Conditionally Specified Bivariate Kummer-Gamma Distribution

DAYA K. NAGAR, EDWIN ZARRAZOLA AND ALEJANDRO ROLDÁN-CORREA
Instituto de Matemáticas
Universidad de Antioquia
Calle 67, No. 53–108, Medellín
COLOMBIA

Abstract: - The Kummer-gamma distribution is an extension of gamma distribution and for certain values of parameters slides to a bimodal distribution. In this article, we introduce a bivariate distribution with Kummer-gamma conditionals and call it the *conditionally specified bivariate Kummer-gamma distribution/bivariate Kummer-gamma conditionals distribution*. Various representations are derived for its product moments, marginal densities, marginal moments, conditional densities, and conditional moments. We also discuss several important properties including, entropies, distributions of sum, and quotient. Most of these representations involve special functions such as the Gauss and the confluent hypergeometric functions. The bivariate Kummer-gamma conditionals distribution studied in this article may serve as an alternative to many existing bivariate models with support on $(0,\infty)\times(0,\infty)$.

Key-Words: - Bivariate distribution; confluent hypergeometric function; gamma distribution; gamma function; Gauss hypergeometric function; Kummer-gamma distribution.

Received: December 31, 2020. Revised: April 10, 2021. Accepted: April 30, 2021. Published: May 20, 2021.

1 Introduction

There are several bivariate distributions proposed in the statistical literature, see Arnold, Castillo and Sarabia [1], Balakrishnan and Lai [2], Kotz, Balakrishnan and Johnson [3], Hutchinson and Lai [4], and Mardia [5] for good reviews. For more recent work, the reader is referred to Bran-Cardona, Orozco-Castañeda and Nagar [6], Bakouch et al. [7], Bondesson [8], Chen, Tzeng and Lin [9], Franco, Vivo and Kundu [10], Ghosh [11], Gupta and Nadarajah [12], Gupta and Nagar [13, 14], Gupta, Orozco-Castañeda and Nagar [15], Mathai [16], Nadarajah and Kotz [17, 18], Nagar, Arashi and Nadarajah [19], Nagar, Nadarajah and Okorie [20], Orozco-Castañeda, Nagar and Gupta [21], and Semenikhine, Furman and Su [22]. These distributions have attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e., the proportions of brands of some consumer product that are bought by customers, proportions of the electorate voting for the candidate in a two-candidate election and the dependence between two soil strength parameters, and hydrology. They've also been widely used as a prior in Bayesian statistics.

Bivariate distributions have also been applied in areas such as physics, economics, engineering, risk analysis, and medicine. For interesting real life applications the reader is referred to Alsayed and Manzi [23], Danaher and Smith [24], Li et al. [25], Makanda and Shaw [26], Takeuchi [27], and references therein.

Arnold and Strauss [28] considered the most general class of bivariate distributions such that both sets of conditional densities are exponential (also see Nadarajah and Choi [29]). They called their bivariate distribution the *bivariate exponential conditionals (BEC) distribution*. The conditionally specified bivariate gamma distribution is given in Kotz, Balakrishnan and Johnson [30] (also see Arnold, Castillo and Sarabia [1], Nadarajah [31]). For a review on the construction of bivariate distributions by using the conditional approach, the reader is referred to Arnold, Castillo and Sarabia [1], and Balakrishnan and Lai [2, Chapter 6].

In this article, we introduce a bivariate distribution of positive random variables X and Y such that conditional densities of $X\mid y$ and $Y\mid x$ are Kummer-gamma. The bivariate distribution defined in this article is closely connected to gamma and Kummer-gamma distributions, and, therefore we first define these distributions.

The random variable X is said to have a gamma distribution with parameters (α,θ), denoted as $X\sim Ga(\alpha,\theta)$, if its probability density function (pdf) is given by

$$\frac{x^{\alpha-1}\exp\left(-x/\theta\right)}{\theta^{\alpha}\Gamma(\alpha)},\ x>0,$$

where $\alpha>0$ and $\theta>0$.

The random variable X is said to have a Kummer-gamma distribution with parameters (α,γ,ξ), denoted as $X\sim KG(\alpha,\gamma,\xi)$, if its pdf is given by

$$\frac{x^{\alpha-1}\exp\left(-\xi x\right)(1+x)^{-\gamma}}{\Gamma(\alpha)\psi\left(\alpha,\alpha-\gamma+1;\xi\right)},\ x>0,$$

where $\alpha>0$, $\xi>0$, $-\infty<\gamma<\infty$, and ψ is the confluent hypergeometric functions of the second kind (also known as Kummer's function of the second kind, Tricomi function, or Gordon function), see Luke [32]. The Kummer-gamma distribution is an extension of gamma distribution and for $\alpha<1$ (and certain values of γ) yields bimodal distribution. For properties and results on gamma and Kummer-gamma distributions the reader is referred to Gupta and Nagar [33], Gupta, Cardeño and Nagar [34], Johnson, Kotz and Balakrishnan [35], and Koudou [36].

Consider the bivariate distribution of positive random variables X and Y defined by the pdf

$$\begin{aligned} f(x,y;\alpha,\beta,\nu,\sigma,\phi) &= K(\alpha,\beta,\nu,\sigma,\phi)x^{\alpha-1}y^{\beta-1}(x+y)^{\nu} \\ &\times \exp\left[-\left(\frac{x}{\sigma}+\frac{y}{\phi}\right)\right], \; x>0,\; y>0, \end{aligned} \tag{1}$$

where $\alpha>0$, $\beta>0$, $\sigma>0$, $\phi>0$ and $\nu\geq 0$. The normalizing constant $K(\alpha,\beta,\nu,\sigma,\phi)$, by using Lemma A.1, is given by

$$\begin{aligned} [K(\alpha,\beta,\nu,\sigma,\phi)]^{-1} &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\phi^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \\ &\times {}_2F_1\left(\alpha,\alpha+\beta+\nu;\alpha+\beta;1-\frac{\phi}{\sigma}\right),\; \frac{\phi}{\sigma}\leq 1 \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\sigma^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \\ &\times {}_2F_1\left(\beta,\alpha+\beta+\nu;\alpha+\beta;1-\frac{\sigma}{\phi}\right),\; \frac{\sigma}{\phi}< 1. \end{aligned}$$

From (1), by integrating suitably, the marginal densities of X and Y can be derived as (see Theorem 2.2 and Theorem 2.3),

$$\begin{aligned} &K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\beta)x^{\alpha+\beta+\nu-1}\exp\left(-\frac{x}{\sigma}\right) \\ &\quad\times \psi\left(\beta,\beta+\nu+1;\frac{x}{\phi}\right),\quad x>0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\alpha)y^{\alpha+\beta+\nu-1}\exp\left(-\frac{y}{\phi}\right) \\ &\quad\times \psi\left(\alpha,\alpha+\nu+1;\frac{y}{\sigma}\right),\quad y>0, \end{aligned} \tag{3}$$

respectively. Observe that the densities of X and Y are not gamma and have an additional factor containing the confluent hypergeometric function and can be treated close allies of the gamma distribution. By using (1) and (2), the conditional density of $X\,|\,y$ is obtained as

$$\frac{x^{\alpha-1}y^{-(\alpha+\nu)}(y+x)^{\nu}\exp(-x/\sigma)}{\Gamma(\alpha)\psi(\alpha,\alpha+\nu+1;y/\sigma)},\quad x>0. \tag{4}$$

Likewise, the use of (1) and (3) yields the conditional density of $Y\,|\,x$ as

$$\frac{y^{\beta-1}x^{-(\beta+\nu)}(x+y)^{\nu}\exp(-y/\phi)}{\Gamma(\beta)\psi(\beta,\beta+\nu+1;x/\phi)},\quad y>0. \tag{5}$$

From (4) and (5), it is clear that conditional distributions of $X\,|\,y$ and $Y\,|\,x$ are Kummer-gamma. Thus, we call the bivariate distribution defined by the density (1) the *conditionally specified bivariate Kummer-gamma (CSBKG) distribution*. This distribution can also be referred to as the *bivariate Kummer-gamma conditionals distribution*. Further, we will write $(X,Y)\sim \text{CSBKG}(\alpha,\beta,\nu,\sigma,\phi)$ if the joint density of X and Y is given by (1). The CSBKG model may serve as an alternative to many existing bivariate distributions with support on $(0,\infty)\times(0,\infty)$ and can have possible applications in areas such as cross-over trials, life testing, hydrology, reliability theory, renewal processes, and stochastic routing problems (see Nadarajah and Kotz [18]).

