Willmott Fuzzy Implication in Fuzzy Databases

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Abstract: The object of the research are fuzzy functional dependencies on given relation scheme, and the question of their obtaining using the classical and innovated techniques. The attributes of the universal set are associated to the elements of the unit interval, and are turned into fuzzy formulas in this way. We prove that the dependency (which is treated as a fuzzy formula with respect to appropriately chosen valuation) is valid whenever it agrees with the attached two-elements fuzzy relation instance. The opposite direction of the claim is proven to be incorrect in this setting. Generalizing things to sets of attributes, we prove that particular fuzzy functional dependency follows form a set of fuzzy dependencies (in both, the world of two-element and the world of arbitrary fuzzy relation instances) if and only if the dependency is valid with respect to valuation anytime the set of fuzzy formulas agrees with the valuation. The results derived in paper show that the classical techniques in the procedure for generating new fuzzy dependencies may be replaced by the resolution ones, and hence automated. The research is conducted with respect to Willmott fuzzy implication operator.

Key–Words: Willmott fuzzy implication, functional dependencies, fuzzy formulas, fuzzy relations

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1 Introduction

The concept of the clause whose truth depends on truth of some set of clauses is discussed in [1]. Actually, it was proven that in the case when each clause of some set of clauses is more that half true, and the most reliable clause is correct with value a while the clause with the most opposite correctness is b, then, it is guaranteed that all off the logical consequences obtained by repeated applications of the inference rules (resolution rules) are correct with the degree of validity between a and b.

The importance of functional dependencies comes from the fact that any database designer has to remove (at some moment) the redundancy that necessarily occurs in the relations. The concept is based on the equality relation, that it, on the fact that $X \to Y$ (X determines Y), if $t\left[X\right] = t^{'}\left[X\right]$ yields $t\left[Y\right] = t^{'}\left[Y\right]$. Obviously, being based on equalities, the concept is not applicable to similarity-based relations. In [2], authors discuss the problem in terms of semantics and formal definitions. Thus, the definition of functional dependency becomes fuzzy, and turns into $X \to_F Y$ if the similarity between $t\left[X\right]$ and $t^{'}\left[X\right]$ implies the similarity between $t\left[Y\right]$ and $t^{'}\left[Y\right]$. The concept is fully developed in [2], including both, precise and imprecise dependencies by accepting the linguistic strength of the dependency.

An interesting approach was discussed in [3], where the author gives the definition of fuzzy functional dependency based on the semantic distance. In particular, a fuzzy functional dependency $X \to_F Y$ is said to hold in a fuzzy instance r, if

$$SS\left(t_i\left[X\right], t_j\left[X\right]\right) \leq SS\left(t_i\left[Y\right], t_j\left[Y\right]\right)$$

for all t_i, $t_j \in r$, where $SS\left(f_1, f_2\right)$ is the semantic similarity (the complement of $SD\left(f_1, f_2\right)$), for

$$SD\left(f_1, f_2\right) = \max_{x \in dom(A_i)} \left|f_1\left(X\right) - f_2\left(X\right)\right|,$$

on attribute A_i, and fuzzy values $f_1\left(X\right)$, $f_2\left(X\right)$.

The approaches to include fuzzy data into classical database theory assume: the use of fuzzy membership values [4] (the fuzzy membership attribute defines the membership degree of some tuple in the relation instance), [5] (the membership value defines the strength of the dependency between attributes), possibility distribution [6]-[8] (fuzzy-known values are represented in terms of possibility distributions of values contained in the unit interval), and similarity relations [9]-[11].

Regarding some additional fuzzy functional concepts, we recall [12] (described via membership function of the element of fuzzy relation), [13] (conformance between tuples), etc.

Further investigations on relations between theory of fuzzy functional dependencies (based on similarities), and fuzzy logic, are discussed in [14]. They rely on [2].

The research is then generalized in [15] to include not only fuzzy functional, but also fuzzy multivalued dependencies. In particular, the authors offer and algorithm to generally and automatically decide whether or not a fuzzy functional or a fuzzy multivalued dependency follows from a set of dependencies.

Finally, the research is raised to vague level in [16], where was shown that the inference rules for new vague functional dependencies are sound and complete. The use of various similarity relations (measures) between vague values, vague sets, and attributes (sets of attributes) is also discussed. We notice that several concepts of vague similarity relations are present in [17], [18] and [19]. There are also papers like [20] and [21].

The authors in [22], for example, propose a vague relational database model, and define a new kind of vague functional dependency called α-vague functional dependency. The inference rules (similar to Armstrong's axioms for the classical case) are also presented.

The research on properties of vague relational databases is continued in [23], where is
rigorously proved that reflexive, augmentation, transitivity, pseudo-transitivity, and decomposition inference rules are sound. Applications in terms of Lukasiewicz fuzzy implication operator are provided in [24] (see also, [25], [26]).

The interesting results are discussed in [27] and [28].

The goal of the present research is to relate fuzzy functional dependencies to fuzzy formulas (and vice versa).

Motivated by Lee's research [1] (noted above), we assign $[0, 1]$-value to attributes, and hence turn fuzzy functional dependencies into fuzzy formulas. Our research follows similarity-based work [2], and thus relies on the approach developed in [9]-[11]. We actualize the question of automated obtaining of new fuzzy functional dependencies utilizing the techniques of resolution theory. Thus, if dependency is in line with given fuzzy relation instance (see, Theorem 1), we prove that it also agrees with the attached valuation (when written in the form of fuzzy formula).

The opposite claim is proven not to be valid (Theorem 2).

The concept is generalized to sets of attributes, where is proven that particular fuzzy functional dependency is a logical consequence of some set of dependencies (in areas of two-element and arbitrary fuzzy relations) if and only if the dependency (taken in the form of fuzzy formula) is valid with respect to valuation (interpretation) when the set of fuzzy formulas is true with respect to same valuation (Theorems 3-5).

The research is done utilizing the Willmott fuzzy implication operator [29], and it shows that the traditional methods (inference rules), which are classically applied to generate new dependencies, may be replaced by the automated ones (following from resolution techniques).

2 Preliminaries

Let t_i be the i-th tuple in any relation.

It can be represented in the form $(d_{i1}, ..., d_{im})$, where d_{ij} is a subset of the domain D_j of the attribute A_j.

In particular, we assume that $d_{ij} \neq \emptyset$.

A similarity relation s_j is a function $s_j : D_j \times D_j \to [0, 1]$, having the properties: $s_j(a, a) = 1$ (reflexive), $s_j(a, b) = s_j(b, a)$ (symmetry), $s_j(a, c) \geq \max\left\{\min_{b \in D_j} \{s_j(a, b), s_j(b, c)\}\right\}$ (max-min transitivity).

The identity relation is then a special case of the similarity relation.

A fuzzy relation instance r is defined as a subset of the set $2^{D_1} \times 2^{D_2} \times ... \times 2^{D_m}$.

So, a fuzzy tuple t is any member of fuzzy relation instance r and the set $2^{D_1} \times 2^{D_2} \times ... \times 2^{D_m}$.

The conformance of the attribute A_k (on the domain D_k) for any two tuples t_i and t_j (the elements of some fuzzy relation instance r), is denoted by $\varphi(A_k[t_i, t_j])$, and is given by

$$\begin{aligned}
&\varphi(A_k[t_i, t_j]) \\
&= \min\left\{\min_{x \in d_i}\left\{\max_{y \in d_j}\{s(x, y)\}\right\},\right. \\
&\left.\min_{x \in d_j}\left\{\max_{y \in d_i}\{s(x, y)\}\right\}\right\},
\end{aligned}$$

where d_i is the value of the attribute A_k on tuple t_i, d_j is the value of A_k on t_j, and $s(x, y)$ is a similarity relation between values x and y.

