A New Exponentially Fitted Numerical Integration Scheme for Solving Singularly Perturbed Two Point Boundary Value Problems

MOHAMMAD JAVED ALAM
National Institute of Technology,
Jamshedpur
Department of Mathematics
India-831014
javed.2014rsma002@nitjsr.ac.in

HARI SHANKAR PRASAD
National Institute of Technology,
Jamshedpur
Department of Mathematics
India-831014
hsprasad.math@nitjsr.ac.in

RAKESH RANJAN
National Institute of Technology,
Jamshedpur
Department of Mathematics
India-831014
90.ranjan@gmail.com

Abstract: This article is concerned with an exponentially fitted numerical integration method based on uniform mesh for solving singularly perturbed two point boundary value problems. Exact and approximate rule of integration with finite difference approximation of first derivatives are used to derive a three term scheme. Theory of singular perturbation is used to introduce a fitting factor in the derived scheme. Thomas algorithm is employed to solve the resulting tridiagonal system of equations. Convergence of the proposed method is also analyzed. Solutions of several linear and nonlinear example problems are presented in terms of maximum absolute errors (MAE) to show the applicability of the proposed scheme. It is easily observed that the proposed method is able to approximate the solution very well.

Key–Words: Singular perturbation problems, Boundary value problems, Stability and convergence, Numerical Integration

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1 Introduction

Differential equations where the highest order derivative is multiplied by an arbitrarily small parameter known as the singular perturbation parameter. These problems are arise frequently in the applied sciences and engineering, typical examples include Navier-Stokes flow problems involving high Reynolds number [1,2,3], mathematical models of liquid crystal materials and chemical reactions, control theory, electrical networks[4,5,6]. These problems have been received a significant amount of attention in past and recent years. It is a known fact that the solution of these types of problems exhibit sharp boundary layers or interior layers when the value of is taken very small, i.e., the solution exhibits rapid variation near the layer region while in regular region it varies slowly. Typically thin transition layers are present where the solutions can jump abruptly, while away from the layers the solution exhibits regular behaviour and varies slowly. So it becomes extremely difficult to solve singularly perturbed problems as severe complications have to be confronted to obtain numerical solutions accurately. Thus solution techniques which are more efficient with simpler computations are needed to solve singular perturbation problems. Readers may consult the books: [7,8,9,10,11] for a detailed analytical discussion on various methods to solve singular perturbation problems. Also, recent books [12,13,14,3,15]and their corresponding references present some numerical methods and their convergence analysis. In the recent past, the authors in [16,17] have suggested exponentially fitted finite difference methods on uniform mesh for solving model equation of the form (1). Reddy and Mohapatra[16] have presented an efficient numerical method with exponentially fitted factor to obtain the solution of singularly perturbed two point boundary value problems exhibiting boundary layer at one end point (either left or right). Gbsl Soujanya et al.[18] have developed an exponentially fitted non-symmetric finite difference method to solve singularly perturbed problems with layer behaviour using Numerovs method. Articles[19,20,21,22,23] propose different numerical approaches combining fitted mesh methods and fitted operator methods employed by several researchers for solving SPPs where as Kadalbajoo and Kumar[24] presents a detailed outline on the numerical methods for solving SPPs. But these existing numerical methods are mostly based on fitted operator techniques or use reasonable theoretical information regarding the solutions which forms a limitation of these approaches. Ranjan and Prasad[25] have presented an efficient method of numerical integration for a class of singularly perturbed two point boundary value problem at one end point (either left or right). Ranjan, Prasad and Alam[26] have developed a simple method of numerical integration for a class of singularly perturbed two point boundary value problems at one end point (either left or right). Ranjan and Prasad[27] have propose a fitted finite difference scheme for solving singularly perturbed two point boundary value problems having boundary layer at left or right end points.

The main purpose of this paper is to present computationally a new exponentially fitted numerical integration scheme for solving singularly perturbed two-point boundary value problems(SPTPBVP) having boundary layer at left or right end points of the interval considered. In this paper we have presented fitted schemes using the usual rule of evaluating exact and approximate value of the definite integral with finite difference approximation of derivatives to solve a class of SPTPBVP. The computational results show that the present method is capable of producing accurate results with minimal computational effort when perturbation parameter for any fixed value of the mesh size.

The rest of the paper is organized as follows: Some assumptions on the statement of the continuous problem is given in Section 2. In Subsection 2.1 and Subsection 2.2, we present in detail the construction of the numerical method having boundary layer at left and right end points of the underlying interval respectively. A study of the convergence analysis is presented in Section 3. In Section 4, some numerical examples are presented to show the applicability and the effectiveness of the proposed method. The numerical results are reported with the maximum absolute error in tables. Finally, the conclusion is given in Section 5. The paper ends with the references.