In this article, in Section 2, we study several properties such as marginal and conditional distributions, joint moments, correlation, and mixture representation of the bivariate distribution defined by the density (1). We also derive distributions of $X+Y$ and $X/(X+Y)$, where $(X,Y)\sim \text{CSBKG}(\alpha,\beta,\nu,\sigma,\phi)$. In Section 4, entropies such as Renyi and Shannon are derived for the conditionally specified bivariate Kummer-gamma distribution defined in this article. Finally, in the Appendix, several known results used in this article are given.

2 Properties

In this section we study several properties of the conditionally specified bivariate Kummer-gamma distribution defined in Section 1.

First, we briefly discuss the shape of (1) for $\sigma=\phi$. The first derivatives of $\ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)$ with respect to x and y are

$$\begin{aligned} f_x(x,y) &= \frac{\partial \ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)}{\partial x} \\ &= \frac{\alpha-1}{x}+\frac{\nu}{x+y}-\frac{1}{\sigma} \end{aligned} \tag{6}$$

and

$$\begin{aligned} f_y(x,y) &= \frac{\partial \ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)}{\partial y} \\ &= \frac{\beta-1}{y}+\frac{\nu}{x+y}-\frac{1}{\sigma} \end{aligned} \tag{7}$$

respectively. Setting (6) and (7) to zero, one can compute stationary point of (1) as (a,b), $a=(\alpha-1)(\alpha+\beta+\nu-2)\sigma/(\alpha+\beta-2)$, $b=(\beta-1)(\alpha+\beta+\nu-2)\sigma/(\alpha+\beta-2)$. Computing second order derivatives of $\ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)$, from (6) and (7), we get

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial^2 \ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)}{\partial x^2} \\ &= -\frac{\alpha-1}{x^2}-\frac{\nu}{(x+y)^2}, \end{aligned} \tag{8}$$

$$f_{xy}(x,y) = \frac{\partial^2 \ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)}{\partial x \partial y} = -\frac{\nu}{(x+y)^2}, \tag{9}$$

and

$$f_{yy}(x,y) = \frac{\partial^2 \ln f(x,y;\alpha,\beta,\nu,\sigma,\sigma)}{\partial y^2} = -\frac{\beta-1}{y^2} - \frac{\nu}{(x+y)^2}. \tag{10}$$

Further, from (8), (9) and (10), we get

$$f_{xx}(a,b) = -\frac{(\alpha+\beta-2)^2+(\alpha-1)\nu}{(\alpha-1)(\alpha+\beta+\nu-2)^2\sigma^2},$$
$$f_{xy}(a,b) = -\frac{\nu}{(\alpha+\beta+\nu-2)^2\sigma^2},$$
$$f_{yy}(a,b) = -\frac{(\alpha+\beta-2)^2+(\beta-1)\nu}{(\beta-1)(\alpha+\beta+\nu-2)^2\sigma^2}$$

and

$$f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 = \frac{(\alpha+\beta-2)^3}{(\alpha-1)(\beta-1)(\alpha+\beta+\nu-2)^3\sigma^4}.$$

Now, observe that

- If $\alpha > 1$, $\beta > 1$ and $\alpha+\beta+\nu > 2$, then $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 > 0$, $f_{xx}(a,b) < 0$ and $f_{yy}(a,b) < 0$ and therefore (a,b) is a maximum point.
- If $\alpha > 1$, $\beta > 1$ and $0 < \alpha+\beta+\nu < 2$, then $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 < 0$, and therefore (a,b) is a saddle point.
- If $\alpha > 1$, $\beta < 1$, $\alpha+\beta < 2$ and $0 < \alpha+\beta+\nu < 2$, then $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 < 0$, and therefore (a,b) is a saddle point.
- If $\alpha > 1, \beta < 1, \alpha+\beta < 2$ and $\alpha+\beta+\nu > 2$, then $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 > 0$, and therefore (a,b) is a relative maximum.
- If $\alpha < 1, \beta > 1, \alpha+\beta < 2$ and $\alpha+\beta+\nu > 2$, then $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2 > 0$, and therefore (a,b) is a relative maximum.

Figure 1 illustrates the shape of the pdf (1) for selected values of α,β,ν,σ and ϕ. Here one can appreciate the wide range of shapes that result from the bivariate density defined by (1).

For a non-negative integer ν, we can write (1) as a linear combination of the product of gamma densities; that is

$$\sum_{j=0}^{\nu} C_j \binom{\nu}{j} f_X(x;\alpha+j,\sigma) f_Y(y;\beta+\nu-j,\phi), \tag{11}$$

where $x > 0$ and $y > 0$ with

$$C_j = K(\alpha,\beta,\nu,\sigma,\phi)\sigma^{\alpha+j}\phi^{\beta+\nu-j}\Gamma(\alpha+j)\Gamma(\beta+\nu-j).$$

Further, for $\nu = 0$, the random variables X and Y are independent each having gamma distribution. Thus, the distribution defined by the density (1) is a bivariate generalization of the gamma distribution.

A distribution is said to be negatively likelihood ratio dependent if the density $f(x,y)$ satisfies

$$f(x_1,y_1)f(x_2,y_2) \le f(x_1,y_2)f(x_2,y_1)$$

(Lehmann [37], Tong [38]). In the case of conditionally specified bivariate Kummer-gamma distribution the above inequality reduces to

$$(x_1+y_1)(x_2+y_2) < (x_1+y_2)(x_2+y_1), \quad \nu > 0$$

which clearly holds. Hence, for $\nu > 0$, the bivariate distribution defined by the density (1) is negatively likelihood ratio dependent.

