Inextensible flows of spacelike curves according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$

W. M. Mahmoud[1], Alaa Hassan Noreldeen[2]
[1,2]Department of Mathematics
Aswan University
Faculty of Science
Aswan, EGYPT

Abstract: - In this paper, we study inextensible flows of spacelike curves lying fully on a spacelike surface Ω according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. We give necessary and sufficient conditions for this inextensible flows which are expressed as a partial differential equation involving the equiform curvature functions in 4-dimensional Minkowski space $\mathbb{R}_1^4$. Finally we give an application of inextensible flows of spacelike curves in $\mathbb{R}_1^4$.

Key-Words: - Curvature flows, equiform frame, inextensible, Minkowski space-time.
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1 Introduction

The flow of a curve or surface is said to be inextensible if, in the former case, the arc-length is preserved, and in the latter case, if the intrinsic Curvature is preserved [10, 11, 15, 18, 19].

Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from the motion [1, 7, 8]. Some relevant studies can be found in [3] and [4].

In this paper, we derive a general formulation for inextensible flows of spacelike curves according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. In Section 2, we clarify the basic conceptions of 4-dimensional Minkowski space $\mathbb{R}_1^4$and give of equiform Fremet frame that will be used during this work. In Section 3, we using the equiform frame to present the necessary and sufficient conditions for the inextensible flow as a partial differential equation involving the equiform curvature functions in 4-dimensional Minkowski space $\mathbb{R}_1^4$. In Section 4, we give an application of inextensible flows of spacelike curves in $\mathbb{R}_1^4$.

2 Preliminaries

The 4-dimensional Minkowski space $\mathbb{R}_1^4$ is the 3- dimensional Euclidean space $\mathbb{R}_1^4$provided with the metric

$$\xi = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 ,$$

Where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{R}_1^4$. Any arbitrary vector $\upsilon \in \mathbb{R}_1^4$ can have one of three Lorentzian clause depicts; it can be a spacelike, timelike or lightlike if $\mathfrak{F}(\nu, \nu)$ is positive, negative or zero respectively. Similarly, any arbitrary curve $\xi = \xi(s)$ can locally spacelike, timelike, or lightlike if $\mathfrak{F}\left(\dot{\zeta}(s), \dot{\zeta}(s)\right)$is positive, negative or zero respectively [14].

For any$u, v, w \in \mathbb{R}_1^4$, the vector product in 4-dimensional Minkowski space $\mathbb{R}_1^4$is defined by [17]:

$$u \wedge v \wedge w = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

Let $\zeta: I \subset \mathbb{R} \to \mathbb{R}_1^4$ be a regular spacelike curve parametrized by arc-length in $\mathbb{R}_1^4$ with timelike second binormal. Then the Frenet formulas along ζ can be given as [5, 16]:

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \\ \dot{b_1}(s) \\ \dot{b_2}(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & 0 & 0 \\ -k_1(s) & 0 & k_2(s) & 0 \\ 0 & -k_2(s) & 0 & k_3(s) \\ 0 & 0 & k_3(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b_1(s) \\ b_2(s) \end{pmatrix} \quad (1)$$

where $\left(\cdot = \frac{d}{ds}\right)$, $\{t, n, b_1, b_2\}, k_1, k_2$ and k_3 are the moving Frenet frame and the natural curvature functions respectively. Additionally, the vectors t, n, b_1 and b_2 satisfying the equations $\mathfrak{F}(t,t) = \mathfrak{F}(n,n) = \mathfrak{F}(b_1,b_1) = -\mathfrak{F}(b_2,b_2) = 1$, and $\mathfrak{F}(t,n) = \mathfrak{F}(t,b_1) = \mathfrak{F}(t,b_2) = \mathfrak{F}(n,b_1) = \mathfrak{F}(n,b_2) = \mathfrak{F}(b_1,b_2) = 0$.

For any arbitrary spacelike curve $\zeta = \zeta(s)$ in 4-dimensional Minkowski space $\mathbb{R}_1^4$ satisfying Eq. (1), the Frenet apparatus of ζ can be formed as follows [17]:

$$t = \frac{\dot{\zeta}}{\|\dot{\zeta}\|},$$

$$n = \frac{\|\dot{\zeta}\|^2 \cdot \ddot{\zeta} - \mathfrak{F}(\dot{\zeta}, \ddot{\zeta}) \cdot \dot{\zeta}}{\left\| \|\dot{\zeta}\|^2 \cdot \ddot{\zeta} - \mathfrak{F}(\dot{\zeta}, \ddot{\zeta}) \cdot \dot{\zeta} \right\|},$$

$$b_1 = \mu n \wedge t \wedge b_2 ,$$

$$b_2 = \frac{\mu(t \wedge n \wedge \dddot{\zeta})}{\|t \wedge n \wedge \dddot{\zeta}\|},$$