2 Statement of the problem

In this paper, we consider the second order singularly perturbed problem of the form:

$$\varepsilon y''(x)+\alpha(x)y'(x)+\beta(x)y(x)=\gamma(x);\; 0\le x\le 1 \quad (1)$$

subject to the interval and boundary conditions

$$y(0)=\eta\,\text{and}\,y(1)=\delta \quad (2)$$

where ε $(0<\varepsilon<<1)$ is a perturbation parameter and $\eta,\ \delta$ are known finite constants. Also it is assumed that $\alpha(x),\ \beta(x),\ \gamma(x)$ are sufficiently smooth and bounded functions on $[0,\ 1]$ along with $\beta(x)\le 0$ throughout of the interval $[0,\ 1]$. If we assume that $\alpha(x)\ge W>0$ throughout the interval $[0,\ 1]$, where W is a positive constant, the equation (1) along with (2) has a unique solution $y(x)$ with boundary layer at $x=0$ i.e. at left end point of the interval for small values of ε, while the boundary layer will be present in the neighbourhood of $x=1$ if $\alpha(x)\le W<0$ throughout the interval $[0,\ 1]$, where W is a negative constant.

The operator $L_\tau=\varepsilon\frac{d^2}{dx^2}+\alpha(x)\frac{d}{dx}+\beta(x)$ in (1) satisfies the following minimum principle [17].

Lemma 2.1. Suppose $\omega\,(x)$ represents a smooth function satisfying the conditions $\omega\,(0)\ge 0,\ \omega\,(1)\ge 0$. Then $L_\tau\omega\,(x)\le 0,\ \forall x\in(0,1)$ implies $\omega\,(x)\ge 0,\ \forall x\in[0,1]$.

proof : Let $m\in[0,1]$ be such that $\omega\,(m)<0$ and $\omega\,(m)=\min\limits_{x\in[0,1]}\omega\,(x)$. Clearly $m\notin\{0,1\}$, therefore $\omega'\,(m)=0$ and $\omega''\,(m)\ge 0$. Hence, we obtain

$$L_\tau\omega\,(m)=\varepsilon\omega''\,(m)+\alpha(m)\omega'\,(m)+\beta(m)\omega\,(m)>0,$$

which contradicts our assumption. Hence it is proved that $\omega\,(m)\ge 0$ and thus $\omega\,(x)\ge 0\,\forall x\in[0,1]$.

Lemma 2.2. Let $y(x)$ be the solution of the problem (1) and (2) then we have

$$\|y\|\le a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|).$$

where $\|\cdot\|$ is the L_∞ norm given by $\|y\|=\max\limits_{0\le x\le 1}|y(x)|$.

proof : Let $\omega^{\pm}\,(x)$ be two barrier functions defined by

$$\omega^{\pm}\,(x)=a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\pm y(x)$$

Then this implies

$$\begin{aligned}\omega^{\pm}\,(0)&=a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\pm y(0)\\&=a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\pm\eta_0\ \text{since}, y(0)=\eta(0)=\eta_0\\&\ge 0\end{aligned}$$

$$\begin{aligned}\omega^{\pm}\,(1)&=a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\pm y(1)\\&=a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\pm\delta\ \text{since}, y(1)=\delta\\&\ge 0\end{aligned}$$

$$\begin{aligned}\Rightarrow\ L_\tau\omega^{\pm}\,(x)&=\varepsilon(\omega^{\pm}\,(x))''+\alpha(x)(\omega^{\pm}\,(x))'+\beta(x)\omega^{\pm}\,(x)\\&=\beta(x)\left[a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\right]\pm L_\tau y(x)\\&=\beta(x)\left[a_2^{-1}\,\|\gamma\|+\max(|\eta_0|,\,|\delta|)\right]\pm\gamma(x)\ \text{using}\ (1)\end{aligned}$$

As $\beta(x)\le -a_2<0$ implies $\beta(x)a_2^{-1}\le -1$ and since $\|\gamma\|\ge\gamma(x)$, we have

$$\Rightarrow\ L_\tau\omega^{\pm}\,(x)\le(-\|\gamma\|\pm\gamma(x))+\beta(x)\max(|\eta_0|,\,|\delta|)\le 0,\ \forall x\in[0,1].$$

Thus using the minimum principle we obtain, $\omega^{\pm}\,(x)\ge 0\,,\ \forall x\in[0,1]$. Now for computing the error that has occurred in our numerical approximations, the derivative of the solution $y(x)$ should possess a boundedness which remains valid for all $x\in(0,\,1]$.

With the help of Lemma 2.1, the required estimate is obtained.