Theorem 2.1. *Let $(X,Y) \sim \text{CSBKG}(\alpha,\beta,\nu,\sigma,\phi)$, and define $W = X/(X+Y)$ and $S = X+Y$. Then, the density of S is given by*

$$K(\alpha,\beta,\nu,\sigma,\phi)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}s^{\alpha+\beta+\nu-1}\exp\left(-\frac{s}{\phi}\right) \times {}_1F_1\left(\alpha;\alpha+\beta;\left(\frac{1}{\phi}-\frac{1}{\sigma}\right)s\right),\ s > 0$$

and the density of W is given by

$$K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\alpha+\beta+\nu) \times \frac{w^{\alpha-1}(1-w)^{\beta-1}}{[w/\sigma+(1-w)/\phi]^{\alpha+\beta+\nu}},\ 0 < w < 1,$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

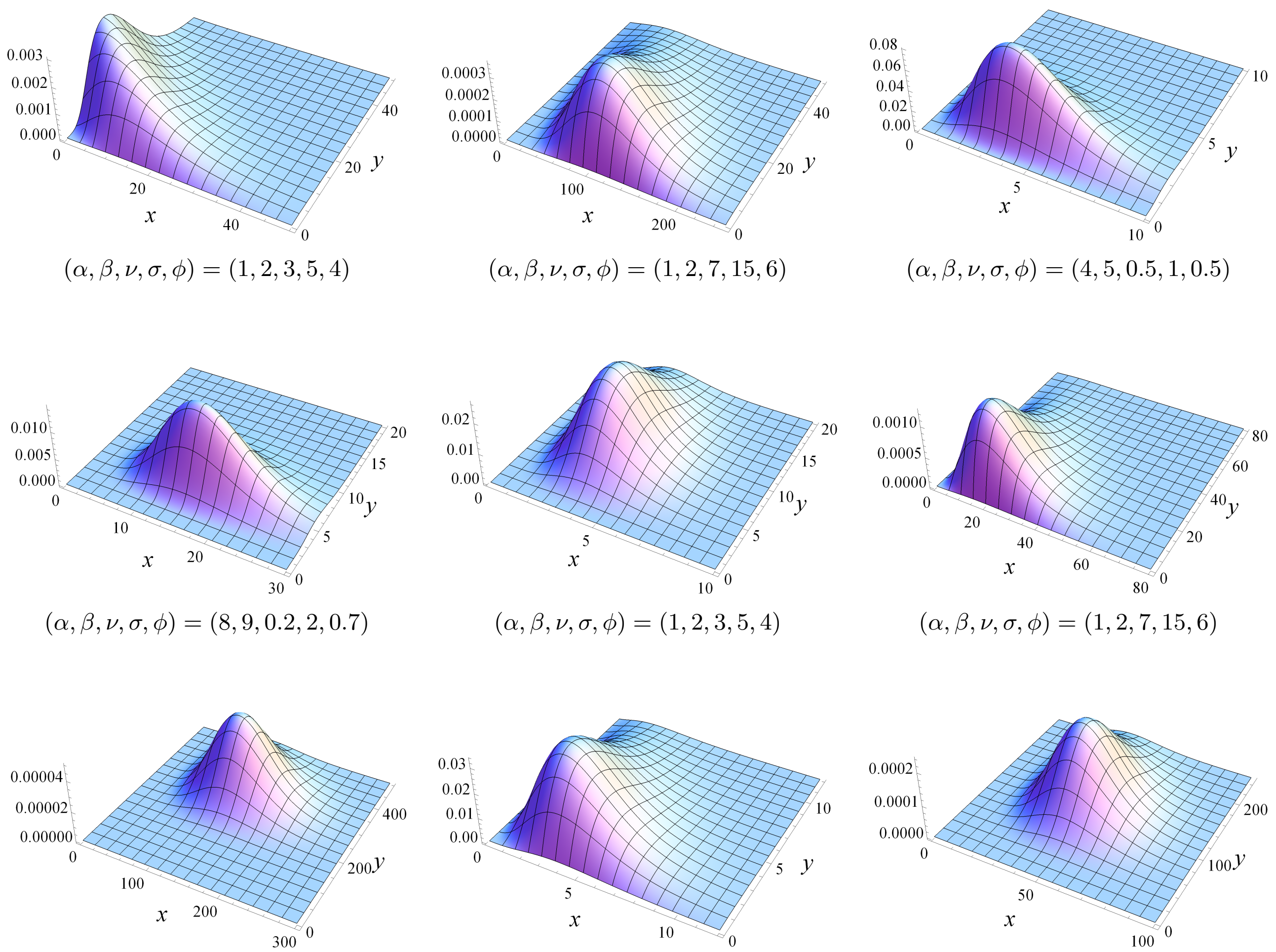
Proof. Substituting $x = ws$ and $y = s(1-w)$ with the Jacobian $J(x,y \to w,s) = s$, in the joint density of X and Y, we obtain the joint density of W and S as

$$K(\alpha,\beta,\nu,\sigma,\phi)s^{\alpha+\beta+\nu-1}w^{\alpha-1}(1-w)^{\beta-1} \times \exp\left[-\left(\frac{w}{\sigma}+\frac{1-w}{\phi}\right)s\right], \tag{12}$$

where $0 < s < 1$ and $0 < w < 1$. Now, integrating appropriately by using the integral representation of confluent hypergeometric function (A.6), we obtain marginal densities of S ans W. □

Corollary 2.1.1. *The density of $R = X/Y$ is given by*

$$K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\alpha+\beta+\nu) \times \frac{r^{\alpha-1}(1+r)^{\nu}}{(r/\sigma+1/\phi)^{\alpha+\beta+\nu}},\ r > 0.$$



$(\alpha,\beta,\nu,\sigma,\phi)=(1,2,3,5,4)$ $(\alpha,\beta,\nu,\sigma,\phi)=(1,2,7,15,6)$ $(\alpha,\beta,\nu,\sigma,\phi)=(4,5,0.5,1,0.5)$

$(\alpha,\beta,\nu,\sigma,\phi)=(8,9,0.2,2,0.7)$ $(\alpha,\beta,\nu,\sigma,\phi)=(1,2,3,5,4)$ $(\alpha,\beta,\nu,\sigma,\phi)=(1,2,7,15,6)$

$(\alpha,\beta,\nu,\sigma,\phi)=(4,5,0.5,1,0.5)$ $(\alpha,\beta,\nu,\sigma,\phi)=(8,9,0.2,2,0.7)$ $(\alpha,\beta,\nu,\sigma,\phi)=(10,5,6,4,15)$

Figure 1: Plots of the pdf (1) for some selected values of parameters.

Proof. Use the transformation $W=R/(1+R)$ in the density of W given in the Theorem 2.1. □

Corollary 2.1.2. *If $\sigma=\phi$, then, the density of S is given by*

$$\frac{s^{\alpha+\beta+\nu-1}\exp\left(-s/\sigma\right)}{\sigma^{\alpha+\beta+\nu}\Gamma(\alpha+\beta)},\ s>0.$$

The density of W is given by

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}w^{\alpha-1}(1-w)^{\beta-1},\ 0<w<1$$

and the density of R is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{r^{\alpha-1}}{(1+r)^{\alpha+\beta}},\ r>0.$$

Proof. Substitute $\sigma=\phi$ in the Theorem 2.1 and Corollary 2.1.1. □

Corollary 2.1.3. *Let X and Y be independent $X\sim Ga(\alpha,\sigma)$ and $Y\sim Ga(\beta,\phi)$. Then, the density of S is given by*

$$\frac{s^{\alpha+\beta-1}}{\sigma^{\alpha}\phi^{\beta}\Gamma(\alpha+\beta)}\exp\left(-\frac{s}{\phi}\right)\times{}_1F_1\left(\alpha;\alpha+\beta;\left(\frac{1}{\phi}-\frac{1}{\sigma}\right)s\right),\ s>0.$$

The density of W is given by

$$\frac{\Gamma(\alpha+\beta)}{\sigma^{\alpha}\phi^{\beta}\Gamma(\alpha)\Gamma(\beta)}\frac{w^{\alpha-1}(1-w)^{\beta-1}}{[w/\sigma+(1-w)/\phi]^{\alpha+\beta}},\ 0<w<1$$

and the density of R is

$$\frac{\Gamma(\alpha+\beta)}{\sigma^{\alpha}\phi^{\beta}\Gamma(\alpha)\Gamma(\beta)}\frac{r^{\alpha-1}}{(r/\sigma+1/\phi)^{\alpha+\beta}},\ r>0.$$