Consequently, the definition of conformance is generalized to describe the similarities between tuples on sets of attributes. In particular, the conformance of tuples t_i and t_j on set of attributes X is denoted by $\varphi(X[t_i, t_j])$, and is defined by

$$\varphi\left(X\left[t_i,t_j\right]\right)=\min_{A\in X}\left\{\varphi\left(A\left[t_i,t_j\right]\right)\right\}.$$

Thus, if $X\supseteq Y$, then $\varphi\left(Y\left[t_i,t_j\right]\right)\geq\varphi\left(X\left[t_i,t_j\right]\right)$ for any t_i, $t_j\in r$. If $X=\{A_1,...,A_n\}$, and $\varphi\left(A_k\left[t_i,t_j\right]\right)\geq\theta$ for all $k\in\{1,2,...,n\}$, then $\varphi\left(X\left[t_i,t_j\right]\right)\geq\theta$ for any t_i and t_j in r, $\theta\in[0,1]$. Furthermore, if $\varphi\left(X\left[t_i,t_j\right]\right)\geq\theta$, and $\varphi\left(X\left[t_j,t_k\right]\right)\geq\theta$, then $\varphi\left(X\left[t_i,t_k\right]\right)\geq\theta$ for any t_i, t_j and t_k in r, $\theta\in[0,1]$.

If r is a fuzzy relation instance on scheme $R\left(A_1,...,A_n\right)$, where U is the universal set of attributes ($U=\{A_1,...,A_m\}$), and X, Y are subsets of U, then the fuzzy relation instance r is said to satisfy the fuzzy functional dependency $X\overset{\theta}{\rightarrow}_F Y$ if for every pair of tuples t_1 and t_2 in r,

$$\varphi\left(Y\left[t_1,t_2\right]\right)\geq\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}.$$

θ belongs to $[0,1]$ and describes the linguistic strength of the dependency.

A formula $f\in S$, where S is a set of fuzzy formulas, is said to be satisfied (incorrect) with respect to interpretation I, if $T(f)\geq 0.5$ ($T(f)\leq 0.5$) under I. If $T(f)\leq 0.5$ with regard to all possible I's, the formula f is called unsatisfiable.

Suppose that $D_1 : L_1\vee D_1'$ and $D_2 : L_2\vee D_2'$ are two disjuncts, where L_1 and L_2 are contra pair of literals, i.e., $L_2 : \neg L_1$. Assume that D_1' and D_2' do not contain any additional pair of contra literals. Then, the disjunct $D_1'\vee D_2'$ is called the resolvent between disjuncts D_1 and D_2 with the key word L_1.

If S is a set of clauses, then the resolution of S, denoted by $Res(S)$, is the set consisting of members of S together with all of the resolvents of pairs of members in S. The n-th resolution of S ($Res^n(S)$), is defined for $n\geq 0$ by

$$Res^0(S)=S,$$
$$Res^{n+1}(S)=Res\left(Res^n(S)\right).$$

If R is a relation scheme, $r=\{t_1,t_2\}$ is an instance, and θ is a value in $[0,1]$, the valuation $i_{r,\theta}(A)$ of the attribute A in R is defined by

$$\begin{cases} >0.5, & \varphi\left(A\left[t_1,t_2\right]\right)\geq\theta; \\ \leq 0.5, & \varphi\left(A\left[t_1,t_2\right]\right)<\theta. \end{cases}$$

Consequently, the fuzzy formula

$$\left(A_1\wedge...\wedge A_m\right)\Rightarrow\left(B_1\wedge...\wedge B_n\right)$$

is assigned to the fuzzy functional dependency $X\overset{\theta}{\rightarrow}_F Y$, where $X=\{A_1,...,A_m\}$, $Y=\{B_1,...,B_n\}$.

Through the rest of the paper we shall assume that the fuzzy implication is actually the Willmott fuzzy implication $W(a,b)$ [29], defined by

$$W(a,b)=\min\Big\{\max\{1-a,b\},\max\{a,1-a\},\max\{b,1-b\}\Big\},$$

and that conjunction $C(a,b)$ and disjunction $D(a,b)$ are given by $C(a,b)=\min\{a,b\}$ and $D(a,b)=\max\{a,b\}$ (see, [30] and [31]).

3 Results

Theorem 1. *Suppose that $X\overset{\theta}{\rightarrow}_F Y$ is a fuzzy functional dependency given on the relation scheme R, and r a two-element fuzzy relation instance on R. Let $r=\{t_1,t_2\}$. If the fuzzy functional dependency $X\overset{\theta}{\rightarrow}_F Y$ is satisfied by the instance r, then, the fuzzy formula attached to $X\overset{\theta}{\rightarrow}_F Y$ is satisfied by the valuation $i_{r,\theta}$.*

Proof. Consider the Willmott fuzzy implication

$$W(a,b)=\min\Big\{\max\{1-a,b\},\max\{a,1-a\},\max\{b,1-b\}\Big\}.$$

Assume that the fuzzy relation instance r satisfies fuzzy functional dependency $X\overset{\theta}{\rightarrow}_F Y$.

So, we assume that

$$\varphi\left(Y\left[t_1,t_2\right]\right)\geq\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\},$$

where $X=\{A_1,...,A_m\}$, $Y=\{B_1,...,B_n\}$.

Suppose contrary, that the claim of theorem is not fulfilled, that is, that the fuzzy formula

$$\left(A_1\wedge...\wedge A_m\right)\Rightarrow\left(B_1\wedge...\wedge B_n\right)$$

is not satisfied with respect to $i_{r,\theta}$.

This means that

$$\frac{1}{2} \geq i_{r,\theta}\Big((A_1 \wedge ... \wedge A_m) \Rightarrow (B_1 \wedge ... \wedge B_n)\Big) = \tag{1}$$

$$\min\Big\{ \max\Big\{1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m), i_{r,\theta}(B_1 \wedge ... \wedge B_n)\Big\}, \max\Big\{i_{r,\theta}(A_1 \wedge ... \wedge A_m), 1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m)\Big\}, \max\Big\{i_{r,\theta}(B_1 \wedge ... \wedge B_n), 1 - i_{r,\theta}(B_1 \wedge ... \wedge B_n)\Big\}\Big\}.$$

If the minimum is

$$\max\Big\{1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m), i_{r,\theta}(B_1 \wedge ... \wedge B_n)\Big\}, \tag{2}$$

then the maximum must be at most $\frac{1}{2}$.

If the maximum is $1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m)$, we obtain

$$1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2},$$

i.e.,

$$i_{r,\theta}(A_1 \wedge ... \wedge A_m) \geq \frac{1}{2}.$$

Hence, $i_{r,\theta}(A_k) \geq \frac{1}{2}$ for all $k \in \{1, 2, ..., m\}$, that is $\varphi(A_k[t_1, t_2]) \geq \theta$ for all $k \in \{1, 2, ..., m\}$.

Now, it follows that

$$\varphi(X[t_1, t_2]) = \min\{\varphi(A_1[t_1, t_2]), ..., \varphi(A_m[t_1, t_2])\} \geq \theta$$

Since we assume that the dependency $X \xrightarrow{\theta}_F Y$ is satisfied by r, we obtain

$$\min\{\varphi(B_1[t_1, t_2]), ..., \varphi(B_n[t_1, t_2])\} = \varphi(Y[t_1, t_2]) \geq \min\{\theta, \varphi(X[t_1, t_2])\} \geq \min\{\theta, \theta\} = \theta.$$

It follows that $\varphi(B_k[t_1, t_2]) \geq \theta$ for all $k \in \{1, 2, ..., n\}$.