2.1 Description of the method for left-end boundary layer problems

First, equation (1) can be written in the given form:

$$\begin{aligned}&\left[A(x)y'(x)\right]'-A'(x)y'(x)+y'(x)+C(x)y(x)\\&=H(x);\quad 0\le x\le\ 1\end{aligned}\quad (3)$$

$$\text{where}\ A(x)=\frac{\varepsilon}{\alpha(x)},\ C(x)=\frac{\beta(x)}{\alpha(x)},\ H(x)=\frac{\gamma(x)}{\alpha(x)}.\quad (4)$$

Consider the partition $0=x_0<x_1<x_2,...,<x_{N-1}<x_N=1$ of the interval $[0,\ 1]$ with the uniform mesh size h defined by the relation $x_i=x_0+ih,$ for $i=0,1,2,...,N.$ Integrate equation (3) in the interval $[x_i,\ x_{i+1}]$ and rearrange to get:

$$\begin{aligned}&A(x_{i+1})y'(x_{i+1})-A(x_i)y'(x_i)-A'(x_{i+1})\\&y(x_{i+1})+A'(x_i)y(x_i)+\int_{x_i}^{x_{i+1}}A''(x)y(x)dx+\\&y(x_{i+1})-y(x_i)+\int_{x_i}^{x_{i+1}}C(x)y(x)dx=\\&\int_{x_i}^{x_{i+1}}H(x)dx\end{aligned}\quad (5)$$

Now, apply trapezoidal rule of Integration on the equation (5) to get:

$$\begin{array}{l} A(x_{i+1})y'(x_{i+1}) - A(x_i)y'(x_i) - A'(x_{i+1})y(x_{i+1}) \\ +\frac{h}{2}\left[A''(x_i)y(x_i) + A''(x_{i+1})y(x_{i+1})\right] \\ +A'(x_i)y(x_i) + \frac{h}{2}\left[C(x_i)y(x_i) + C(x_{i+1})y(x_{i+1})\right] \\ +y(x_{i+1}) - y(x_i) = \frac{h}{2}\left[H(x_i) + H(x_{i+1})\right] \end{array} \tag{6}$$

Use the first order backward/forward finite difference approximations: $y'(x_{i+1}) = \frac{(y(x_{i+1})-y(x_i))}{h}$ and$y'(x_i) = \frac{(y(x_i)-y(x_{i-1}))}{h}$and notations: $y(x_i) = y_i$, $y(x_{i+1}) = y_{i+1}$, $A(x_i) = A_i$, $C(x_i) = C_i, \alpha(x_i) = \alpha_i$ etc. in equation (6) , we obtain the scheme:

$$\begin{array}{l} \frac{1}{h}\left[A_{i+1}y_{i+1} - A_{i+1}y_i - A_i y_i + A_i y_{i-1}\right] - \\ A'_{i+1}y_{i+1} + A'_i y_i + \frac{h}{2}\left[A''_i y_i + A''_{i+1}y_{i+1}\right] - y_i + \\ y_{i+1} + \frac{h}{2}\left[C_i y_i + C_{i+1}y_{i+1}\right] = \frac{h}{2}\left[H_i + H_{i+1}\right] \end{array} \tag{7}$$

Now, fitting factor $\sigma(\rho)$ is introduced into equation (7) to get:

$$\begin{array}{l} \frac{\sigma(\rho)}{\rho}\left[\left\{\frac{1}{\alpha_{i+1}}y_{i+1} - \frac{1}{\alpha_{i+1}}y_i - \frac{1}{\alpha_i}y_i + \frac{1}{\alpha_i}y_{i-1}\right\} - \right. \\ h\left\{\left(\frac{1}{\alpha_{i+1}}\right)' y_{i+1} - \left(\frac{1}{\alpha_i}\right)' y_i\right\} + \\ \left.\frac{h^2}{2}\left\{\left(\frac{1}{\alpha_i}\right)'' y_i + \left(\frac{1}{\alpha_{i+1}}\right)'' y_{i+1}\right\}\right] + y_{i+1} - y_i + \\ \frac{h}{2}\left[C_i y_i + C_{i+1}y_{i+1}\right] = \frac{h}{2}\left[H_i + H_{i+1}\right] \end{array} \tag{8}$$

where $\rho = \frac{h}{\varepsilon}$ and $\sigma(\rho)$ is the fitting factor which can be determined in such a way that the solution of equation (8) converges uniformly to the solution equations (1) and (3). Taking limits as $h \to 0$, we obtain

$$\begin{array}{l} \frac{\sigma(\rho)}{\rho\,\alpha(0)}\lim\limits_{h\to 0}\left[y(ih+h) - 2y(ih) + y(ih-h)\right] + \\ \lim\limits_{h\to 0}\left[y(ih+h) - y(ih)\right] = 0 \end{array} \tag{9}$$

under the assumption that the expression $\left[A'_i y_i - A'_{i+1}y_{i+1}\right] + \frac{1}{2}\left[A''_i y_i + A''_{i+1}y_{i+1}\right] + \frac{1}{2}\left[C_i y_i + C_{i+1}y_{i+1}\right]$ and $\frac{1}{2}\left[H_i + H_{i+1}\right]$ are bounded.