$$k_1 = \frac{\left\| \|\dot{\zeta}\|^2 \cdot \ddot{\zeta} - \mathfrak{F}(\dot{\zeta}, \ddot{\zeta}) \cdot \dot{\zeta} \right\|}{\|\dot{\zeta}\|^4},$$

$$k_2 = \frac{\|t \wedge n \wedge \dddot{\zeta}\| \cdot \|\dot{\zeta}\|}{\left\| \|\dot{\zeta}\|^2 \cdot \ddot{\zeta} - \mathfrak{F}(\dot{\zeta}, \ddot{\zeta}) \cdot \dot{\zeta} \right\|},$$

$$k_3 = \frac{\mathfrak{F}(\zeta^{(4)}, b_2)}{\|t \wedge n \wedge \dddot{\zeta}\| \cdot \|\dot{\zeta}\|},$$

where μ is taken ± 1 to make the determinant of matrix $\{t, n, b_1, b_2\}$ equal ± 1.

Let be a regular spacelike curve lying fully on a spacelike surface Ω in Minkowski 4-space $\mathbb{R}_1^4$. We define the equiform parameter of $\zeta(s)$ by $\sigma = \int k_1 \, ds$, where $\rho = \frac{1}{k_1}$ is the radius of curvature ofthe curve ζ. Then, we have $\rho = \frac{ds}{d\sigma}$

Let $\mathcal{D}$ be a homothetic with the center in the origin and the coefficient λ. If we put $\zeta^* = \mathcal{D}(\zeta)$, then it follows

$$s^* = \lambda s \text{ and } \rho^* = \lambda \rho,$$

where s^* is the arc-length parameter of ζ^* and ρ^* is the radius of curvature of this curve. Hence, σ is an equiform invariant parameter of ζ [6]. We recall $[T, \eta, \xi_1, \xi_2]$ be the moving equiform Frenet frame of the curve ζ where $T(\sigma) = \rho t(s)$ is the equiform tangent vector, $\eta(\sigma) = \rho n(s)$ is the equiform principal normal vector $\xi_1(\sigma) = \rho b_1(s)$r, is the equiform first binormal vector and $\xi_2(\sigma) = \rho b_2(s)$ is the equiform second binormal vector. Additionally, the equiform curvatures of the curve $\zeta = \zeta(\sigma)$ are defined by $k_1(\sigma) = \dot{\rho}, k_2(\sigma) = \left(\frac{k_2}{k_1}\right)$ and $k_3(\sigma) = \left(\frac{k_3}{k_1}\right)$. Thus, the equiform Frenet formulas in $\mathbb{R}_1^4$ have the following frame [2]:

$$\begin{pmatrix} T'(\sigma) \\ \eta'(\sigma) \\ \xi_1'(\sigma) \\ \xi_2'(\sigma) \end{pmatrix} = \begin{pmatrix} k_1(\sigma) & 1 & 0 & 0 \\ -1 & k_1(\sigma) & k_2(\sigma) & 0 \\ 0 & -k_2(\sigma) & k_1(\sigma) & k_3(\sigma) \\ 0 & 0 & k_3(\sigma) & k_1(\sigma) \end{pmatrix} \begin{pmatrix} T(\sigma) \\ \eta(\sigma) \\ \xi_1(\sigma) \\ \xi_2(\sigma) \end{pmatrix} \quad (2)$$

where $\left(' = \frac{d}{d\sigma}\right)$, $\mathfrak{F}(T,T) = \mathfrak{F}(\eta,\eta) = \mathfrak{F}(\xi_1,\xi_1) = -\mathfrak{F}(\xi_2,\xi_2) = \rho^2$ and $\mathfrak{F}(T,\eta) = \mathfrak{F}(T,\xi_1) = \mathfrak{F}(T,\xi_2) = \mathfrak{F}(\eta,\xi_1) = \mathfrak{F}(\eta,\xi_2) = 0$.

3 Main Results

Through out this paper, we assume that the one parameter family of spacelike curves on a spacelike surface Ω in $\mathbb{R}_1^4$ is

$$\zeta : [0, l] \times [0, w] \to \Omega \subset \mathbb{R}_1^4,$$

where l is the arc-length of initial curve. Let u be the curve parametrization variable $0 < u < l$. Then, the arc-length of ζ is given by

$$s(u) = \int_0^u \left\| \frac{\partial \zeta}{\partial u} \right\| du , \quad (3)$$

where $\left\|\frac{\partial \zeta}{\partial u}\right\| = \left\|\mathcal{D}(\frac{\partial \zeta}{\partial u}, \frac{\partial \zeta}{\partial u}\right\|^{\frac{1}{2}}$. The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u},$$

where $v = \left\|\frac{\partial \zeta}{\partial u}\right\|$. the arc-length parameter is $ds = v du$. We can be represented any flows of spacelike curve ζ as

$$\frac{\partial \zeta}{\partial t} = \mathfrak{R}_1 T + \mathfrak{R}_2 \eta + \mathfrak{R}_3 \xi_1 + \mathfrak{R}_4 \xi_2 , \quad (4)$$