In other words, $i_{r,\theta}(B_k) > \frac{1}{2}$ for all $k \in \{1, 2, ..., n\}$.

Since the minimum in (2) is $1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m)$, we obtain that $i_{r,\theta}(B_1 \wedge ... \wedge B_n)$ is also at most $\frac{1}{2}$.

Hence, $i_{r,\theta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}$, so $i_{r,\theta}(B_k) \leq \frac{1}{2}$ for at least one B_k, $k \in \{1, 2, ..., n\}$.

This is a contradiction.

So, it must be $i_{r,\theta}(B_1 \wedge ... \wedge B_n) > \frac{1}{2}$.

We obtain, $i_{r,\theta}(B_k) > \frac{1}{2}$, and $\varphi(B_k[t_1, t_2]) \geq \theta$ for all $k \in \{1, 2, ..., n\}$.

If the minimum in (2) is $i_{r,\theta}(B_1 \wedge ... \wedge B_n)$, then $i_{r,\theta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}$, i.e., $i_{r,\theta}(B_k) \leq \frac{1}{2}$ for some $k \in \{1, 2, ..., n\}$.

Since in this scenario

$$1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m) \leq i_{r,\theta}(B_1 \wedge ... \wedge B_n),$$

it follows that

$$1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2},$$

that is, $i_{r,\theta}(A_1 \wedge ... \wedge A_m) \geq \frac{1}{2}$.

As in the previous case, it follows that $\varphi(X[t_1, t_2]) \geq \theta$ and then, $\varphi(Y[t_1, t_2]) \geq \theta$.

In other words, $i_{r,\theta}(B_k) > \frac{1}{2}$ for all $k \in \{1, 2, ..., n\}$.

This is the contradiction.

Suppose that the minimum in (1) is

$$\max\Big\{i_{r,\theta}(A_1 \wedge ... \wedge A_m), 1 - i_{r,\theta}(A_1 \wedge ... \wedge A_m)\Big\}. \tag{3}$$

Let the maximum in (3) be $i_{r,\theta}(A_1 \wedge ... \wedge A_m)$.

Then, $i_{r,\theta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2}$, i.e., $i_{r,\theta}(A_k) \leq \frac{1}{2}$ for some $k \in \{1, 2, ..., m\}$.

Hence, $\varphi(A_k[t_1, t_2]) < \theta$ for some $k \in \{1, 2, ..., m\}$. This means that

$$\varphi(X[t_1, t_2]) = \min\{\varphi(A_1[t_1, t_2]), ..., \varphi(A_m[t_1, t_2])\} < \theta.$$

Note that,

$$\varphi\left(Y\left[t_1,t_2\right]\right)\geq\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}.$$

Moreover,

$$\begin{aligned}&1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\\ \leq& i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\leq\frac{1}{2}.\end{aligned}$$

We obtain, $i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\geq\frac{1}{2}$, i.e., $\varphi\left(X\left[t_1,t_2\right]\right)\geq\theta$.

This is a contradiction.

Now, suppose that the maximum in (3) is

$$1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right).$$

We have,

$$\begin{aligned}&i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\\ \leq&1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\leq\frac{1}{2}.\end{aligned}$$

Hence, $i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\leq\frac{1}{2}$, i.e., $i_{r,\theta}\left(A_k\right)\leq\frac{1}{2}$ for some $k\in\{1,2,...,n\}$, or $\varphi\left(A_k\left[t_1,t_2\right]\right)<\theta$ for some $k\in\{1,2,...,m\}$.

Furthermore, $i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\geq\frac{1}{2}$, i.e., $i_{r,\theta}\left(A_k\right)\geq\frac{1}{2}$ for all $k\in\{1,2,...,m\}$, or $\varphi\left(A_k\left[t_1,t_2\right]\right)\geq\theta$ for all $k\in\{1,2,...,m\}$.

This is a contradiction.

Finally, suppose that the minimum in (1) is

$$\begin{aligned}\max&\Big\{i_{r,\theta}\left(B_1\wedge...\wedge B_n\right),\\ &1-i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)\Big\}.\end{aligned}$$

The case is symmetric to the case (3), so it necessarily leads to contradiction.

The assumption that the claim of theorem is not fulfilled leads to contradiction.

Consequently, the theorem follows. □

Now, we prove that the opposite claim of the claim given by Theorem 1 is not always true.

Theorem 2. *Suppose that the assumptions of Theorem 1 hold. If the fuzzy formula attached to $X\overset{\theta}{\rightarrow}_F Y$ is valid under the action of $i_{r,\theta}$, then, the fuzzy functional dependency $X\overset{\theta}{\rightarrow}_F Y$ is not necessarily correct with respect to r.*

Proof. By our assumption,

$$\begin{aligned}\frac{1}{2}<&i_{r,\theta}\Big(\left(A_1\wedge...\wedge A_m\right)\Rightarrow\\ &\left(B_1\wedge...\wedge B_n\right)\Big)=\end{aligned}\tag{4}$$

$$\begin{aligned}&\min\Big\{\max\Big\{1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right),\\ &i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)\Big\},\\ &\max\Big\{i_{r,\theta}\left(A_1\wedge...\wedge A_m\right),\\ &1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\Big\},\\ &\max\Big\{i_{r,\theta}\left(B_1\wedge...\wedge B_n\right),\\ &1-i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)\Big\}\Big\}.\end{aligned}$$

Let the minimum in (4) be

$$\begin{aligned}\max&\Big\{1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right),\\ &i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)\Big\}.\end{aligned}$$

If the maximum is $1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)$, then

$$\begin{aligned}&i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)\\ \leq&1-i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)>\frac{1}{2}.\end{aligned}$$

It follows that $i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)<\frac{1}{2}$.

Hence, $i_{r,\theta}\left(A_k\right)<\frac{1}{2}$ for some $k\in\{1,2,...,m\}$, or $\varphi\left(A_k\left[t_1,t_2\right]\right)<\theta$.

Consequently, $\varphi\left(X\left[t_1,t_2\right]\right)<\theta$.

Since

$$\begin{aligned}&i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)+\\ &i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)\leq1,\end{aligned}$$

and $i_{r,\theta}\left(A_1\wedge...\wedge A_m\right)>\frac{1}{2}$, we obtain that $i_{r,\theta}\left(B_1\wedge...\wedge B_n\right)<\frac{1}{2}$.

Consequently, $\varphi\left(Y\left[t_1,t_2\right]\right)<\theta$.

We have,

$$\begin{aligned}&\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}\\ =&\varphi\left(X\left[t_1,t_2\right]\right).\end{aligned}$$

Since $\varphi\left(X\left[t_1,t_2\right]\right)<\theta$, and also $\varphi\left(Y\left[t_1,t_2\right]\right)<\theta$, it is possible to happen that

$$\begin{aligned}&\varphi\left(Y\left[t_1,t_2\right]\right)\\<&\varphi\left(X\left[t_1,t_2\right]\right)\\=&\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}.\end{aligned}$$

Thus, $X \overset{\theta}{\rightarrow}_F Y$ is not necessarily satisfied by r.
The cases

$$\max\Big\{i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right),\ 1-i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)\Big\}$$

and

$$\max\Big\{i_{r,\theta}\left(B_1\wedge\ldots\wedge B_n\right),\ 1-i_{r,\theta}\left(B_1\wedge\ldots\wedge B_n\right)\Big\}$$

which may appear in (4) are symmetric, and mutually equivalent hence. So, we consider only one of them.

For the sake of clarity, assume that the minimum in (4) is

$$\max\Big\{i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right),\ 1-i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)\Big\}.$$

If the maximum is $i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)$, then

$$\begin{aligned}&1-i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)\\\leq&\, i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)>\frac{1}{2}.\end{aligned}$$

Thus, the right inequality yields that $i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)>\frac{1}{2}$.