It is well known that the solutions of equation (1) with equation (2) is of the following form (cf.[10], pp.22-26):

$$y(x) = y_0(x) + \frac{\alpha(0)}{\alpha(x)}\left(\alpha - y_0(0)\right)e^{-\int_0^x\left(\frac{\alpha(x)}{\varepsilon} - \frac{\beta(x)}{\alpha(x)}\right)dx} + o(\varepsilon) \tag{10}$$

where $y_0(x)$ is the solution of the reduced problem:

$$\alpha(x)y_0'(x) + \beta(x)y_0(x) = \gamma(x);\ y_0(1) = \delta \tag{11}$$

Under the consideration of Taylor's series expansions for $\alpha(x)$ and $\beta(x)$ about the point $'x = 0'$ upto their first terms only, the equation(10) becomes:

$$y(x) = y_0(x) + \left(\alpha - y_0(0)\right)e^{-\left(\frac{\alpha(0)}{\varepsilon} - \frac{\beta(0)}{\alpha(0)}\right)x} + o(\varepsilon) \tag{12}$$

Further, considering equation (12) at the point $x = x_i = ih$, $i = 0, 1, 2, ..., N$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} y(ih) = y_0(0) + \left(\alpha - y_0(0)\right)e^{-\left(\frac{\alpha^2(0)-\varepsilon\beta(0)}{\alpha(0)}\right)i\rho} + o(\varepsilon) \tag{13}$$

where $\rho = h/\varepsilon$.

Using the equation (13) for $y(ih-h)$, $y(ih)$, $y(ih+h)$ in equation (9) and then simplifying, we get the value of the fitting factor as

$$\sigma(\rho) = \frac{\alpha(0)\rho}{2}\left[\frac{e^{-\left(\frac{\alpha^2(0)-\varepsilon\beta(0)}{\alpha(0)}\right)\frac{\rho}{2}}}{\sinh\left(\left(\frac{\alpha^2(0)-\varepsilon\beta(0)}{\alpha(0)}\right)\frac{\rho}{2}\right)}\right] \tag{14}$$

Finally, by making use of equation (8) and $\sigma(\rho)$ given by equation (14), we can get the following three-term recurrence relationship of the form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = R_i,\ (i = 1, 2, 3, ..., N-1) \tag{15}$$

where

$$\begin{array}{l} E_i = \frac{\sigma A_i}{h} \\ F_i = 1 + \frac{\sigma(A_{i+1}+A_i)}{h} - \sigma A'_i - \frac{h\sigma A''_i}{2} - \frac{hC_i}{2} \\ G_i = 1 + \frac{\sigma A_{i+1}}{h} - \sigma A'_{i+1} + \frac{h\sigma A''_{i+1}}{2} + \frac{hC_{i+1}}{2} \\ R_i = \frac{h}{2}\left[H_i + H_{i+1}\right] \end{array} \tag{16}$$

Equation (15) gives a system of $(N-1)$ equations with $(N-1)$ unknowns y_1 to y_{N-1}. These $(N-1)$ equations together with the equation (2) are sufficient to solve the tri-diagonal system by using Thomas Algorithm also called 'Discrete Invariant Imbedding algorithm'.

2.2 Description of the method for right-end boundary layer problems

Now, integrating equation (3) in $[x_{i-1}, x_i]$ and rearranging we obtain:

$$\begin{array}{l} A(x_i)y'(x_i) - A(x_{i-1})y'(x_{i-1}) - A'(x_i)y(x_i) + \\ A'(x_{i-1})y(x_{i-1}) + \int_{x_{i-1}}^{x_i} A''(x)y(x)dx + y(x_i) \\ -y(x_{i-1}) + \int_{x_{i-1}}^{x_i} C(x)y(x)dx = \int_{x_{i-1}}^{x_i} H(x)dx \end{array} \tag{17}$$