Proof. Substitute $\nu=0$ in the Theorem 2.1 and the Corollary 2.1.1. □

By using the above theorem and (A.5), it is straightforward to show that

$$E(S^r) = \frac{\phi^r \Gamma(\alpha+\beta+\nu+r)}{\Gamma(\alpha+\beta+\nu)} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+r; \alpha+\beta; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

if $\phi \le \sigma$ and

$$E(S^r) = \frac{\sigma^r \Gamma(\alpha+\beta+\nu+r)}{\Gamma(\alpha+\beta+\nu)} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+r; \alpha+\beta; 1-\sigma/\phi)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\sigma/\phi)},$$

if $\phi > \sigma$. Further, by using the density of W given in Theorem 2.1, we derive $E(W^r)$ as

$$E(W^r) = K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\alpha+\beta+\nu) \times \int_0^1 \frac{w^{\alpha+r-1}(1-w)^{\beta-1}}{[w/\sigma+(1-w)/\phi]^{\alpha+\beta+\nu}}\, dw.$$

Now, writing

$$\frac{w}{\sigma}+\frac{1-w}{\phi} = \begin{cases} \left[1-\left(1-\frac{\phi}{\sigma}\right)w\right]\frac{1}{\phi}, & \frac{\phi}{\sigma}\le 1 \\ \left[1-\left(1-\frac{\sigma}{\phi}\right)(1-w)\right]\frac{1}{\sigma}, & \frac{\sigma}{\phi}<1, \end{cases}$$

and integrating w by using (A.7), we get

$$\begin{aligned} E(W^r) &= \frac{\Gamma(\alpha+r)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+r)} \times \frac{{}_2F_1(\alpha+r, \alpha+\beta+\nu+r; \alpha+\beta; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)}, \quad \phi/\sigma \le 1, \\ &= \frac{\Gamma(\beta+r)\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha+\beta+r)} \times \frac{{}_2F_1(\beta+r, \alpha+\beta+\nu+r; \alpha+\beta; 1-\sigma/\phi)}{{}_2F_1(\beta, \alpha+\beta+\nu; \alpha+\beta; 1-\sigma/\phi)}, \quad \phi/\sigma > 1. \end{aligned}$$

In next two theorems, we derive marginal distributions of X and Y. It is interesting to note that these marginal distributions do not belong to the gamma family and differ by an additional factor containing the confluent hypergeometric function ψ. Figure 2 and Figure 3 illustrate the shape of the marginal denisity of X for selected of $\alpha, \beta, \nu, \sigma$ and ϕ. It can be observed that, for certain values of parameters, the marginal density of X tends to symmetry. A graphical comparison of the marginal density of X with the gamma density is also given in Figure 2 and Figure 3. It can also be seen that, compared to the gamma density, the curves of the marginal density of X are taller and have thinner tails. Further, the median of the marginal density of X is smaller than that of the gamma density.

Theorem 2.2. *If* $(X,Y) \sim$ CSBKG$(\alpha,\beta,\nu,\sigma,\phi)$*, then the marginal density of* X *is given by*

$$K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\beta)x^{\alpha+\beta+\nu-1}\exp\left(-\frac{x}{\sigma}\right) \times \psi\left(\beta,\beta+\nu+1,\frac{x}{\phi}\right),\ x>0,$$

where ψ *is the confluent hypergeometric function of the second kind.*

Proof. To find the marginal density of X, we integrate (1) with respect to y to get

$$K(\alpha,\beta,\nu,\sigma,\phi)x^{\alpha-1}\exp\left(-\frac{x}{\sigma}\right) \times \int_0^\infty y^{\beta-1}(x+y)^\nu \exp\left(-\frac{y}{\phi}\right) \mathrm{d}y.$$

Substituting $z = y/x$ with $\mathrm{d}y = x\,\mathrm{d}z$ above, one obtains

$$K(\alpha,\beta,\nu,\sigma,\phi)x^{\alpha+\beta+\nu-1}\exp\left(-\frac{x}{\sigma}\right) \times \int_0^\infty z^{\beta-1}(1+z)^\nu \exp\left(-\frac{xz}{\phi}\right) \mathrm{d}z.$$

Now, the desired result is obtained by using (A.8). □

Theorem 2.3. *If* $(X,Y) \sim$ CSBKG$(\alpha,\beta,\nu,\sigma,\phi)$*, then the marginal density of* Y *is given by*

$$K(\alpha,\beta,\nu,\sigma,\phi)\Gamma(\alpha)y^{\alpha+\beta+\nu-1}\exp\left(-\frac{y}{\phi}\right) \times \psi\left(\alpha,\alpha+\nu+1,\frac{y}{\sigma}\right),\ y>0.$$

Proof. Similar to the proof of the Theorem 2.2. □

Using the above theorem, the conditional density function of X given $Y = y$, $y > 0$, is obtained as

$$\frac{\exp(-x/\sigma)x^{\alpha-1}(x+y)^\nu}{\Gamma(\alpha)y^{\alpha+\nu}\psi(\alpha,\alpha+\nu+1,y/\sigma)},\ x>0.$$

Similarly, using Theorem 2.2, the conditional density function of Y given $X = x$, $x > 0$, is derived as

$$\frac{\exp(-y/\phi)y^{\beta-1}(x+y)^\nu}{\Gamma(\beta)x^{\beta+\nu}\psi(\beta,\beta+\nu+1,x/\phi)},\ y>0.$$

Further, using conditional densities given above, we derive

$$E(X^r \mid y) = \frac{y^r\Gamma(\alpha+r)\psi(\alpha+r,\alpha+\nu+r+1,y/\sigma)}{\Gamma(\alpha)\psi(\alpha,\alpha+\nu+1,y/\sigma)}$$

and

$$E(Y^r \mid x) = \frac{x^r\Gamma(\beta+r)\psi(\beta+r,\alpha+\nu+r+1,x/\phi)}{\Gamma(\beta)\psi(\beta,\beta+\nu+1,x/\phi)}.$$

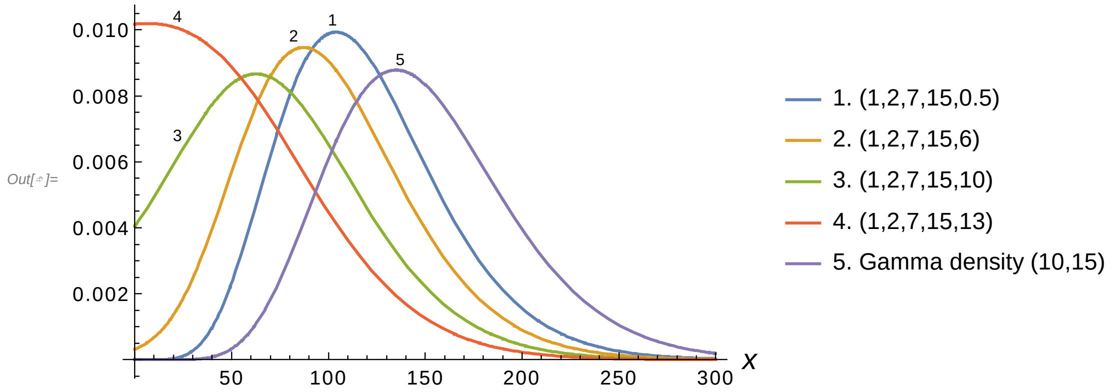


Figure 2: Comparison of the marginal density of X with the gamma density. Plots 1, 2, 3 and 4 are drawn by using the density of X given in Theorem 2.2 for $(\alpha, \beta, \nu, \sigma, \phi) = (1, 2, 7, 15, \phi)$ with $\phi/\sigma < 1$, and the plot of the gamma density with parameters $(1 + 2 + 7, 15)$ is given by 5.