On the other side, the left-hand side inequality implies that $i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right)\geq\frac{1}{2}$.

This is the contradiction.

This means that the only possible case, which makes sense in (4), is the case

$$\max\Big\{1-i_{r,\theta}\left(A_1\wedge\ldots\wedge A_m\right),\ i_{r,\theta}\left(B_1\wedge\ldots\wedge B_n\right)\Big\}.$$

As we already discussed, this case does not necessarily yield the correctness of $X \overset{\theta}{\rightarrow}_F Y$ under r.

This completes the proof. □

The following result does not depend on the selection of fuzzy implication operator.

Theorem 3. *Let $X \overset{\theta}{\rightarrow}_F Y$ be a fuzzy functional dependency on given scheme R. Suppose that F is a set of fuzzy functional dependencies on R. Then, $X \overset{\theta}{\rightarrow}_F Y$ is a consequence of the set F if and only if $X \overset{\theta}{\rightarrow}_F Y$ follows from the set F in the world of two-elements relations.*

Proof. $(\Rightarrow)$ Clearly, if the dependency $X \overset{\theta}{\rightarrow}_F Y$ is a consequence of F, then, it is understood that $X \overset{\theta}{\rightarrow}_F Y$ is a consequence of F in arbitrary fuzzy relation instances.

So, if this implication is true (note that it is obvious that the set of two-element fuzzy relation instances is a subset of the general set of instances), it becomes clear that this direction of the claim follows immediately. $(\Leftarrow)$ Suppose that the set F determines

$X \overset{\theta}{\rightarrow}_F Y$ in the world of two-tuples relations.

Suppose also the opposite, that $X \overset{\theta}{\rightarrow}_F Y$ does not follow from F.

This means that some fuzzy relation instance r, violates dependency $X \overset{\theta}{\rightarrow}_F Y$.

By the very definition of the functional dependency, it follows that the condition

$$\varphi\left(Y\left[t_1,t_2\right]\right)\geq\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}$$

is not true for all t_1, $t_2 \in r$.

In other words,

$$\varphi\left(Y\left[t_1,t_2\right]\right)<\min\left\{\theta,\varphi\left(X\left[t_1,t_2\right]\right)\right\}$$

for some t_1, $t_2 \in r$.

Denote, $r_1=\{t_1,t_2\}$.

Now, r_1 is an example of the two-tuples instance which violates $X \overset{\theta}{\rightarrow}_F Y$.

This is the contradiction.

Hence, the theorem follows. □

Theorem 4. *Let $X \overset{\theta}{\rightarrow}_F Y$ be a fuzzy functional dependency on scheme R, and F a set of fuzzy functional dependencies on R. If some fuzzy relation instance r on R satisfies F, and violates $X \overset{\theta}{\rightarrow}_F Y$, then, there exists some two-elements fuzzy relation instance $r_1\subseteq r$, such that r satisfies F, and violates $X \overset{\theta}{\rightarrow}_F Y$.*

Proof. Let r satisfies F, and violates $X \overset{\theta}{\rightarrow}_F Y$.

By Theorem 3, this means that the assumption that $X \overset{\theta}{\rightarrow}_F Y$ is a consequence of F is not true, which means that there is a fuzzy relation instance (it is exactly the instance r), such that r satisfies F and violates $X \xrightarrow{\theta}_F Y$.

Hence, by Theorem 3, $X \xrightarrow{\theta}_F Y$ does not follow from F in the world of two-element relations.

More precisely, there exists some two-element relation r_1 which agrees with the set F, and fails to satisfy the dependency $X \xrightarrow{\theta}_F Y$.

Clearly, r_1 may be considered as a subset of r.

Namely, r violates $X \xrightarrow{\theta}_F Y$, so there are tuples t_1 and t_2 in r, such that

$$\varphi\left(Y\left[t_1, t_2\right]\right) < \min\left\{\theta, \varphi\left(X\left[t_1, t_2\right]\right)\right\}.$$

If we put $r_1 = \{t_1, t_2\}$, then r_1 is a desired two-element relation.

This completes the proof. $\square$

The main results of this research are given by the following two theorems.

Theorem 5. *Let $X \xrightarrow{\theta}_F Y$ be a fuzzy functional dependency on scheme R, and F a set of fuzzy functional dependencies. If $X \xrightarrow{\theta}_F Y$ follows from F in the world of two-element fuzzy relation instances, then, the fuzzy formula attached to $X \xrightarrow{\theta}_F Y$ is valid.*

Proof. In order to prove the correctness of the claim, we shall show assume that each of the domains of the attributes has only two elements. Moreover, we shall assume that the domains are the same, and that they are given by set $\{p, q\}$.

The dependencies from the set F together with the given dependency determine the set of corresponding linguistic strengths.

Since the set is finite one, it has the minimal element. If such the element is denoted by $\theta^{'}$, we shall take and fix any value which is strictly smaller than $\theta^{'}$.

Denote it by $\theta^{''}$.

The similarity relations on each of the universe of discourses will be assumed to take value $\theta^{''}$ at (p, q) and (q, p).

In order to prove the claim of theorem, we shall apply the contraposition law. Thus, we shall prove that the negated claim of theorem implies the negated assumption.

So, suppose that the fuzzy formula

$$(A_1 \wedge ... \wedge A_m) \Rightarrow (B_1 \wedge ... \wedge B_n)$$

is not correct, where $X = \{A_1, ..., A_m\}$,
$Y = \{B_1, ..., B_n\}$.

Thus,

$$\begin{aligned}\frac{1}{2} \geq & i_{r,\beta}\Big((A_1 \wedge ... \wedge A_m) \Rightarrow \\ & (B_1 \wedge ... \wedge B_n) \Big) = \end{aligned} \tag{5}$$

$$\begin{aligned}
&\min\Big\{ \max\Big\{1 - i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right), \\
&i_{r,\beta}\left(B_1 \wedge ... \wedge B_n\right)\Big\}, \\
&\max\Big\{i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right), \\
&1 - i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right)\Big\}, \\
&\max\Big\{i_{r,\beta}\left(B_1 \wedge ... \wedge B_n\right), \\
&1 - i_{r,\beta}\left(B_1 \wedge ... \wedge B_n\right)\Big\}\Big\}
\end{aligned}$$

for some $i_{r,\beta}$, where r has two-elements, and $0 \leq \beta \leq 1$.

Let $r = \{t_1, t_2\}$.

First, we shall consider the set of those attributes A, for which $i_{r,\beta}(A) > \frac{1}{2}$.

We shall prove that this set is not empty, and that it does not coincide with the set of all attributes.

We shall prove this in two steps.

First, suppose contrary, that $i_{r,\beta}(A) \leq \frac{1}{2}$ for all A's.

Note that this fact gives that $i_{r,\beta}(C_1 \wedge ... \wedge C_p) \leq \frac{1}{2}$ for any set of attributes $\{C_1, ..., C_p\}$.

Suppose the minimum in (5) is

$$\begin{aligned}&\max\Big\{i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right), \\ &1 - i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right)\Big\}.\end{aligned}$$

The case

$$\begin{aligned}&\max\Big\{i_{r,\beta}\left(B_1 \wedge ... \wedge B_n\right), \\ &1 - i_{r,\beta}\left(B_1 \wedge ... \wedge B_n\right)\Big\}\end{aligned}$$

is symmetric case, so we consider only the first one.