Evaluate the integrals in equation (17) using trapezoidal rule of Integration to get:

$$\begin{array}{l} A(x_i)y'(x_i) - A(x_{i-1})y'(x_{i-1}) - A'(x_i)y(x_i) + \\ A'(x_{i-1})y(x_{i-1}) + y(x_i) + \frac{h}{2}\left[A''(x_{i-1})y(x_{i-1}) + \right. \\ \left. A''(x_i)y(x_i)\right] - y(x_{i-1}) + \frac{h}{2}\left[C(x_{i-1})y(x_{i-1}) + \right. \\ \left. C(x_i)y(x_i)\right] = \frac{h}{2}\left[H(x_{i-1}) + H(x_i)\right] \end{array} \tag{18}$$

Using the finite difference approximations of first derivatives: $y'(x_i) = \frac{(y(x_{i+1})-y(x_i))}{h}$ and $y'(x_{i-1}) = \frac{(y(x_i)-y(x_{i-1}))}{h}$ and the following notations: $y(x_i) = y_i$, $y(x_{i-1}) = y_{i-1}$, $A(x_i) = A_i, C(x_i) = C_i$, $\alpha(x_i) = \alpha_i$ etc. in equation (18), we obtain the scheme:

$$\begin{array}{l} \frac{1}{h}\left[A_i y_{i+1} - A_i y_i - A_{i-1}y_i + A_{i-1}y_{i-1}\right] - \\ A'_i y_i + A'_{i-1}y_{i-1} + \frac{h}{2}\left[A''_{i-1}y_{i-1} + A''_i y_i\right] + y_i - \\ y_{i-1} + \frac{h}{2}\left[C_{i-1}y_{i-1} + C_i y_i\right] = \frac{h}{2}\left[H_{i-1} + H_i\right] \end{array} \tag{19}$$

noindent Introducing the fitting factor $\sigma(\rho)$ into equation (19), we obtain

$$\begin{array}{l} \frac{\sigma(\rho)}{h}\left[A_i y_{i+1} - A_i y_i - A_{i-1}y_i + A_{i-1}y_{i-1} + \right. \\ \left. h\left[A'_{i-1}y_{i-1} - A'_i y_i\right] + \frac{h^2}{2}\left[A''_{i-1}y_{i-1} + A''_i y_i\right]\right] + \\ y_i - y_{i-1} + \frac{h}{2}\left[C_{i-1}y_{i-1} + C_i y_i\right] = \frac{h}{2}\left[H_{i-1} + H_i\right] \end{array} \tag{20}$$

$y_{x_0} = \eta$; $y_{x_N} = \delta$ And, the fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of equation (20) converges uniformly to the solution equations (1) and (2). Now, taking limits as $h \to 0$ we obtain

$$\begin{aligned}&\lim_{h\to 0} \sigma(\rho)A(ih)\left[y(ih+h) - 2y(ih) + y(ih-h)\right]\\ &+\lim_{h\to 0}\left[y(ih) - y(ih-h)\right] = 0\end{aligned} \tag{21}$$

where $\left[A'_{i-1}y_{i-1} - A'_iy_i\right] + \frac{1}{2}\left[A''_{i-1}y_{i-1} + A''_iy_i\right] + \frac{1}{2}\left[C_{i-1}y_{i-1} + C_iy_i\right]$ and $\frac{1}{2}\left[H_i + H_{i-1}\right]$ are bounded functions.

It is well known that the solutions of equations (1) with (2) is of the following form (cf.[10], pp.22-26):

$$\begin{aligned}&y(x) = y_0(x) + \tfrac{\alpha(1)}{\alpha(x)}\left(\alpha - y_0(1)\right)e^{\int_x^1\left(\frac{\alpha(x)}{\varepsilon} - \frac{\beta(x)}{\alpha(x)}\right)dx}\\ &+o(\varepsilon)\end{aligned} \tag{22}$$

where $y_0(x)$ represents the solution of the reduced problem:

$$\alpha(x)y_0'(x) + \beta(x)y_0(x) = \gamma(x);\ y_0(1) = \eta \tag{23}$$

Expanding $\alpha(x)$ and $\beta(x)$ in equation (21) with the help of the Taylor's series about the point $'x = 1'$ and restricting to their first terms, we obtain:

$$y(x) = y_0(x) + \left(\gamma - y_0(1)\right)e^{\left(\frac{\alpha(1)}{\varepsilon} - \frac{\beta(1)}{\alpha(1)}\right)(1-x)} + o(\varepsilon) \tag{24}$$

Further, considering equation (23) at the point $x = x_i = ih,\ i = 0, 1, 2, ..., N$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} y(ih) = y_0(0) + \left(\gamma - y_0(1)\right)e^{\left(\frac{\alpha^2(1) - \varepsilon\beta(1)}{\alpha(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} + o(\varepsilon) \tag{25}$$

where $\rho = h/\varepsilon$.