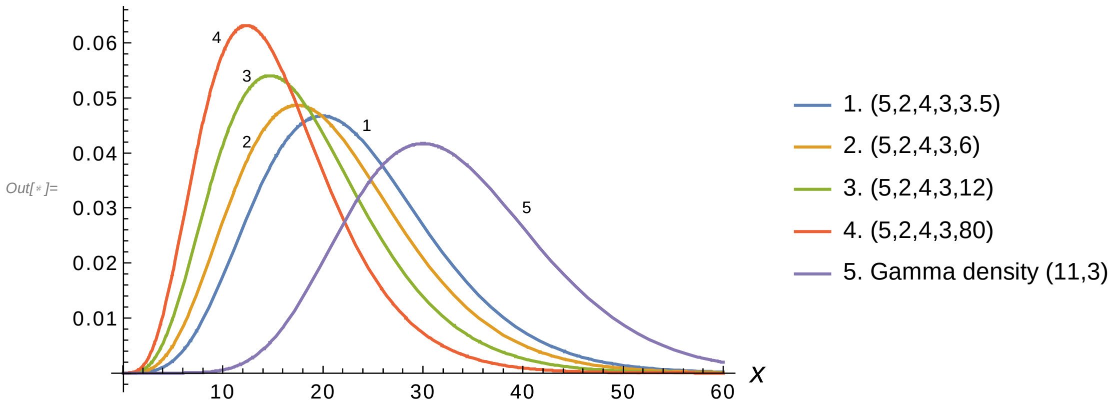


Figure 3: comparison of the marginal density of X with the gamma density. Plots 1, 2, 3 and 4 are drawn by using the density of X given in theorem 2.2 for $(\alpha, \beta, \nu, \sigma, \phi) = (5, 2, 4, 3, \phi)$ with $\sigma/\phi < 1$, and the plot of the gamma density with parameters $(5 + 2 + 4, 3)$ is given by 5.

Also, using (1), the joint (r, s)-th moment is obtained as

$$\begin{aligned} \mathrm{E}(X^r Y^s) &= K(\alpha, \beta, \nu, \sigma, \phi) \int_0^\infty \int_0^\infty x^{\alpha+r-1} y^{\beta+s-1} \\ &\quad (x+y)^\nu \exp\left[-\left(\frac{x}{\sigma} + \frac{y}{\phi}\right)\right] \mathrm{d}y\, \mathrm{d}x \\ &= \frac{K(\alpha, \beta, \nu, \sigma, \phi)}{K(\alpha + r, \beta + s, \nu, \sigma, \phi)}. \end{aligned}$$

Further, substituting for $K(\alpha, \beta, \nu, \sigma, \phi)$, one gets

$$\begin{aligned} &\mathrm{E}(X^r Y^s) \\ &= \phi^{r+s} \frac{\Gamma(\alpha + r)\Gamma(\beta + s)\Gamma(\alpha + \beta + \nu + r + s)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + r + s)} \\ &\quad \times \frac{{}_2F_1(\alpha+r, \alpha+\beta+\nu+r+s; \alpha+\beta+r+s; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)}, \\ &\qquad \phi/\sigma \le 1, \end{aligned}$$

$$\begin{aligned} &= \sigma^{r+s} \frac{\Gamma(\alpha + r)\Gamma(\beta + s)\Gamma(\alpha + \beta + \nu + r + s)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + r + s)} \\ &\quad \times \frac{{}_2F_1(\beta+s, \alpha+\beta+\nu+r+s; \alpha+\beta+r+s; 1-\sigma/\phi)}{{}_2F_1(\beta, \alpha + \beta + \nu; \alpha + \beta; 1 - \sigma/\phi)}, \\ &\qquad \sigma/\phi < 1, \end{aligned}$$

where $\alpha + r > 0$, $\beta + s > 0$ and $\alpha + \beta + \nu + r + s > 0$. Now, substituting appropriately, we obtain, for $\phi/\sigma \le 1$,

$$\begin{aligned} &\mathrm{E}[(XY)^h] \\ &= \phi^{2h} \frac{\Gamma(\alpha + h)\Gamma(\beta + h)\Gamma(\alpha + \beta + \nu + 2h)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + \nu)\Gamma(\alpha + \beta + 2h)} \\ &\quad \times \frac{{}_2F_1(\alpha+h, \alpha+\beta+\nu+2h; \alpha+\beta+2h; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha + \beta + \nu; \alpha + \beta; 1 - \phi/\sigma)}, \end{aligned}$$

$$\mathrm{E}(X) = \phi \frac{\alpha(\alpha+\beta+\nu)}{\alpha+\beta} \times \frac{{}_2F_1(\alpha+1, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

$$\mathrm{E}(Y) = \phi \frac{\beta(\alpha+\beta+\nu)}{\alpha+\beta} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

$$\mathrm{E}(X^2) = \phi^2 \frac{\alpha(\alpha+1)(\alpha+\beta+\nu)(\alpha+\beta+\nu+1)}{(\alpha+\beta)(\alpha+\beta+1)} \times \frac{{}_2F_1(\alpha+2, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

$$\mathrm{E}(Y^2) = \phi^2 \frac{\beta(\beta+1)(\alpha+\beta+\nu)(\alpha+\beta+\nu+1)}{(\alpha+\beta)(\alpha+\beta+1)} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

$$\mathrm{E}(XY) = \phi^2 \frac{\alpha\beta(\alpha+\beta+\nu)(\alpha+\beta+\nu+1)}{(\alpha+\beta)(\alpha+\beta+1)} \times \frac{{}_2F_1(\alpha+1, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)},$$

$$\mathrm{Var}(X) = \phi^2 \frac{\alpha(\alpha+\beta+\nu)}{\alpha+\beta} \Bigg[\frac{(\alpha+1)(\alpha+\beta+\nu+1)}{\alpha+\beta+1} \times \frac{{}_2F_1(\alpha+2, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} - \frac{\alpha(\alpha+\beta+\nu)}{\alpha+\beta} \times \left\{ \frac{{}_2F_1(\alpha+1, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} \right\}^2 \Bigg],$$

$$\mathrm{Var}(Y) = \phi^2 \frac{\beta(\alpha+\beta+\nu)}{\alpha+\beta} \Bigg[\frac{(\beta+1)(\alpha+\beta+\nu+1)}{\alpha+\beta+1} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} - \frac{\beta(\alpha+\beta+\nu)}{\alpha+\beta} \times \left\{ \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} \right\}^2 \Bigg],$$

and

$$\mathrm{Cov}(X,Y) = \phi^2 \frac{\alpha\beta(\alpha+\beta+\nu)}{\alpha+\beta} \Bigg[\frac{\alpha+\beta+\nu+1}{\alpha+\beta+1} \times \frac{{}_2F_1(\alpha+1, \alpha+\beta+\nu+2; \alpha+\beta+2; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} - \frac{\alpha+\beta+\nu}{\alpha+\beta} \times \frac{{}_2F_1(\alpha+1, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} \times \frac{{}_2F_1(\alpha, \alpha+\beta+\nu+1; \alpha+\beta+1; 1-\phi/\sigma)}{{}_2F_1(\alpha, \alpha+\beta+\nu; \alpha+\beta; 1-\phi/\sigma)} \Bigg].$$

Notice that $\mathrm{E}(XY)$, $\mathrm{E}(X^2)$, $\mathrm{E}(Y^2)$, $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ involve ${}_2F_1(a, b; c; z)$ which can be computed by using a suitable software. Tables for correlations between X and Y can also be prepared by using the definition

$$\mathrm{corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

3 Entropies

In this section, Renyi and Shannon entropies are derived for the conditionally specified bivariate Kummer-gamma distribution defined in this article.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. The well-known Shannon entropy $H_{SH}(f)$ introduced by Shannon [39] is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \log f(x)\, \mathrm{d}\mu. \tag{13}$$

One of the main extensions of the Shannon entropy was defined by Rényi [40]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\log G(\eta)}{1-\eta} \qquad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \tag{14}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^{\eta} d\mu.$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η, while the Shannon entropy (13) is obtained from (14) for $\eta \uparrow 1$. These entropies have been used in information theory, science and engineering. For details see Nadarajah and Zografos [41], Zografos [42], and Zografos and Nadarajah [43].