It can happen that either

$$\begin{aligned}&i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right) \\ \leq & 1 - i_{r,\beta}\left(A_1 \wedge ... \wedge A_m\right) \leq \frac{1}{2}\end{aligned}$$

or

$$1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2}.$$

Both of these scenarios yield the contradiction $i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2}$ and $i_{r,\beta}(A_1 \wedge ... \wedge A_m) \geq \frac{1}{2}$.

So, it follows that the minimum in (5) is

$$\max\Big\{1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m), i_{r,\beta}(B_1 \wedge ... \wedge B_n)\Big\}.$$

We obtain that,

$$1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}$$

or

$$i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq 1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2}.$$

In any case, it follows that $1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq \frac{1}{2}$, i.e., $i_{r,\beta}(A_1 \wedge ... \wedge A_m) \geq \frac{1}{2}$.

Thus,

$$\min\{i_{r,\beta}(A_1), ..., i_{r,\beta}(A_m)\} = i_{r,\beta}(A_1 \wedge ... \wedge A_m) \geq \frac{1}{2},$$

so $i_{r,\beta}(A_k) \geq \frac{1}{2}$ for all $k \in \{1, 2, ..., m\}$.

This is a contradiction.

It follows that the set of the attributes A, such that $i_{r,\beta}(A) > \frac{1}{2}$ is not empty.

Second, suppose contrary, that $i_{r,\beta}(A) > \frac{1}{2}$ for all A's.

Reasoning as in the previous case, we see that (5) yields that either

$$1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \leq i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}$$

or

$$i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq 1 - i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}.$$

Hence, in any scenario, $i_{r,\beta}(B_1 \wedge ... \wedge B_n) \leq \frac{1}{2}$.

This means that there exists some B_k, $k \in \{1, 2, ..., n\}$, such that $i_{r,\beta}(B_k) \leq \frac{1}{2}$.

This is the contradiction.

We conclude, the set of those A's, for which $i_{r,\beta}(A) > \frac{1}{2}$, does not coincide with the set of all attributes.

Now, we introduce two-tuple fuzzy relation instance, requiring that its tuples coincide in those A's, for which $i_{r,\beta}(A) > \frac{1}{2}$.

If we denote this instance by r', and its tuples by t' and t'', then we have that $\varphi\left(A\left[t', t''\right]\right) = 1$ if $i_{r,\beta}(A) > \frac{1}{2}$, and $\varphi\left(A\left[t', t''\right]\right) = \theta''$ if $i_{r,\beta}(A) \leq \frac{1}{2}$.

Since each domain of each attribute is given by the set $\{p, q\}$, we may assume that each attribute A ($i_{r,\beta}(A) > \frac{1}{2}$), assumes the value p on both tuples t' and t'', while the attribute A ($i_{r,\beta}(A) \leq \frac{1}{2}$), takes the value p on t', and the value q on t''.

Let

$$(P_1 \wedge ... \wedge P_i) \Rightarrow (Q_1 \wedge ... \wedge Q_j)$$

be the fuzzy formula corresponding to some element of F.

It can happen that either $i_{r,\beta}(P_1 \wedge ... \wedge P_i) \leq \frac{1}{2}$ or $i_{r,\beta}(P_1 \wedge ... \wedge P_i) > \frac{1}{2}$.

In the first case, $i_{r,\beta}(P_k) \leq \frac{1}{2}$ for some $k \in \{1, 2, ..., i\}$, that is, $\varphi\left(P_k\left[t', t''\right]\right) = \theta''$.

Consequently, $\varphi\left(P\left[t', t''\right]\right) = \theta''$, where $P = \{P_1, ..., P_i\}$, $Q = \{Q_1, ..., Q_j\}$, and the dependency is given by $P \xrightarrow{\theta_2}_F Q$ for some strength $\theta_2 \in [0, 1]$.

Since $\varphi\left(Q\left[t', t''\right]\right) \geq \theta''$ always, we have that

$$\varphi\left(Q\left[t', t''\right]\right) \geq \min\left\{\theta_2, \varphi\left(P\left[t', t''\right]\right)\right\}.$$

Thus, $P \xrightarrow{\theta_2}_F Q$ is correct under r'.

Suppose that $i_{r,\beta}(P_1 \wedge ... \wedge P_i) > \frac{1}{2}$.

It is understood that we assume that

$$\frac{1}{2} < i_{r,\beta}\Big((P_1 \wedge ... \wedge P_i) \Rightarrow (Q_1 \wedge ... \wedge Q_j)\Big).$$

As earlier, we obtain that

$$1 - i_{r,\beta}(P_1 \wedge ... \wedge P_i) \le i_{r,\beta}(Q_1 \wedge ... \wedge Q_j) > \frac{1}{2}.$$

or

$$i_{r,\beta}(Q_1 \wedge ... \wedge Q_j) \le 1 - i_{r,\beta}(P_1 \wedge ... \wedge P_i) > \frac{1}{2}.$$

In the second case it follows that $i_{r,\beta}(P_1 \wedge ... \wedge P_i) < \frac{1}{2}$.

This is the contradiction.

In the first case, $i_{r,\beta}(Q_1 \wedge ... \wedge Q_j) > \frac{1}{2}$ yields that $\varphi\left(Q\left[t', t''\right]\right) = 1$.

Hence,

$$\varphi\left(Q\left[t', t''\right]\right) \ge \min\left\{\theta_2, \varphi\left(P\left[t', t''\right]\right)\right\}.$$

We conclude, $P \xrightarrow{\theta_2}_F Q$ is correct with respect to r'.

In view of the contraposition law, the claim of theorem will follow if we prove that $X \xrightarrow{\theta}_F Y$ is violated by r'.

By (5),

$$1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \le i_{r,\beta}(B_1 \wedge ... \wedge B_n) \le \frac{1}{2}.$$

or

$$i_{r,\beta}(B_1 \wedge ... \wedge B_n) \le 1 - i_{r,\beta}(A_1 \wedge ... \wedge A_m) \le \frac{1}{2}.$$

In any case, $i_{r,\beta}(B_1 \wedge ... \wedge B_n) \le \frac{1}{2}$, i.e., $\varphi\left(Y\left[t', t''\right]\right) = \theta''$.

Note that,

$$\min\left\{\theta, \varphi\left(X\left[t', t''\right]\right)\right\} \ge \min\left\{\theta, \theta''\right\} = \theta''.$$

Hence,

$$\varphi\left(Y\left[t', t''\right]\right) < \min\left\{\theta, \varphi\left(X\left[t', t''\right]\right)\right\}.$$

This completes the proof. □

Now, it is not hard to check that the opposite claim of the claim given by Theorem 5 holds also true.

Theorem 6. *Let $X \xrightarrow{\theta}_F Y$ be a fuzzy functional dependency on scheme R, and F a set of fuzzy functional dependencies. If the fuzzy formula associated to $X \xrightarrow{\theta}_F Y$ is correct, then the dependency $X \xrightarrow{\theta}_F Y$ is a logical consequence of the set F with respect to two-element fuzzy relations.*

4 Applications

As it is known, new dependencies are classically derived with the use of the corresponding list of inference rules.

The second way, where the results derived above may be applied, is the one via resolution rules.

Example 1. Suppose that P, Q, R, S and L are some attributes on some scheme R_1. Let F be the set of fuzzy functional dependencies on R_1, whose elements are: $\{P\} \xrightarrow{\theta_1}_F \{Q\}$, $\{Q, R\} \xrightarrow{\theta_2}_F \{S\}$ and $\{S\} \xrightarrow{\theta_3}_F \{L\}$. Let $\{P, R\} \xrightarrow{\theta}_F \{L\}$ be the fuzzy functional dependency on R_1, with $\theta = \min\{\theta_i : i \in \{1,2,3\}\}$. Prove that the set F determines $\{P, R\} \xrightarrow{\theta}_F \{L\}$.