Using the equation (25) for $y(ih-h),\ y(ih),\ y(ih+h)$ in equation (21) and then simplifying, we get the value of the fitting factor as

$$\sigma(\rho) = \frac{\alpha(1)\rho}{2}\left[\frac{e^{-\left(\frac{\alpha^2(1)-\varepsilon\beta(1)}{\alpha(1)}\right)\frac{\rho}{2}}}{\sinh\left(\left(\frac{\alpha^2(1)-\varepsilon\beta(1)}{\alpha(1)}\right)\frac{\rho}{2}\right)}\right] \tag{26}$$

Finally, by making use of equation (26) and $\sigma(\rho)$ given by equation (20), we can get the following three-term recurrence relationship of the form:

$$E_iy_{i-1} - F_iy_i + G_iy_{i+1} = R_i,\ (i = 1, 2, 3, ..., N-1) \tag{27}$$

where

$$\begin{aligned}&E_i = -1 + \tfrac{\sigma A_{i-1}}{h} + \sigma A'_{i-1} + \tfrac{h\sigma A''_{i+1}}{2} + \tfrac{hC_{i-1}}{2}\\ &F_i = -1 + \tfrac{\sigma(A_{i-1}+A_i)}{h} + \sigma A'_i - \tfrac{h\sigma A''_i}{2} - \tfrac{hC_i}{2}\\ &G_i = \tfrac{\sigma A_i}{h}\\ &R_i = \tfrac{h}{2}\left[H_i + H_{i+1}\right]\end{aligned}$$

Equation (27) gives a system of $(N-1)$ equations with $(N-1)$ unknowns y_1 to y_{N-1}. These $(N-1)$ equations together with the equation (2) are sufficient to solve the tri-diagonal system by using Thomas Algorithm also called 'Discrete Invariant Imbedding algorithm'.

3 Convergence

In this section, we discuss the convergence analysis of the method. Writing the tri-diagonal system of equation (27) in matrix-vector form[28], we get

$$DY = M \tag{28}$$

where $D = (u_{i,j}),\ 1 \le i, j \le N-1$ is a tri-diagonal matrix of order $N-1$, with

$$\begin{aligned}&u_{i,i+1} = -\left[\sigma A_{i+1} + h - h\sigma A'_{i+1} + \tfrac{h^2C_{i+1}}{2} + \tfrac{h^2\sigma A''_{i+1}}{2}\right]\\ &u_{i,i} = \left[\sigma A_{i+1} + \sigma A_i + h - h\sigma A'_i - \tfrac{h^2C_i}{2} - \tfrac{h^2\sigma A''_i}{2}\right]\\ &u_{i,i-1} = -\left[\sigma A_i\right]\end{aligned}$$

and $M = (d_i)$ is a column vector with $d_i = -\frac{h^2}{2}\left[H_i + H_{i+1}\right]$, where $i = 1, 2, ..., N-1$ with local truncation error:

$$\tau_i(h) = \frac{h^2}{2}\left[C_i'y_i + C_iy_i' + y_i'' - H_i'\right] + o(h^3 \tag{29}$$

we also have

$$D\overline{Y} - \tau(h) = M \tag{30}$$

where $\overline{Y} = \left(\overline{Y_0}, \overline{Y_1}, \overline{Y_2}, ..., \overline{Y_N}\right)^t$ denotes the actual solution and $\tau(h) = \left(\tau_1(h), \tau_2(h), ..., \tau_N(h)\right)^t$ is the local truncation error. From the equations (28) and (30), we have

$$D\left(\overline{Y} - Y\right) = \tau(h) \tag{31}$$

Thus, we obtained the error equation is

$$DE = \tau(h) \tag{32}$$

where $E = \overline{Y} - Y = \left(e_0, e_1, e_2, ..., e_N\right)^t$.

Let Y_i be the sum of elements of i^{th} row of D, then we have

$$\begin{aligned}&Y_1 = \textstyle\sum_{i=1}^{N-1} u_{1,j} = \sigma A_1\\ &-h\left(\sigma A_1' - \sigma A_2'\right) - \tfrac{h^2}{2}\left(C_1 + C_2 + \sigma A_1'' + \sigma A_2''\right)\end{aligned}$$

$$\begin{aligned}&Y_{N-1} = \textstyle\sum_{j=1}^{N-1} u_{N-1,j} = \sigma A_N\\ &+h\left(1 - \sigma A'_{N-1}\right) - \tfrac{h^2}{2}\left(C_{N-1} + \sigma A''_{N-1}\right)\end{aligned}$$

$$\begin{aligned}&Y_i = \textstyle\sum_{j=1}^{N-1} u_{i,j} = h\left[\sigma A'_{i+1} - \sigma A'_i\right] - \tfrac{h^2}{2}\left[C_i + \right.\\ &\left.C_{i+1} + \sigma A''_i + \sigma A''_{i+1}\right] = hV_i + o(h^2);\ i = 2(1)N-2\end{aligned}$$

where $V_i = \left[\sigma A'_{i+1} - \sigma A'_i\right]$.