First, we give the following lemma useful in deriving these entropies.

Lemma 3.1. *Let* $g(\alpha,\beta,\nu,\sigma,\phi)=\lim_{\eta\to 1} h(\eta)$*, where*

$$h(\eta)=\frac{\mathrm{d}}{\mathrm{d}\eta}{}_2F_1\Big(\eta(\alpha-1)+1,\eta(\alpha+\beta+\nu-2)+2;\eta(\alpha+\beta-2)+2;1-\frac{\phi}{\sigma}\Big).$$

with $\phi/\sigma<1$*. Then,*

$$\begin{aligned}&g(\alpha,\beta,\nu,\sigma,\phi)\\&=\sum_{j=1}^{\infty}\frac{\Gamma(\alpha+j)\Gamma(\alpha+\beta+\nu+j)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+\nu)\Gamma(\alpha+\beta+j)}\\&\quad\frac{(1-\phi/\sigma)^j}{j!}\Big[(\alpha-1)\psi(\alpha+j)\\&\quad+(\alpha+\beta+\nu-2)\psi(\alpha+\beta+\nu+j)\\&\quad+(\alpha+\beta-2)\psi(\alpha+\beta)-(\alpha-1)\psi(\alpha)\\&\quad-(\alpha+\beta+\nu-2)\psi(\alpha+\beta+\nu)\\&\quad-(\alpha+\beta-2)\psi(\alpha+\beta+j)\Big],\end{aligned}\tag{15}$$

where $\psi(\alpha)=\Gamma'(\alpha)/\Gamma(\alpha)$ *is the digamma function.*

Proof. Expanding ${}_2F_1$ in series form, we write

$$\begin{aligned}h(\eta)&=\frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{j=0}^{\infty}\Delta_j(\eta)\frac{(1-\phi/\sigma)^j}{j!}\\&=\sum_{j=0}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_j(\eta)\right]\frac{(1-\phi/\sigma)^j}{j!},\end{aligned}\tag{16}$$

where

$$\begin{aligned}\Delta_j(\eta)&=\frac{\Gamma[\eta(\alpha-1)+1+j]}{\Gamma[\eta(\alpha-1)+1]}\\&\quad\times\frac{\Gamma[\eta(\alpha+\beta+\nu-2)+2+j]}{\Gamma[\eta(\alpha+\beta+\nu-2)+2]}\\&\quad\times\frac{\Gamma[\eta(\alpha+\beta-2)+2]}{\Gamma[\eta(\alpha+\beta-2)+2+j]}.\end{aligned}$$

Now, differentiating the logarithm of $\Delta_j(\eta)$ w.r.t. to η, one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_j(\eta)\\&=\Delta_j(\eta)\Big[(\alpha-1)\psi(\eta(\alpha-1)+1+j)\\&\quad+(\alpha+\beta+\nu-2)\psi(\eta(\alpha+\beta+\nu-2)+2+j)\\&\quad+(\alpha+\beta-2)\psi(\eta(\alpha+\beta-2)+2)\\&\quad-(\alpha-1)\psi(\eta(\alpha-1)+1)\\&\quad-(\alpha+\beta+\nu-2)\psi(\eta(\alpha+\beta+\nu-2)+2)\\&\quad-(\alpha+\beta-2)\psi(\eta(\alpha+\beta-2)+2+j)\Big].\end{aligned}\tag{17}$$

Finally, substituting (17) in (16) and taking $\eta\to 1$, one obtains the desired result. □

Theorem 3.1. *For the bivariate distribution defined by the pdf (1), the Rényi and the Shannon entropies are given by*

$$\begin{aligned}H_R(\eta,f)&=\frac{1}{1-\eta}\Big[\eta\ln K(\alpha,\beta,\nu,\sigma,\phi)\\&\quad+[\eta(\alpha+\beta+\nu-2)+2]\ln\Big(\frac{\phi}{\eta}\Big)\\&\quad+\ln\Gamma[\eta(\alpha-1)+1]+\ln\Gamma[\eta(\beta-1)+1]\\&\quad+\ln\Gamma[\eta(\alpha+\beta+\nu-2)+2]\\&\quad-\ln\Gamma[\eta(\alpha+\beta-2)+2]\\&\quad+\ln{}_2F_1\Big(\eta(\alpha-1)+1,\eta(\alpha+\beta+\nu-2)+2;\\&\qquad\eta(\alpha+\beta-2)+2;1-\frac{\phi}{\sigma}\Big)\Big]\end{aligned}$$

and

$$\begin{aligned}H_{SH}(f)=&-\ln K(\alpha,\beta,\nu,\sigma,\phi)\\&-[(\alpha-1)\psi(\alpha)+(\beta-1)\psi(\beta)\\&+(\alpha+\beta+\nu-2)\psi(\alpha+\beta+\nu)\\&-(\alpha+\beta-2)\psi(\alpha+\beta)]\\&-\frac{g(\alpha,\beta,\nu,\sigma,\phi)}{{}_2F_1\left(\alpha,\alpha+\beta+\nu;\alpha+\beta;1-\phi/\sigma\right)},\end{aligned}$$

respectively, where $\psi(\alpha)=\Gamma'(\alpha)/\Gamma(\alpha)$ *is the digamma function and* $g(\alpha,\beta,\nu,\sigma,\phi)$ *is given by (15).*

Proof. For $\eta>0$ and $\eta\neq 1$, using the joint density of X and Y given by (1), we have

$$\begin{aligned}G(\eta)&=\int_0^\infty\int_0^\infty f^\eta(x,y;\alpha,\beta;\nu;\sigma,\phi)\,\mathrm{d}y\,\mathrm{d}x\\&=[K(\alpha,\beta,\nu,\sigma,\phi)]^\eta\\&\quad\times\int_0^\infty\int_0^\infty x^{\eta(\alpha-1)}y^{\eta(\beta-1)}\,(x+y)^{\eta\nu}\\&\quad\exp\left[-\eta\left(\frac{x}{\sigma}+\frac{y}{\phi}\right)\right]\,\mathrm{d}y\,\mathrm{d}x\\&=\frac{[K(\alpha,\beta,\nu,\sigma,\phi)]^\eta}{K(\eta(\alpha-1)+1,\eta(\beta-1)+1,\eta\nu,\sigma/\eta,\phi/\eta)}\\&=[K(\alpha,\beta,\nu,\sigma,\phi)]^\eta\frac{\Gamma[\eta(\alpha+\beta+\nu-2)+2]}{\Gamma[\eta(\alpha+\beta-2)+2]}\\&\quad\times\frac{\Gamma[\eta(\alpha-1)+1]\Gamma[\eta(\beta-1)+1]}{(\phi/\eta)^{-\eta(\alpha+\beta+\nu-2)-2}}\\&\quad\times{}_2F_1\Big(\eta(\alpha-1)+1,\eta(\alpha+\beta+\nu-2)+2;\\&\qquad\eta(\alpha+\beta-2)+2;1-\frac{\phi}{\sigma}\Big),\end{aligned}$$

where the last line has been obtained by using (A.9). Now, taking logarithm of $G(\eta)$ and using (14) we get $H_R(\eta,f)$. The Shannon entropy is obtained from $H_R(\eta,f)$ by taking $\eta\uparrow 1$ and using L'Hopital's rule. □