Where $\Re_1, \Re_2, \Re_3$ and $\Re_4$ are smooth functions of arc-length and time [5, 9, 12, 13, 16].
Let the arc-length variation be

$$s(u,t) = \int_0^u v\, du\,, \tag{5}$$

In 4-dimensional Minkowski space $\mathbb{R}_1^4$, the requirement that the spacelike curve be not subject to any elongation or compression by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0 \tag{6}$$

Definition 3.1.A curve evolution $\zeta(u.t)$ and its flow $\frac{\partial \zeta}{\partial t}$ on the spacelike surface Ω in $\mathbb{R}_1^4$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial \zeta}{\partial u}\right\| = 0\,. \tag{7}$$

Theorem 3.1. Let$\frac{\partial \zeta}{\partial t} = \Re_1 T + \Re_2 \eta + \Re_3 \xi_1 + \Re_4 \xi_2$, be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. The flow is inextensible if and only if

$$\frac{\partial \Re_1}{\partial u} = v(\Re_2 - \Re_1 k_1)\,, \tag{8}$$

where$\Re_1, \Re_2, \Re_3$ and $\Re_4$ are smooth functions of arc-length and time.

Proof. Assume that $\frac{\partial \zeta}{\partial t}$ is inextensible flow in $\mathbb{R}_1^4$. Then

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du$$
$$= \int_0^u \left(\frac{\partial \Re_1}{\partial u} + \Re_1 v k_1 - \Re_2 v\right) du$$
$$= 0\,. \tag{9}$$

Substituting Eq. (7) in Eq. (9), we have Eq. (8) holds, this complete the proof.

Now, we restrict ourselves to arc-length parametrized curves. That is $v = 1$ and the local coordinate s corresponding to the curve arc-length.

Theorem 3.2. Let $\frac{\partial \zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. We have

$$\frac{\partial T}{\partial t} = \left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right)\eta + \left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\xi_1 + \left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\xi_2\,, \tag{10}$$

where $\Re_1, \Re_2, \Re_3$ and $\Re_4$ are smooth functions of arc-length and time.

Proof. Using the definition of ζ, we have

$$\frac{\partial T}{\partial t} = \left(\frac{\partial \Re_1}{\partial s} + \Re_1 k_1 - \Re_2\right)T + \left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right)\eta + \left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\xi_1 + \left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\xi_2\,, \tag{11}$$

Substituting Eq. (8) in Eq. (10), then the Eq. (11) holds, which complete the proof.

Lemma 3.1. Let $\frac{\partial \zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$, then the evolution of k_1 satisfy the partial differential equation

$$\frac{\partial k_1}{\partial t} = 0\,. \tag{12}$$

Proof. Assume that $\frac{\partial \zeta}{\partial t}$ is inextensible flow in $\mathbb{R}_1^4$. Thus, we have

$$\frac{\partial}{\partial s}\frac{\partial T}{\partial t} = -\left[\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right] T + \left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) + k_1\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) - k_2\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\right]\eta + \left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) + k_1\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + k_3\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\xi_1 + \left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right) + k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + k_1\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\xi_2 \qquad (13)$$

On the other hand, From Eq. (2) we have

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial s} = \frac{\partial}{\partial t}(k_1 T + \eta) = \frac{\partial k_1}{\partial t} T + k_1 \frac{\partial T}{\partial t} + \frac{\partial \eta}{\partial t}. \qquad (14)$$

From Eqs. (13) and (14), we see that – 7 = 0. This complete the proof.

Theorem 3.3. Let $\frac{\partial \zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. We have

$$\frac{\partial \eta}{\partial t} = \left[\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right] T + \left[\frac{\partial}{\partial t}\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) + k_3\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\xi_1 + \left[\frac{\partial}{\partial t}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right) + k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\right]\xi_2, \qquad (15)$$

where $\Re_1, \Re_2, \Re_3$ and $\Re_4$ are smooth functions of arc-length and time.