Since $0 < \varepsilon << 1$; for sufficiently small h the matrix D is irreducible and mono-tone. Hence D^{-1} exists and $D^{-1} \ge 0$. Hence, from the error equation (32) we have

$$E = D^{-1}\tau(h) \tag{33}$$

$$\|E\| \le \left\|D^{-1}\right\|\,\|\tau(h)\|$$

Let $u_{k,i}$ be the $(k,i)^{th}$ elements of D^{-1}: Since $\overline{u}_{k,i} \ge 0$, from the theory of matrices we have,

$$\sum_{i=1}^{N-1} \overline{u}_{k,i}Y_i = 1;\ k = 1, 2, ..., N-1 \tag{34}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \overline{u}_{k,i} \leq \frac{1}{\min\limits_{0\leq i\leq N-1} Y_i} = \frac{1}{hV_{i_0}} \leq \frac{1}{h\left\|V_{i_0}\right\|}$$

for some i_0 lies between 1and $N-1$.
Now, we define $\left\|D^{-1}\right\| = \max\limits_{0\leq i\leq N-1} \sum_{i=1}^{N-1}\left|\overline{u}_{k,i}\right|$ and $\|\tau(h)\| = \max\limits_{0\leq i\leq N-1} |\tau(h)|$.
Therefore, from the equations (29), (33) and (34), we obtain

$$e_j = \sum_{i=1}^{N-1} \overline{u}_{k,i}\tau_i(h);\ j = 1(1)N-1$$

and therefore

$$|e_j| \leq \frac{kh^2}{h\left|V_{i_0}\right|};\ j = 1(1)N-1 \tag{35}$$

where $k = \left[\frac{C_i'}{2}|y_i| + \frac{C}{2}|y_i'| + \frac{1}{2}|y_i''| - \frac{H_i'}{2}\right]$ is constant independent of h.
Therefore, using the definitions and equation (35), we have

$$\|E\| = o(h)$$

This implies that the scheme (15) derived for the solution of left layer problems is of first order convergence on uniform mesh.

4 Numerical illustrations

In this section, the numerical results of some test problems are chosen.

4.1 Numerical example problems with left-end boundary layer

To demonstrate the applicability of proposed method computationally for left-end boundary layer problems, we have considered the following one linear and one non-linear model test problems:
Example 01 : Consider the following constant coefficient non-homogeneous singular perturbation problem from [8]:

$\varepsilon y''(x) + y'(x) = 1 + 2x;\ x \in [0,1]$
with boundary conditions $y(0) = 0$ and $y(1) = 1$.

The exact solution of this example is given by: $y(x) = x\,(x + 1 - 2\varepsilon) + \frac{(2\varepsilon-1)\left(1-e^{-x/\varepsilon}\right)}{\left(1-e^{-1/\varepsilon}\right)}$ which has a boundary layer at the left side of the domain near $x = 0$. Clearly, the MAE presented in Table-3 for problem-1 show that the present scheme is capable of producing uniformly convergent solution in case when ε tends to zero for any fixed value of the step size $h = 1/N$. The comparison in MAE for Problem 1 with the existing methods in [16,17,18] for various values of ε and grid point N is presented in Tables 1 and 2.
Example 02 : Consider the following non-linear singular perturbation problem from ([11], p. 463, Eq.(9.7.1))

$\varepsilon y''(x) + y'(x) + e^{(y(x))} = 0;\ x \in [0,\ 1]$
with boundary conditions $y(0) = 0$ and $y(1) = 0$.

The linear problem concerned to this example is:

$$\varepsilon y''(x) + 2y'(x) + \tfrac{2}{x+1}y(x) = \tfrac{2}{x+1}\left[\ln\left(\tfrac{2}{x+1}\right) - 1\right];\quad x \in [0,\ 1]$$

The uniform valid approximation of Bender and Orszag ([11], P. 463, Eq. (9.7.6)) is $y(x) = \ln\left(\frac{2}{x+1}\right) - \ln(2)e^{-2x/\varepsilon}$, which has a boundary layer of thickness $o(\varepsilon)$ near $x = 0$. Clearly, the MAE presented in Table-5 for problem-2 show that the present scheme is capable of producing uniformly convergent solution in case when ε tends to zero for any fixed value of the step size $h = 1/N$. The comparison in MAE for Problem 1 with the existing methods[29,17] for various values of ε and grid point N is presented in Tables 4.