4 Conclusion

We have considered a bivariate distribution of positive random variables X and Y defined by (1). It has been shown that conditional distributions of $X|y$ and $Y|x$ are Kummer-gamma thereby naming the bivariate distribution defined by (1) the *conditionally specified bivariate Kummer-gamma (CSBKG) distribution*. The CSBKG distribution may serve as an alternative to many existing bivariate distributions defined on $(0,\infty)\times(0,\infty)$. By using standard definitions and results, several properties of this distribution have also been derived. The distributions of $X+Y$ and $X/(X+Y)$ have also been obtained by using transformation of variables. It is interesting to note that most of the results (including the normalizing constant) derived in this article involve the well know Gauss' hypergeometric function/confluent hypergeometric function studied extensively in the literature making the CSBKG model mathematically and statistically interesting. The matrix variate generalization of the CSBKG model is also a captivating topic to explore.

Appendix

The Pochammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$. The generalized hypergeometric function of scalar argument is defined by

$$ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \quad \text{(A.1)}$$

where a_i, $i=1,\ldots,p$; b_j, $j=1,\ldots,q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (A.1) are available in the literature, see Luke [32]. From (A.1) it is easy to see that

$$ {}_1F_1(a;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \quad \text{(A.2)}$$

and

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad |z|<1. \quad \text{(A.3)}$$

Also, under suitable conditions, we have (Luke [32, Eq. 3.6(10)]),

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;zy)\,\mathrm{d}z = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \times {}_{p+1}F_{q+1}(a_1,\ldots,a_p,\alpha;b_1,\ldots,b_q,\alpha+\beta;y). \quad \text{(A.4)}$$

and (Luke [32, Eq. 3.6(13)]),

$$\int_0^\infty \exp(-\delta z)z^{\alpha-1}{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;zy)\,\mathrm{d}z = \delta^{-\alpha}\Gamma(\alpha) \times {}_{p+1}F_q(a_1,\ldots,a_p,\alpha,\alpha;b_1,\ldots,b_q;\delta^{-1}y). \quad \text{(A.5)}$$

The integral representations of the confluent hypergeometric function (first kind) and the Gauss hypergeometric function are given as

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,\mathrm{d}t, \quad \text{(A.6)}$$

and

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times\int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-zt)^b}\,\mathrm{d}t, \quad \text{(A.7)}$$

respectively, where $\mathrm{Re}(a)>0$ and $\mathrm{Re}(c-a)>0$. Note that, the series expansions for ${}_1F_1$ and ${}_2F_1$ given in (A.2) and (A.3) can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt|<1$, in (A.6) and (A.7) and integrating t.

The integral representations of the confluent hypergeometric function of the second kind is defined by the integral

$$\psi(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-zt)t^{a-1}(1+t)^{b-a-1}\,\mathrm{d}t, \quad \mathrm{Re}(a)>0. \quad \text{(A.8)}$$

For properties and further results on these functions the reader is referred to Luke [32].

Lemma A.1. *Let*

$$C(\alpha,\beta,\nu,\sigma,\phi) = \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\nu \exp\left[-\left(\frac{x}{\sigma}+\frac{y}{\phi}\right)\right]\mathrm{d}x\,\mathrm{d}y, \quad \text{(A.9)}$$

where $\alpha>0$, $\beta>0$, $\sigma>0$, $\phi>0$ *and* $\alpha+\beta+\nu>0$. *Then*

$$C(\alpha,\beta,\nu,\sigma,\phi) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\phi^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \times {}_2F_1\Big(\alpha,\alpha+\beta+\nu;\alpha+\beta;1-\frac{\phi}{\sigma}\Big)$$

for $\phi/\sigma<1$ *and*

$$C(\alpha,\beta,\nu,\sigma,\phi) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\nu)}{\sigma^{-(\alpha+\beta+\nu)}\Gamma(\alpha+\beta)} \times {}_2F_1\Big(\beta,\alpha+\beta+\nu;\alpha+\beta;1-\frac{\sigma}{\phi}\Big)$$

for $\sigma/\phi<1$.

Proof. Substituting $s = x + y$ and $r = x/(x + y)$ with $\mathrm{d}x\mathrm{d}y = s\mathrm{d}s\mathrm{d}r$ in (A.9) and integrating s, one gets

$$C(\alpha,\beta,\nu,\sigma,\phi) = \Gamma(\alpha+\beta+\nu)\int_0^1 r^{\alpha-1}(1-r)^{\beta-1} \left(\frac{r}{\sigma}+\frac{1-r}{\phi}\right)^{-(\alpha+\beta+\nu)} \mathrm{d}r. \qquad \text{(A.10)}$$

Now, writing

$$\left(\frac{r}{\sigma}+\frac{1-r}{\phi}\right)^{-(\alpha+\beta+\nu)} = \phi^{\alpha+\beta+\nu}\left[1-r\left(1-\frac{\phi}{\sigma}\right)\right]^{-(\alpha+\beta+\nu)}, \ \frac{\phi}{\sigma}<1$$

and integrating r by using (A.7) we get the result. Further if $\phi/\sigma > 1$, then $\sigma/\phi < 1$ and we write

$$\left(\frac{r}{\sigma}+\frac{1-r}{\phi}\right)^{-(\alpha+\beta+\nu)} = \sigma^{\alpha+\beta+\nu}\left[1-(1-r)\left(1-\frac{\sigma}{\phi}\right)\right]^{-(\alpha+\beta+\nu)}, \ \frac{\sigma}{\phi}<1$$

and (A.10) becomes

$$C(\alpha,\beta,\nu,\sigma,\phi) = \sigma^{\alpha+\beta+\nu}\Gamma(\alpha+\beta+\nu)\int_0^1 r^{\alpha-1}(1-r)^{\beta-1} \left[1-(1-r)\left(1-\frac{\sigma}{\phi}\right)\right]^{-(\alpha+\beta+\nu)} \mathrm{d}r.$$

Now, application of (A.7) yield the desired result. □

Acknowledgments

The research work of Daya K. Nagar and Edwin Zarrazola was supported by the Sistema Universitario de Investigación, Universidad de Antioquia [project no. 2019-25374].

The authors would like to thank three anonymous referees for their valuable comments which helped to improve the article.

References:

[1] B. C. Arnold, E. Castillo and J. M. Sarabia, *Conditional Specification of Statistical Models*, Springer-Verlag, New York, 1999.

[2] N. Balakrishnan and Chin-Diew Lai, *Continuous Bivariate Distributions*. Second edition. Springer, Dordrecht, 2009.

[3] S. Kotz, N. Balakrishnan and N. L. Johnson, *Continuous Multivariate Distributions, Vo. 1*, Second Edition, John Wiley & Sons, New York, 2000.