Proof. I. Put $X = \{P\}$, $Y = \{Q\}$, $W = \{R\}$, $Z = \{S\}$, $T = \{L\}$.

It should be recognized that the elements of the set F can be written in the form: $X \xrightarrow{\theta_1}_F Y$, $W \cup Y \xrightarrow{\theta_2}_F Z$ and $Z \xrightarrow{\theta_3}_F T$.

Writing the set F in this form, we can easily find out which of the inference rules are to be applied.

Indeed, following [2, p. 168], the dependencies $X \xrightarrow{\theta_1}_F Y$ and $W \cup Y \xrightarrow{\theta_2}_F Z$, together with the pseudo-transitivity rule, give us $W \cup Y \xrightarrow{\theta'}_F Z$, where $\theta' = \min\{\theta_i : i \in \{1,2\}\}$.

Furthermore, the dependencies $W \cup Y \xrightarrow{\theta'}_F Z$ and $Z \xrightarrow{\theta_3}_F T$ together with the classical transitivity rule yield the dependency $W \cup Y \xrightarrow{\theta''}_F T$, where $\theta'' = \min\left\{\theta', \theta_3\right\}$.

Having in mind our sets $X = \{P\}$, $W = \{R\}$, $T = \{L\}$, and the fact that

$$\begin{aligned}\theta'' &= \min\left\{\theta', \theta_3\right\} \\ &= \min\{\min\{\theta_i : i \in \{1,2\}\}, \theta_3\} \\ &= \min\{\theta_i : i \in \{1,2,3\}\} = \theta,\end{aligned}$$

we arrive to the dependency $\{P, R\} \xrightarrow{\theta}_F \{L\}$. □

Proof. II. In view of the results derived in this research, we may proceed in the following way.

According to our results,it is enough to prove that the fuzzy formula $\Gamma : P \wedge R \Rightarrow L$ is valid.

Clearly, we first assign the fuzzy formulas: $F_1 : P \Rightarrow Q$, $F_2 : Q \wedge R \Rightarrow S$ and $F_3 : S \Rightarrow L$ to the dependencies contained in the set F.

Our task is reduced in a way that it is enough to show that the fuzzy formula $F^{'} : F_1 \wedge F_2 \wedge F_3 \wedge \neg\Gamma$ is not valid.

Namely, the fact that F_1, F_2, F_3 are known to be valid, together with the fact that $F^{'}$ is not valid, would mean that $\neg\Gamma$ is incorrect, which would immediately yield that Γ must be true.

To use the resolution rules, we transform F into conjunctive normal form F^{*}, where

$$F^{*} = \Big\{ \neg P \vee Q, \neg Q \vee \neg R \vee S, \\ \neg S \vee L, P, R, \neg L \Big\}.$$

The following resolvents hold true: $\neg P \vee Q$ and P give Q.

$\neg S \vee L$ and $\neg L$ give $\neg S$.

Furthermore, $\neg Q \vee \neg R \vee S$ and R give $\neg Q \vee S$.

Now, $\neg Q \vee S$ and Q give S.

So, S and $\neg S$ exclude each other.

The obtained condition means that $\neg\Gamma$ does not hold, i.e., that Γ is true. □

5 Conclusion

The paper is devoted to equivalence established between the similarity-based fuzzy relation databases theory (taken through fuzzy functional dependencies), and a fragment of fuzzy logic theory (via fuzzy logic operators).

In view of the subject and goal, the attributes are assigned explicit values in the unit interval $[0, 1]$, and are treated as fuzzy formulas in this way. Consequently, and generally, any fuzzy functional dependency in the case at hand is turned into a fuzzy formula.

Looking back at the derived results, one notes that the attached fuzzy formula is satisfied by the appropriate valuation, anytime the given dependeny agrees with the initial fuzzy relation (Theorem 1).

However, the opposite statement is not necessarily true (Theorem 2).

The research is widen to include sets of fuzzy functional dependencies (Theorems 3-5), where the connection between arbitrary fuzzy relations and the correctness of the associated fuzzy formulas is achieved by making use of two-element fuzzy relation instances.

As becomes clear from the example above, the obtained equivalence is making possible the application of the resolution techniques (treated as relevant inference rules). Thus, one's work on purely theoretical aspects of the dependency theory may be substituted by the one, which is more appropriate to machine language.

Regarding the future research, we note that an analogous study is planned to be conducted upon fuzzy multivalued, or more generally, vague functional (multivalued) dependencies theories.

6 Remarks

Regarding the electrical and computer engineering applications of this study (automatic control, robotics and industry), we provide the following example.

Consider the attributes "Name", "Age", "Education", "Salary" and "Consumption" on some fuzzy relation scheme R_1.

Denote by U the universal set of attributes.

Suppose that domains of the above attributes are given by the following sets:

$$\begin{aligned} P^{'} &= \{Sanela, Dzenan, Amela, Zenan\}, \\ Q^{'} &= \{young, 45, old\}, \\ R^{'} &= \{bachelor, master\}, \\ S^{'} &= \{250EUR, 1500EUR\}, \\ L^{'} &= \{low, average, high\}. \end{aligned}$$

As it is usual, we introduce similarity relations on given domains. Such relations (see, Section 1) must be reflexive, symmetric and must satisfy the max-min transitivity condition.

Let $s_1 : P^{'} \times P^{'} \to [0, 1]$ be the similarity relation on $P^{'}$ given by: $s_1(x, x) = 1$ for all $x \in P^{'}$, and $s_1(x, y) = s_1(y, x) = 0$ if $x, y \in P^{'}$, $x \neq y$.

Hence, we have for example:
$s_1(Sanela, Sanela) = 1$, while
$s_1(Amela, Dzenan) = 0$.

Note that s_1 defined in this way is obviously reflexive and symmetric. It is also max-min transitive. Namely,

$$\begin{aligned}
&\max_{q\in P'}\left\{\min\left\{s_1(Sanela,q),s_1(q,Amela)\right\}\right\}\\
&=\max\left\{\min\left\{s_1(Sanela,Sanela),\right.\right.\\
&s_1(Sanela,Amela)\left.\right\},\\
&\min\left\{s_1(Sanela,Dzenan),\right.\\
&s_1(Dzenan,Amela)\left.\right\},\\
&\min\left\{s_1(Sanela,Amela),\right.\\
&s_1(Amela,Amela)\left.\right\},\\
&\min\left\{s_1(Sanela,Zenan),\right.\\
&s_1(Zenan,Amela)\left.\left.\right\}\right\}\\
&=\max\left\{\min\{1,0\},\min\{0,0\},\right.\\
&\min\{0,1\},\min\{0,0\}\left.\right\}\\
&=\max\{0,0,0,0\}\\
&=0=s_1(Sanela,Amela).
\end{aligned}$$

It follows that

$$\begin{aligned}
&s_1(Sanela,Amela)\\
&\geq\max_{q\in P'}\left\{\min\left\{s_1(Sanela,q),s_1(q,Amela)\right\}\right\},
\end{aligned}$$

so the max-min transitivity condition holds true in this particular case.

In order to show that the condition

$$\begin{aligned}
&s_1(x,z)\\
&\geq\max_{q\in P'}\left\{\min\left\{s_1(x,q),s_1(q,z)\right\}\right\}
\end{aligned}\tag{6}$$

is always valid, we note the following facts.

If x and z are mutually different, then each minimum which appears on the right hand side of (6) is 0.

More precisely, q can be equal either to x or to z. Nevertheless, either $s_1(x,q)$ or $s_1(q,z)$ is 0.

If $q\neq x$ and $q\neq z$, then $s_1(x,q)=0$, $s_1(q,z)=0$, so the corresponding minimum is once again 0.