Proof. Let $\frac{\partial \zeta}{\partial t}$ is inextensible flow of a spacelike curve ζ in $\mathbb{R}_1^4$. From Eq. (14), we have

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}\frac{\partial T}{\partial s} - k_1 \frac{\partial T}{\partial t} - \frac{\partial k_1}{\partial t} T. \qquad (16)$$

Substituting Eqs. (10), (12) and (13) in Eq. (16), we get

$$\frac{\partial \eta}{\partial t} = -\left[\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right] T + \left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) - k_2\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\right]\eta$$

$$+\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)+k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)+k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right]\xi_1+\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)+k_3\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)\right]\xi_2$$

Then, one can easily say that

$$\mathfrak{F}\left(\eta,\frac{\partial \eta}{\partial t}\right)=0\,.$$

Asa consequence of the above equation

$$\frac{\partial \eta}{\partial t}=-\left[\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right]T+\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)+k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)+k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right]\xi_1+\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)+k_3\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)\right]\xi_2$$

Theorem 3.4. Let $\frac{\partial \zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. Then

$$\frac{\partial \xi_1}{\partial t}=\frac{-2}{k_2}\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)\right]T+\frac{1}{k_2}\left[2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)+\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)-k_2\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)\right]-k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)+k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)+k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right]\right]\eta+\frac{1}{k_2}[k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1+\mathfrak{R}_4 k_3\right)+k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right)+k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right]]$$

$$+\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right) + k_3\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right)\right] - \left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3 + \mathfrak{R}_4 k_1\right)\xi_2 \qquad (17)$$

where $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3$ and $\mathfrak{R}_4$are smooth functions of arc-length and time.

Proof. Assume that $\frac{\partial \zeta}{\partial t}$ is inextensible flow of a spacelike curveζ in $\mathbb{R}_1^4$. We have

$$\frac{\partial}{\partial s}\frac{\partial \eta}{\partial t} = -\left[2\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right) + k_1\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3\right)\right]T$$
$$+\left[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3 k_2\right) - k_2\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right)\right] + \frac{\partial \mathfrak{R}_2}{\partial s}\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3 k_2 - k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right) + k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right) + k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right] + k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3 k_2\right) - k_2\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right)\right]\right]\eta$$
$$+\left[k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1-\mathfrak{R}_3 k_2\right) - k_2\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right)\right] + \frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right) + k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3 k_2\right) + k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right) + k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right) + k_2\left(\frac{\partial \mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2 k_1 - \mathfrak{R}_3 k_2\right) + k_3\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right)\right] + k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathfrak{R}_4}{\partial s}+\mathfrak{R}_3 k_3+\mathfrak{R}_4 k_1\right) + k_3\left(\frac{\partial \mathfrak{R}_3}{\partial s}+\mathfrak{R}_2 k_2+\mathfrak{R}_3 k_1 + \mathfrak{R}_4 k_3\right)\right]\right]\xi_1$$

$$+\left[k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right)\right.\right.$$
$$+k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right)$$
$$\left.+k_3\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right]$$
$$+\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right.$$
$$\left.+k_3\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right)\right]$$
$$+k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right.$$
$$+k_3\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1\right.$$
$$\left.\left.\left.+\mathfrak{R}_4k_3\right)\right]\right]\xi_2. \qquad (18)$$

Underthe assumption of spacelike curve ζ, we have

$$\frac{\partial}{\partial s}\frac{\partial\eta}{\partial t}=-\left[k_1\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right)\right]T$$
$$-\left[\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.$$
$$\left.-\mathfrak{R}_3k_2\right]\eta$$
$$+\left[k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.\right.\right.$$
$$\left.-\mathfrak{R}_3k_2\right)$$
$$+k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.$$
$$\left.-\mathfrak{R}_3k_2\right)$$
$$+k_3\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)$$
$$-\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1\right.$$
$$\left.\left.\left.+\mathfrak{R}_4k_3\right)\right]-\frac{\partial k_2}{\partial t}\right]\xi_1$$

$$+\left[k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right.\right.$$
$$+k_3\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1\right.$$
$$\left.\left.+\mathfrak{R}_4k_3\right)\right]$$
$$\left.-\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right]\xi_2$$
$$+k_2\frac{\partial\xi_2}{\partial t}. \qquad (19)$$

Then, it is follow that

$$k_2\frac{\partial\xi_2}{\partial t}=-2\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.\right.$$
$$\left.\left.-\mathfrak{R}_3k_2\right)\right]T$$
$$+\left[\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.\right.\right.$$
$$\left.-\mathfrak{R}_3k_2\right)$$
$$-k_2\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1\right.$$
$$\left.\left.+\mathfrak{R}_4k_3\right)\right]$$
$$+2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.$$
$$\left.-\mathfrak{R}_3k_2\right)$$
$$-k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1\right.\right.$$
$$\left.+\mathfrak{R}_4k_3\right)$$
$$+k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1\right.$$
$$\left.-\mathfrak{R}_3k_2\right)$$
$$\left.\left.+k_3\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right]\right]\eta$$

