametric representation of a surface pencil with a common line of curvature. Computer-Aided Design 43 (2011), 1110-1117.

[6] E. Bayram, F. Güler and E. Kasap, Parametric representation of a surface pencil with a common asymptotic curve. Computer-Aided Design 44 (2012), 637-643.

[7] R. A. Abdel-Baky, Time-like surfaces with a common asymptotic Curve in Minkowski 3-Space E_1^3, Riv. Mat. Univ. Parma 8(2) (2017), 379-395.

[8] E. Kasap, F. T. Akyildiz, Surfaces with common geodesic in Minkowski 3-space. AMC 177(1) (2006), 260-270.

[9] C. Y. Li, R. H. Wang and C. G. Zhu, An approach for designing a developable surface through a given line of curvature. Computer-Aided Design 45 (2013), 621-627.

[10] S. Hananoi and S. Izumiya, Normal developable surfaces of surfaces along curves, Proceedings of the Royal Society of Edinburgh, 147(A) (2017) 177-203.

[11] T. Körpinar, Tangent bimagnetic curves in terms of inextensible flows in space, Int. J. Geom. Methods in M., 16 (2019), 1950018-3202.

[12] Cai-Yun Li and Chun-Gang Zhu, Construction of the spacelike constant angle surface family in Minkowski 3-space, AIMS Mathematics 5(6) (2020), 6341-6354.

[13] T. Körpinar and Y. Ünlütürk, An approach to energy and elastic for curves with extended Darboux frame in Minkowski space. AIMS Mathematics, 5(2) (2020), 1025-1034.

[14] E. Bayram, F. Güler and E. Kasap, Parametric representation of a surface pencil with a common asymptotic curve. Computer-Aided Design 44 (2012), 637-643.

[15] S. Yılmaz, Determination of space-like curves by vector differential equations in Minkowski space E_1^3. J Adv Res Pure Math. 1(1) (2009), 10–14.