Thus, for $x\neq z$, the inequality (6) is correct.

On the other hand, for $x=z$, we have that $s_1(x,z)=1$, so the inequality (6) is again valid, having in mind that the maximum on the right hand side of (6) is not larger than 1.

This shows that the relation s_1 (given as the identity relation on P') is an example of the similarity relation.

We fix this relation on P' in the sequel.

Define $s_2:Q'\times Q'\to[0,1]$ by
$s_2(young,young)=s_2(45,45)=s_2(old,old)=1$, and $s_2(young,45)=s_2(45,young)=0.5$,
$s_2(45,old)=s_2(old,45)=0.5$, $s_2(young,old)=s_2(old,young)=0.5$.

If $x=z$, the inequality (6) holds immediately true for s_2.

In order to show that s_2 is a similarity relation on Q', it is enough to consider $s_2(young,45)$, $s_2(young,old)$ and $s_2(45,old)$.

We have,

$$\begin{aligned}
&\max_{q\in Q'}\left\{\min\left\{s_2(young,q),s_2(q,45)\right\}\right\}\\
&=\max\left\{\min\left\{s_2(young,young),\right.\right.\\
&s_2(young,45)\left.\right\},\\
&\min\left\{s_2(young,45),s_2(45,45)\right\},\\
&\min\left\{s_2(young,old),s_2(old,45)\right\},\\
&=\max\left\{\min\{1,0.5\},\min\{0.5,1\},\right.\\
&\min\{0.5,0.5\}\left.\right\}\\
&=\max\{0.5,0.5,0.5\}\\
&=0.5=s_2(young,45).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&s_2(young,old)\\
&\geq\max_{q\in Q'}\left\{\min\left\{s_2(young,q),s_2(q,old)\right\}\right\},\\
&s_2(45,old)\\
&\geq\max_{q\in Q'}\left\{\min\left\{s_2(45,q),s_2(q,old)\right\}\right\}.
\end{aligned}$$

For the similarity relation on R' we take
$s_3:R'\times R'\to[0,1]$. It is introduced by
$s_3(bachelor,master)=s_3(master,bachelor)=0.35$, and $s_3(bachelor,bachelor)=s_3(master,master)=1$.

We deduce,

$$\max_{q\in R'}\left\{\min\left\{s_3(bachelor,q),s_3(q,master)\right\}\right\}$$
$$=\max\left\{\min\left\{s_3(bachelor,bachelor),\right.\right.$$
$$\left.s_3(bachelor,master)\right\},$$
$$\min\left\{s_3(bachelor,master),\right.$$
$$\left.\left.s_3(master,master)\right\}\right\},$$
$$=\max\left\{\min\{1,0.35\},\min\{0.35,1\}\right\}$$
$$=\max\{0.35,0.35\}$$
$$=0.35=s_3(bachelor,master).$$

Therefore, s_3 is a similarity relation on $R^{'}$.

We put $s_4 : S^{'} \times S^{'} \to [0,1]$ to be given by:
$s_4(250EUR,250EUR) =$
$s_4(1500EUR,1500EUR) = 1$, and
$s_4(250EUR,1500EUR) =$
$s_4(1500EUR,250EUR) = 0.2$.

As in the previous case, we find that s_4 is a similarity relation on $S^{'}$.

Finally, we introduce $s_5 : L^{'} \times L^{'} \to [0,1]$ by $s_5(low,low) = s_5(average,average) =$ $s_5(high,high) = 1$, and $s_5(low,average) =$ $s_5(average,low) = 0.6$, $s_5(low,high) =$ $s_5(high,low) = 0.6$, $s_5(average,high) =$ $s_5(high,average) = 0.6$.

Reasoning in the same way as in the case of similarity relations s_1 and s_2, we conclude that s_5 is a similarity relation on $L^{'}$.

Let $r = \{t_1,t_2\}$ be two-elements fuzzy relation instance on R_1, whose tuples t_1 and t_2 are determined in the following way.

The elements of t_1 are: $\{Sanela\}$, $\{young,45\}$, $\{master\}$, $\{250EUR,1500EUR\}$ and $\{low,average\}$. Furthermore, the elements of t_2 are: $\{Dzenan,Zenan\}$, $\{young\}$, $\{bachelor,master\}$, $\{250EUR\}$ and $\{low\}$.

Note that the scenario described by r is realistic and absolutely possible to occur in reality.

Now, we calculate the conformances $\varphi(A[t_1,t_2])$ for $A \in U$. We obtain,

$$\varphi(Name[t_1,t_2])$$
$$=\min\left\{\min_{x\in\{Sanela\}}\left\{\max_{y\in\{Dzenan,Zenan\}}\left\{\right.\right.\right.$$
$$\left.\left.s_1(x,y)\right\}\right\},$$
$$\min_{x\in\{Dzenan,Zenan\}}\left\{\max_{y\in\{Sanela\}}\left\{\right.\right.$$
$$\left.\left.\left.s_1(x,y)\right\}\right\}\right\}$$
$$=\min\left\{\max\left\{s_1(Sanela,Dzenan),\right.\right.$$
$$\left.s_1(Sanela,Zenan)\right\},$$
$$\min\left\{s_1(Dzenan,Sanela),\right.$$
$$\left.\left.s_1(Zenan,Sanela)\right\}\right\}$$
$$=\min\{\max\{0,0\},\min\{0,0\}\}$$
$$=\min\{0,0\}=0,$$

$$\varphi(Age[t_1,t_2])$$
$$=\min\left\{\min_{x\in\{young,45\}}\left\{\max_{y\in\{young\}}\left\{\right.\right.\right.$$
$$\left.\left.s_2(x,y)\right\}\right\},$$
$$\min_{x\in\{young\}}\left\{\max_{y\in\{young,45\}}\left\{\right.\right.$$
$$\left.\left.\left.s_2(x,y)\right\}\right\}\right\}$$
$$=\min\left\{\min\left\{s_2(young,young),\right.\right.$$
$$\left.s_2(45,young)\right\},$$
$$\max\left\{s_2(young,young),\right.$$
$$\left.\left.s_2(young,45)\right\}\right\}$$
$$=\min\{\min\{1,0.5\},\max\{1,0.5\}\}$$
$$=\min\{0.5,1\}=0.5,$$

$$\begin{aligned}
&\varphi\left(Education\left[t_1,t_2\right]\right)\\
&=\min\left\{\min_{x\in\{master\}}\left\{\max_{y\in\{bachelor,master\}}\left\{\right.\right.\right.\\
&\quad s_3\left(x,y\right)\left.\left.\right\}\right\},\\
&\quad\min_{x\in\{bachelor,master\}}\left\{\max_{y\in\{master\}}\left\{\right.\right.\\
&\quad s_3\left(x,y\right)\left.\left.\left.\right\}\right\}\right\}\\
&=\min\left\{\max\left\{s_3\left(master,bachelor\right),\right.\right.\\
&\quad s_3\left(master,master\right)\left.\right\},\\
&\quad\min\left\{s_3\left(bachelor,master\right),\right.\\
&\quad s_3\left(master,master\right)\left.\left.\right\}\right\}\\
&=\min\left\{\max\left\{0.35,1\right\},\min\left\{0.35,1\right\}\right\}\\
&=\min\left\{1,0.35\right\}=0.35,
\end{aligned}$$