$$+\Big[k_2\frac{\partial}{\partial s}\Big(\frac{\partial\Re_2}{\partial s}+\Re_1+\Re_2k_1-\Re_3k_2\Big) -(k_2^2-1)\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2 +\Re_3k_1+\Re_4k_3\Big) +\frac{\partial}{\partial s}\Big[\frac{\partial}{\partial s}\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 +\Re_4k_3\Big) +k_2\Big(\frac{\partial\Re_2}{\partial s}+\Re_1+\Re_2k_1 -\Re_3k_2\Big) +k_3\Big(\frac{\partial\Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\Big)\Big] +k_1\Big[\frac{\partial}{\partial s}\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 +\Re_4k_3\Big) +\frac{\partial}{\partial s}\Big(\frac{\partial\Re_2}{\partial s}+\Re_1+\Re_2k_1 -\Re_3k_2\Big)\Big] +k_3\Big[\frac{\partial}{\partial s}\Big(\frac{\partial\Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\Big) +k_3\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 +\Re_4k_3\Big)\Big]-\frac{\partial k_2}{\partial t}\Big]\xi_1$$

$$+\Big[k_3\Big[\frac{\partial}{\partial s}\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2+\Re_3k_1+\Re_4k_3\Big) +k_2\Big(\frac{\partial\Re_2}{\partial s}+\Re_1+\Re_2k_1 -\Re_3k_2\Big) +k_3\Big(\frac{\partial\Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\Big)\Big] +\frac{\partial}{\partial s}\Big[\frac{\partial}{\partial s}\Big(\frac{\partial\Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\Big) +k_3\Big(\frac{\partial\Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 +\Re_4k_3\Big)\Big] -\Big(\frac{\partial\Re_4}{\partial s}+\Re_3k_3 +\Re_4k_1\Big)\Big]\xi_2. \quad (20)$$

From the definition of flow

$$\mathfrak{F}\Big(\xi_1,\frac{\partial\xi_1}{\partial t}\Big)=0.$$

Thus the proof of theorem is complete.

Lemma 3.2. Let $\frac{\partial\zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. The evolution of k_2 satisfy the partial differential equation

$$\frac{\partial k_2}{\partial t} = k_2\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right) - (k_2^2-1)\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) + \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) + k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right) + k_3\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right] + k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) + \frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right)\right] + k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right) + k_3\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right)\right]. \quad (21)$$

Proof. It is obvious directly from Theorem(3.4). This complete the proof.

Theorem 3.5. let $\frac{\partial\zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$. Then

$$\frac{\partial\xi_2}{\partial t} = \frac{1}{k_3}\left[\frac{\partial f}{\partial s} - k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right) - g\right]T + \frac{1}{k_3}\left[\frac{\partial g}{\partial s} - \left(k_2+\frac{2}{k_2}\right)\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right) + (k_2^2-1)\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) - \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) + k_2\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right) + k_3\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right)\right] - k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right) + \frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_2}{\partial s}+\mathfrak{R}_1+\mathfrak{R}_2k_1-\mathfrak{R}_3k_2\right)\right] - k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{R}_4}{\partial s}+\mathfrak{R}_3k_3+\mathfrak{R}_4k_1\right) + k_3\left(\frac{\partial\mathfrak{R}_3}{\partial s}+\mathfrak{R}_2k_2+\mathfrak{R}_3k_1+\mathfrak{R}_4k_3\right)\right]\right]\eta$$

$$+\frac{1}{k_3}\Bigg[hk_3 + k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1-\Re_3k_2\right) + k_1\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1-\Re_3k_2\right) + k_3\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\right) + g\right]\Bigg]\xi_1. \quad (22)$$

Where

$$f=-\frac{2}{k_2}\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1-\Re_3k_2\right)\right],$$

$$g=\frac{1}{k_2}\Bigg[2\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1-\Re_3k_2\right) + \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1 - \Re_3k_2\right) - k_2\left(\frac{\partial \Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 + \Re_4k_3\right)\right] - k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 + \Re_4k_3\right) + k_2\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1 - \Re_3k_2\right) + k_3\left(\frac{\partial \Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\right)\right]\Bigg],$$

$$h=\frac{1}{k_2}\Bigg[k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s}+\Re_2k_2+\Re_3k_1+\Re_4k_3\right) + k_2\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2k_1 - \Re_3k_2\right) + k_3\left(\frac{\partial \Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\right)\right] + \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\right) + k_3\left(\frac{\partial \Re_3}{\partial s}+\Re_2k_2+\Re_3k_1 + \Re_4k_3\right)\right] - \left(\frac{\partial \Re_4}{\partial s}+\Re_3k_3+\Re_4k_1\right)\Bigg],$$

and $\Re_1, \Re_2, \Re_3$ and $\Re_4$ are smooth functions of arc-length and time.