[4] T. P. Hutchinson and C. D. Lai, *The Engineering Statistician's Guide to Continuous Bivariate Distributions*, Rumsby Scientific Publishing, Adelaide, 1991.

[5] K. V. Mardia, *Families of Bivariate Distributions*, Griffin's Statistical Monographs and Courses, No. 27. Hafner Publishing Co., Darien, Conn., 1970.

[6] Paula Andrea Bran-Cardona, Johanna Marcela Orozco-Castañeda and Daya Krishna Nagar, Bivariate generalization of the Kummer-beta distribution, *Rev. Colombiana Estadíst.*, **34** (2011), no. 3, 497–512.

[7] Hassan S. Bakouch, Fernando A. Moala , Abdus Saboor and Haniya Samad, A bivariate Kumaraswamy-exponential distribution with application, *Mathematica Slovaca*, **69** (2019), no. 5, 1185–1212.

[8] Lennart Bondesson, On univariate and bivariate generalized gamma convolutions, *J. Statist. Plann. Inference*, **139** (2009), no. 11, 3759–3765.

[9] Li-Shya Chen, I-Shiang Tzeng and Chien-Tai Lin, Bivariate generalized gamma distributions of Kibble's type, *Statistics*, **48** (2014), no. 4, 933–949.

[10] Manuel Franco, Juana-María Vivo and Debasis Kundu, A Generator of bivariate distributions: properties, estimation, and epplications, *Mathematics*, **8** (2020), no. 10, 1776.

[11] Indranil Ghosh, On the reliability for some bivariate dependent beta and Kumaraswamy distributions: a brief survey, *Stoch. Qual. Control*, **34** (2019), no. 2, 115–121.

[12] Arjun K. Gupta and Saralees Nadarajah, Sums, products and ratios for McKay's bivariate gamma distribution, *Math. Comput. Modelling*, **43** (2006), no. 1–2, 185–193.

[13] Arjun K. Gupta and Daya K. Nagar, Matrix variate generalization of a bivariate beta type 1 distribution, *J. Stat. Manag. Syst.*, **12** (2009), no. 5, 873–885.

[14] Arjun K. Gupta and Daya K. Nagar, Some bimatrix beta distributions, *Comm. Statist. Theory Methods*, **41** (2012), no. 5, 869–879.

[15] Arjun K. Gupta, Johanna Marcela Orozco-Castañeda and Daya K. Nagar, Non-central bivariate beta distribution, *Statist. Papers*, **52** (2011), no. 1, 139–152.

[16] A. M. Mathai, An interesting extension of the Dirichlet model, *J. Indian Soc. Probab. Stat.*, **18** (2017), 195–213.

[17] Saralees Nadarajah and Samuel Kotz, Some bivariate beta distributions, *Statistics*, **39** (2005), no. 5, 457–466.

[18] Saralees Nadarajah and Samuel Kotz, Four bivariate distributions with gamma type marginals, *Rocky Mountain J. Math.*, **39** (2009), no. 1, 231–258.

[19] D. K. Nagar, M. Arashi and S. Nadarajah, Bimatrix variate gamma-beta distributions, *Comm. Statist. Theory Methods*, **46** (2017), no. 9, 4464–4483.

[20] Daya K. Nagar, Saralees Nadarajah and Idika E. Okorie, A New Bivariate Distribution with One Marginal Defined on the Unit Interval, *Ann. Data Sci.*, **4** (2017), no 3, 405–420.

[21] Johanna Marcela Orozco-Castañeda, Daya K. Nagar and Arjun K. Gupta, Generalized bivariate beta distributions involving Appell's hypergeometric function of the second kind, *Comput. Math. Appl.*, **64** (2012), no. 8, 2507–2519.

[22] Vadim Semenikhine, Edward Furman and Jianxi Su, On a multiplicative multivariate gamma distribution with applications in insurance, *Risks*, **6** (2018), 79.

[23] Ahmed Alsayed and Giancarlo Manzi, A Comparison of Monotonic Correlation Measures with Outliers, *WSEAS Transactions on Computers*,**18** (2019), Art. #29, pp. 223–230.

[24] P. J. Danaher and M. S. Smith, Modeling multivariate distributions using copulas: applications in marketing, *Mark. Sci.*, **30** (2011), 4–21.

[25] D-Q Li, S-H Jiang, S-B Wu,C-B Zhou and L-M Zhang, Modeling multivariate distributions using Monte Carlo simulation for structural reliability analysis with complex performance function, *Proc. Inst. Mech. Eng. Part O: J. Risk Reliabil.*, **227** (2013), 109–118.

[26] Gilbert Makanda and Sachin Shaw, Numerical analysis of the bivariate local linearization method (BLLM) for partial differential equations in casson fluid flow, *WSEAS Trans. Fluid Mech.*, **14** (2019), Art. #15, pp. 131–141.

[27] T. T. Takeuchi, Constructing a bivariate distribution function with given marginals and correlation: application to the galaxy luminosity function, *Mon. Notices Roy. Astronom. Soc.*, **406** (2010), 1830–1840.

[28] Barry C. Arnold and David Strauss, Bivariate distributions with exponential conditionals, *J. Amer. Statist. Assoc.*, **83** (1988), no. 402, 522–527.

[29] Saralees Nadarajah and Dongseok Choi, Arnold and Strauss's bivariate exponential distribution-products and ratios, *New Zealand J. Math.*, **35** (2006), no. 2, 189–199.

[30] S. Kotz, N. Balakrishnan and Norman L. Johnson, *Continuous Multivariate Distributions. Vol. 1. Models and Applications*. Second edition, Wiley Series in Probability and Statistics: Applied Probability and Statistics, Wiley-Interscience, New York, 2000.

[31] S. Nadarajah, FIM for Arnold and Strauss's bivariate gamma distribution, *Comput. Statist. Data Anal.*, **51** (2006), no. 3, 1584–1590.

[32] Y. L. Luke, *The Special Functions and their Approximations, Vol. 1*, Academic Press, New York, 1969.

[33] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton, 2000.

[34] A. K. Gupta, L. Cardeño D. K. and Nagar, Matrix variate Kummer-Dirichlet vistributions, *J. Appl. Math.*, **1** (2000), no. 3, 117-139.

[35] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions-2*, Second Edition, John Wiley & Sons, New York, 1994.

[36] Angelo Efoevi Koudou, A Matsumoto-Yor property for Kummer and Wishart random matrices, *Statist. Probab. Lett.*, **82** (2012), no. 11, 1903–1907.

[37] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.*, **37** (1966), 1137–1153.

[38] Yung Liang Tong, *Probability Inequalities in Multivariate Distributions*. Probabilities and Mathematical Statistics, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London-Toronto, Ont., 1980.

[39] C. E. Shannon, A mathematical theory of communication, *Bell System Tech. J.*, **27** (1948), 379–423, 623–656.

[40] A. Rényi, On measures of entropy and information, in *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, Vol. I, Univ. California Press, Berkeley, Calif., pp. 547–561 (1961).

[41] S. Nadarajah and K. Zografos, Expressions for Rényi and Shannon entropies for bivariate distributions, *Inform. Sci.*, **170** (2005), no. 2-4, 173–189.

[42] K. Zografos, On maximum entropy characterization of Pearson's type II and VII multivariate distributions, *J. Multivariate Anal.*, **71** (1999), no. 1, 67–75.

[43] K. Zografos S. Nadarajah, Expressions for Rényi and Shannon entropies for multivariate distributions, *Statist. Probab. Lett.*, **71** (2005), no. 1, 71–84.