$$\begin{aligned}
&\varphi\left(Salary\left[t_1,t_2\right]\right)\\
&=\min\left\{\min_{x\in\{250EUR,1500EUR\}}\left\{\max_{y\in\{250EUR\}}\left\{\right.\right.\right.\\
&\quad s_4\left(x,y\right)\left.\left.\right\}\right\},\\
&\quad\min_{x\in\{250EUR\}}\left\{\max_{y\in\{250EUR,1500EUR\}}\left\{\right.\right.\\
&\quad s_4\left(x,y\right)\left.\left.\left.\right\}\right\}\right\}\\
&=\min\left\{\min\left\{s_4\left(250EUR,250EUR\right),\right.\right.\\
&\quad s_4\left(1500EUR,250EUR\right)\left.\right\},\\
&\quad\max\left\{s_4\left(250EUR,250EUR\right),\right.\\
&\quad s_4\left(250EUR,1500EUR\right)\left.\left.\right\}\right\}\\
&=\min\left\{\min\left\{1,0.2\right\},\max\left\{1,0.2\right\}\right\}\\
&=\min\left\{0.2,1\right\}=0.2,
\end{aligned}$$

$$\begin{aligned}
&\varphi\left(Consumption\left[t_1,t_2\right]\right)\\
&=\min\left\{\min_{x\in\{low,average\}}\left\{\max_{y\in\{low\}}\left\{\right.\right.\right.\\
&\quad s_5\left(x,y\right)\left.\left.\right\}\right\},\\
&\quad\min_{x\in\{low\}}\left\{\max_{y\in\{low,average\}}\left\{\right.\right.\\
&\quad s_5\left(x,y\right)\left.\left.\left.\right\}\right\}\right\}\\
&=\min\left\{\min\left\{s_5\left(low,low\right),\right.\right.\\
&\quad s_5\left(average,low\right)\left.\right\},\\
&\quad\max\left\{s_5\left(low,low\right),\right.\\
&\quad s_5\left(low,average\right)\left.\left.\right\}\right\}\\
&=\min\left\{\min\left\{1,0.6\right\},\max\left\{1,0.6\right\}\right\}\\
&=\min\left\{0.6,1\right\}=0.6.
\end{aligned}$$

Now, one is position to consider various fuzzy functional dependencies on scheme R_1. It is also possible to check whether or not fuzzy relation instance r satisfies such dependencies.

Thus, the dependencies: $\{Name\}\xrightarrow{0.95}_F\{Age\}$, $\{Age,Education\}\xrightarrow{0.19}_F\{Salary\}$ and $\{Salary\}\xrightarrow{0.8}_F\{Consumption\}$ are satisfied by r.

Indeed,

$$\begin{aligned}
&\varphi\left(Age\left[t_1,t_2\right]\right)\\
&=0.5>0=\min\left\{0.95,0\right\}\\
&=\min\left\{0.95,\varphi\left(Name\left[t_1,t_2\right]\right)\right\}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\varphi\left(\{Age,Education\}\left[t_1,t_2\right]\right)\\
&=\min\left\{\varphi\left(Age\left[t_1,t_2\right],\varphi\left(Education\left[t_1,t_2\right]\right)\right)\right\}\\
&=\min\left\{0.5,0.35\right\}=0.35,
\end{aligned}$$

so

$$\begin{aligned}
&\varphi\left(Salary\left[t_1,t_2\right]\right)\\
&=0.2>0.19=\min\left\{0.19,0.35\right\}\\
&=\min\left\{0.19,\varphi\left(\{Age,Education\}\left[t_1,t_2\right]\right)\right\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\varphi\left(Consumption\left[t_1,t_2\right]\right)\\
&=0.6>0.2=\min\left\{0.8,0.2\right\}\\
&=\min\left\{0.8,\varphi\left(Salary\left[t_1,t_2\right]\right)\right\}.
\end{aligned}$$

Note that the dependency $\{Name, Education\} \xrightarrow{0.19}_F \{Consumption\}$ is also satisfied by r. Namely,

$$\begin{aligned}&\varphi\left(\{Name, Education\}[t_1, t_2]\right)\\&= \min\left\{\varphi\left(Name[t_1, t_2], \varphi\left(Education[t_1, t_2]\right)\right)\right\}\\&= \min\{0, 0.35\} = 0,\end{aligned}$$

so

$$\begin{aligned}&\varphi\left(Consumption[t_1, t_2]\right)\\&=0.6 > 0 = \min\{0.19, 0\}\\&= \min\left\{0.19, \varphi\left(\{Name, Education\}[t_1, t_2]\right)\right\}.\end{aligned}$$

This particular example shows that for fixed fuzzy relation instance r, we are able to verify whether or not it satisfies given fuzzy functional dependencies, or more importantly, whether or not it satisfies some individual fuzzy functional dependency. In our case we proved that each of the four dependencies listed above are independently satisfied by two-elements fuzzy relation instance r. Same reasoning could be applied in the case of arbitrary fuzzy relation instance r. However, one should note that this method (case by case), is not so suitable generally. Namely, one should recognize that the conclusions derived in this example could be summarized in the form that the dependency $\{Name, Education\} \xrightarrow{0.19}_F \{Consumption\}$ follows from the dependencies $\{Name\} \xrightarrow{0.95}_F \{Age\}$, $\{Age, Education\} \xrightarrow{0.19}_F \{Salary\}$ and $\{Salary\} \xrightarrow{0.8}_F \{Consumption\}$ in the world of two-tuples fuzzy relation instances, and then require more general result, that the same statement remains valid in the case of arbitrary fuzzy relations. More precisely, one should note that the assertion of Example 1 (see, Section 4), could be applied in this particular situation. Indeed, put P, Q, R, S, L to be the attributes ”Name”, ”Age”, ”Education”, ”Salary”, ”Consumption”, respectively. We assume that R_1 is $R_1(P, Q, R, S, L)$. Furthermore, we assume that $\theta_1 = 0.9$, $\theta_2 = 0.19$ and $\theta_3 = 0.8$. We have that $\theta = \min\{\theta_1, \theta_2, \theta_3\} = 0.19$. According to Example 1, the dependencies: $\{P\} \xrightarrow{\theta_1}_F \{Q\}$, $\{Q, R\} \xrightarrow{\theta_2}_F \{S\}$ and $\{S\} \xrightarrow{\theta_3}_F \{L\}$ yield the dependency $\{P, R\} \xrightarrow{\theta}_F \{L\}$ in arbitrary fuzzy relations. Translated into our notation, this means that the dependencies $\{Name\} \xrightarrow{0.95}_F \{Age\}$, $\{Age, Education\} \xrightarrow{0.19}_F \{Salary\}$ and $\{Salary\} \xrightarrow{0.8}_F \{Consumption\}$ determine the dependency $\{Name, Education\} \xrightarrow{0.19}_F \{Consumption\}$.
In other words, this means that the dependency $\{Name, Education\} \xrightarrow{0.19}_F \{Consumption\}$ is valid not only with respect to our particular two-element fuzzy relation r, but also with respect to arbitrary fuzzy relation satisfying $\{Name\} \xrightarrow{0.95}_F \{Age\}$, $\{Age, Education\} \xrightarrow{0.19}_F \{Salary\}$ and $\{Salary\} \xrightarrow{0.8}_F \{Consumption\}$ at the same time. This general result follows from the assertion of Example 1, and, as we have already seen, the assertion of Example 1 follows either manually from the inference rules (proof I), or automatically from the resolution principle (proof II). Thus, thanks to the results derived in this research, our particular problem of showing that given two-element fuzzy relation instance r satisfies the dependency $\{Name, Education\} \xrightarrow{0.19}_F \{Consumption\}$ can be generalized, and, moreover, its solution can be automated. So, it is automatization which enables possible electrical and computer engineering application of the study as well as its application within automatic control, robotics and industry.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Dženan Gušić has conceptualized the research and organized the paper.
Sanela Nesimović derived the results, discussed their applications, and made examples and the literature review.