Proof. Differentiating Eq. (17) with respect to s, we have

$$\frac{\partial}{\partial s}\frac{\partial \xi_1}{\partial t}=\left[\frac{\partial f}{\partial s}+fk_1-g\right]T + \left[\frac{\partial g}{\partial s}+gk_1+f\right]\eta + [gk_2+hk_3]\xi_1 + \left[\frac{\partial h}{\partial s}+hk_1\right]\xi_2. \quad (23)$$

Under the assumption of spacelike curve ζ, we can easily obtain that

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \xi_1}{\partial s} = &\left[k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right)\right.\\
&\left.+ f k_1\right] T\\
&+ \left[(k_2^2 - 1)\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2\right.\right.\\
&\left.+ \Re_3 k_1 + \Re_4 k_3\right)\\
&- k_2 \frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1\right.\\
&\left.- \Re_3 k_2\right)\\
&- \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1\right.\right.\\
&\left.+ \Re_4 k_3\right)\\
&+ k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1\right.\\
&\left.- \Re_3 k_2\right)\\
&\left.+ k_3\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\\
&- k_1\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1\right.\right.\\
&\left.+ \Re_4 k_3\right)\\
&+ \frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1\right.\\
&\left.\left.- \Re_3 k_2\right)\right]\\
&- k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right.\\
&+ k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1\right.\\
&\left.\left.\left.+ \Re_4 k_3\right)\right] + g k_1\right]\eta\\
&-k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right)\right.\\
&+ k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right)\\
&\left.+ k_3\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\xi_1\\
&+ \left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right.\\
&+ k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + \frac{\partial k_3}{\partial t}\\
&\left.+ h k_1\right]\xi_2 + k_3\frac{\partial \xi_2}{\partial t}. \qquad (24)
\end{aligned}
$$

Then, it is follow that

$$
\begin{aligned}
k_3\frac{\partial \xi_2}{\partial t} = &\left[\frac{\partial f}{\partial s} - k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1\right.\right.\\
&\left.\left.- \Re_3 k_2\right) - g\right] T
\end{aligned}
$$

$$+\Bigg[\frac{\partial g}{\partial s}-\left(k_2+\frac{2}{k_2}\right)\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2 k_1-\Re_3 k_2\right)+(k_2^2-1)\left(\frac{\partial \Re_3}{\partial s}+\Re_2 k_2+\Re_3 k_1+\Re_4 k_3\right)-\frac{\partial}{\partial s}\Bigg[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s}+\Re_2 k_2+\Re_3 k_1+\Re_4 k_3\right)+k_2\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2 k_1-\Re_3 k_2\right)+k_3\left(\frac{\partial \Re_4}{\partial s}+\Re_3 k_3+\Re_4 k_1\right)\Bigg]-k_1\Bigg[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s}+\Re_2 k_2+\Re_3 k_1+\Re_4 k_3\right)+\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2 k_1-\Re_3 k_2\right)\Bigg]-k_3\Bigg[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s}+\Re_3 k_3+\Re_4 k_1\right)+k_3\left(\frac{\partial \Re_3}{\partial s}+\Re_2 k_2+\Re_3 k_1+\Re_4 k_3\right)\Bigg]\Bigg]\eta$$

$$+\Bigg[hk_3+k_2\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2 k_1-\Re_3 k_2\right)+k_1\left(\frac{\partial \Re_2}{\partial s}+\Re_1+\Re_2 k_1-\Re_3 k_2\right)+k_3\left(\frac{\partial \Re_4}{\partial s}+\Re_3 k_3+\Re_4 k_1\right)+g\right]\Bigg]\xi_1+\Bigg[\frac{\partial h}{\partial s}-\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s}+\Re_3 k_3+\Re_4 k_1\right)-k_3\left(\frac{\partial \Re_3}{\partial s}+\Re_2 k_2+\Re_3 k_1+\Re_4 k_3\right)-\frac{\partial k_3}{\partial t}\Bigg]\xi_2. \quad (25)$$

Since

$$\mathfrak{F}\left(\xi_2,\frac{\partial \xi_2}{\partial t}\right)=0.$$

Hence, the proof is complete.

Lemma 3.3 let $\frac{\partial \zeta}{\partial t}$ be a smooth inextensible flow of the spacelike curve ζ according to equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$.The evolution of k_3 satisfy the partial differential equation

$$\frac{\partial k_3}{\partial t} = \frac{\partial}{\partial s}\left[\frac{1}{k_2}\left[k_3\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right) + k_2\left(\frac{\partial \Re_2}{\partial s} + \Re_1 + \Re_2 k_1 - \Re_3 k_2\right) + k_3\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right] + \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right) + k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right)\right] - \left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right)\right]\right] - \frac{\partial}{\partial s}\left(\frac{\partial \Re_4}{\partial s} + \Re_3 k_3 + \Re_4 k_1\right) - k_3\left(\frac{\partial \Re_3}{\partial s} + \Re_2 k_2 + \Re_3 k_1 + \Re_4 k_3\right). \quad (26)$$

Proof. It is obvious directly from theorem 3.5. This complete the proof.

4 Application of inextensible flows of spacelike curves in $\mathbb{R}_1^4$

In this section, we give an application of a special case of inextensible flows of spacelike curves in $\mathbb{R}_1^4$. Let $\Re_1 =$ constant $= a \neq 0$, $\Re$_2 =$\Re_3 = 0$ and $\Re_4 = \frac{a}{k_1(s,t)}$. From Eqs. (12), (21) and (23), we have the PDE system

$$\frac{\partial k_1}{\partial t} = 0,$$

$$\frac{\partial k_2}{\partial t} = a\left(\frac{2k_3+1}{k_1^3}\right)\left[2\left(\frac{\partial k_1}{\partial s}\right)^2 - k_1\left(\frac{\partial^2 k_1}{\partial s^2}\right)\right] + \left(\frac{a}{k_1}\right)\left[\left(\frac{\partial^2 k_3}{\partial s^2}\right) + k_3\left[\left(\frac{\partial k_1}{\partial s}\right) + k_2^2\right]\right] - \left(\frac{3a}{k_1^2}\right)\left(\frac{\partial k_1}{\partial s}\right)\left(\frac{\partial k_3}{\partial s}\right),$$

$$\frac{\partial k_3}{\partial t} = a\frac{\partial}{\partial s}\left\{\left(\frac{k_3}{k_2}\right)\left[\frac{\partial}{\partial s}\left(\frac{k_3}{k_1}\right) + k_3\left[\frac{\partial}{\partial s}\left(\frac{1}{k_1}\right) + 1\right] + k_2\right] + \left(\frac{1}{k_2}\right)\frac{\partial}{\partial s}\left[\frac{\partial^2}{\partial s^2}\left(\frac{1}{k_1}\right) + \left(\frac{k_3^2}{k_1}\right)\right] - \left(\frac{1}{k_2}\right)\left[\frac{\partial}{\partial s}\left(\frac{1}{k_1}\right) + 1\right]\right\} - a\frac{\partial^2}{\partial s^2}\left(\frac{1}{k_1}\right) - \frac{ak_3^2}{k_1}. \quad (27)$$

By solving the system (27) numerically, one solution of this system is

$$k_1(s,t) = c_1,$$

$$k_2(s,t) = \frac{a}{c_2 c_3}[1 - \tanh(c_2 s + c_3 t)], \quad (28)$$

$$k_3(s,t) = c_2^2(c_3 - a)\tanh^2(c_2 s + c_3 t),$$

where c_1, c_2 and c_3 are constants such that $c_2, c_3 \neq 0$. The curvatures of the family of spacelike curves C_t as a function of the coordinates s and t are plotted in Figure 1, Figure 2 and Figure 3.

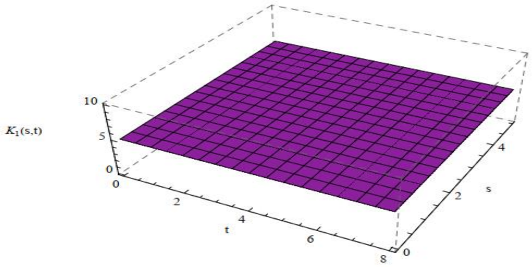

Fig.1: The curvature $k_1(s,t)$ of the family of spacelike curves C_t for $s \in [0,5]$, $t \in [0,8]$ and $c_1 = 0$.

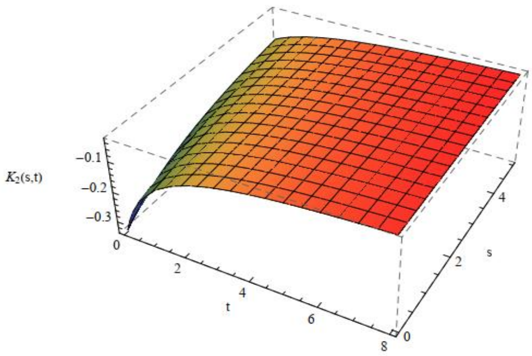

Fig.2: The curvature $k_2(s,t)$ of the family of spacelike curves C_t for $s \in [0,5]$, $t \in [0,8]$, $a = 1$, $c_2 = 4$ and $c_3 = 2$.

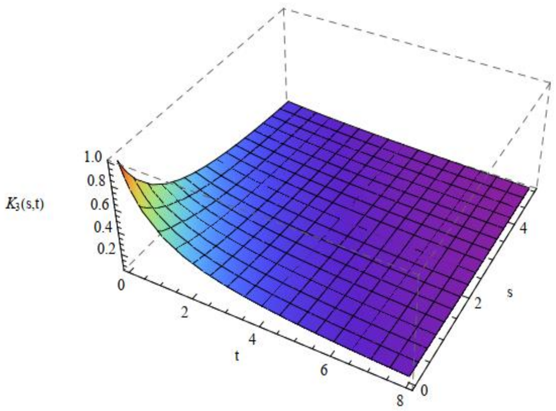

Fig.3: The curvature $k_3(s,t)$ of the family of spacelike curves C_t for $s \in [0,5]$, $t \in [0,8]$, $a = 1, c_2 = 4$ and $c_3 = 2$.

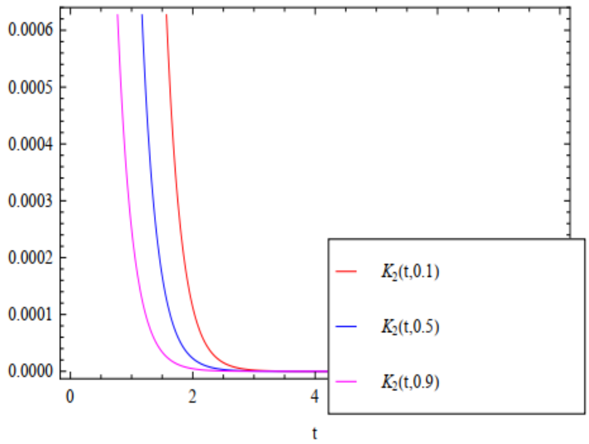

Fig.4: The curvature $k_2(s,t)$ for $s = 0.1, 0.5$ and 0.9.

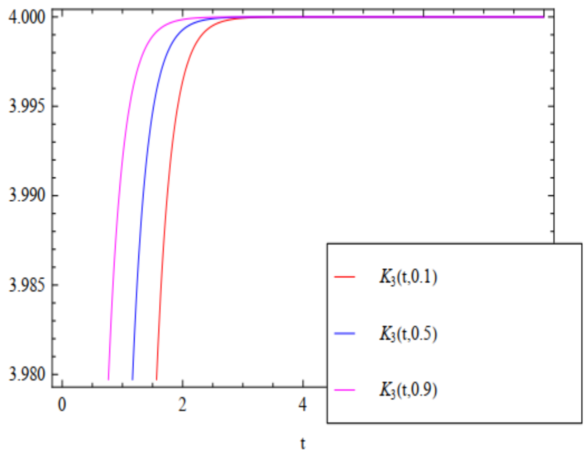

Fig. 5: The curvature $k_3(s,t)$ for $s = 0.1, 0.5$ and 0.9.

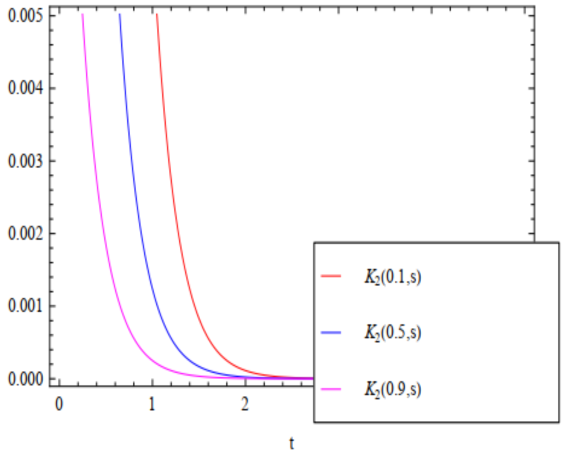

Fig.6: The curvature $k_2(s,t)$ for $t = 0.1, 0.5$ and 0.9

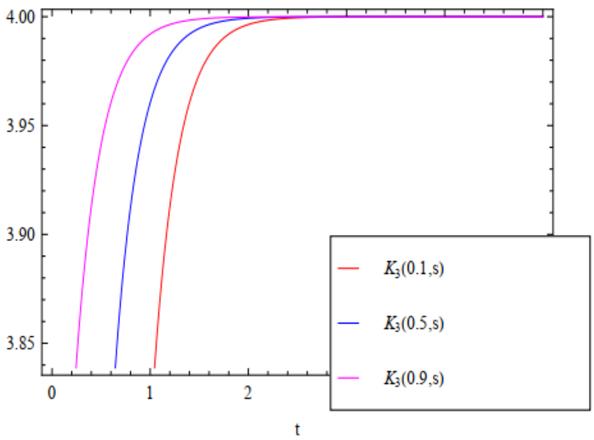

Fig.7: The curvature $k_3(s,t)$ for $t = 0.1, 0.5$

5 Conclusion

As a conclusion of our results, the inextensible flows of spacelike curves lying fully on a spacelike surface Ω according to the equiform frame in 4-dimensional Minkowski space $\mathbb{R}_1^4$ can be expressed as a partial differential equation involving the equiform curvature functions in $\mathbb{R}_1^4$. Also, the advancement equations for the curvatures of the curve in terms of these velocities are derived and found its exact solutions.

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