4.2 Numerical example problems with right-end boundary layer

To demonstrate the applicability of proposed method computationally for right-end boundary layer problems, we have considered the following one linear model test problem:
Example 03 : Consider the following homogeneous singular perturbation problem from [16,17,18]:

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0;\ x \in [0,\ 1]$$

with boundary conditions $y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$ and $y(1) = 1 + 1/e$.
The exact solution is given by: $y(x) = \exp(-x) + \exp\left[(1+\varepsilon)(x-1)/\varepsilon\right]$, which has a boundary layer at the right side of the domain near $x = 1$. Clearly, the MAE presented in Table-8 for problem-3 show that the present scheme is capable of producing uniformly convergent solution in case when ε tends to zero for any fixed value of the step size $h = 1/N$. The comparison in MAE for Problem 1 with the existing methods in [16,17,18]:for various values of ε and grid point N is presented in Tables 6 and 7.

5 Conclusion

We have derived an exponentially fitted tri-diagonal scheme for solving singularly perturbed two-point boundary value problems with boundary layer at one end points(left or right). Derived scheme is applied on four standard model example problems for different values of $N = 1/h$ and perturbation parameter ε. Computational results are presented in tables and compared with the existing

Table 1: Comparison of computational results (MAE) with existing results for various values of ε and N for example problem-1.

	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
$N\downarrow$	Mohapatra and Mahalik[16]	Mohapatra and Reddy[17]	Our Result	Mohapatra and Mahalik[16]	Mohapatra and Reddy[17]	Our Result
16	5.8414e-2	5.8414e−2	1.8746E-04	5.8591e-2	5.8591e-2	0.0000E+00
32	3.0087e-2	3.0087e−2	1.9372E-04	3.0274e-2	3.0274e-2	0.0000E+00
64	1.5169e-3	1.5169e−3	1.9687E-04	1.5389e-2	1.5389e-2	0.0000E+00
128	7.5535e-3	7.5535e−3	1.9842E-04	7.7512e-3	7.7512e-3	0.0000E+00
256	3.6920e-3	3.6920e−3	1.9920E-04	3.8923e-3	3.8923e-3	0.0000E+00
512	1.7547e-3	1.7547e−3	1.9956E-04	1.9497e-3	1.9497e-3	0.0000E+00

Table 2: Comparison of computational results(MAE) with existing results [with fitting factor(w.f.f.) and without fitting factor(w.o.f.f.)] for various values of ε and N for example problem-1.

	$\varepsilon = 10^{-3}$				$\varepsilon = 10^{-5}$			
	SOUJANYA et.al. [18]		Our Result		SOUJANYA et.al. [18]		Our Result	
$N\downarrow$	with f.f	w.o. f.f	with f.f	w.o. f.f	with f.f	w.o. f.f	with f.f	without f.f
8	1.07e-001	15.41	1.75E-03	7.92E-03	1.09e-001	1.56(+3)	1.75E-05	8.00E-05
16	5.67e-002	4.043	1.88E-03	1.57E-02	5.85e-002	390.49	1.88E-05	1.60E-04
32	2.83e-002	1.8208	1.94E-03	3.10E-02	3.02e-002	97.60	1.94E-05	3.20E-04
64	1.34e-002	1.5446	1.97E-03	6.00E-02	1.53e-002	24.434	1.97E-05	6.39E-04
128	6.17e-003	1.1839	1.98E-03	1.13E-01	7.73e-003	6.2991	1.99E-05	1.28E-03

Table 3: Computational results in terms of Maximum absolute errors for different values of N and ε for example problem- 1

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	1.750E-04	1.875E-04	1.937E-04	1.969E-04	1.984E-04	1.992E-04	1.996E-04
10^{-6}	1.788E-06	1.907E-06	1.967E-06	1.967E-06	2.027E-06	2.027E-06	2.027E-06
10^{-8}	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10^{-10}	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10^{-15}	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10^{-25}	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
10^{-30}	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

Table 4: Comparison of computational results(MAE) with existing results for various values of ε and N for example problem-2.

	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
$N\downarrow$	Reddy and Mohapatra[29]	Mohapatra and Reddy[17]	Our Result	Reddy and Mohapatra[29]	Mohapatra and Reddy[17]	Our Result
16	1.962e-2	1.9628e-2	2.7067E-04	1.962e-2	1.9623e-2	2.7067E-04
32	1.031e-2	1.0315e-2	7.4089E-05	1.031e-2	1.0311e-2	7.4089E-05
64	5.284e-3	5.2847e-3	1.9491E-05	5.284e-3	5.2842e-3	1.9491E-05
128	2.675e-3	2.6759e-3	5.0664E-06	2.675e-3	2.6755e-3	5.0664E-06
256	1.344e-3	1.3444e-3	1.6689E-06	1.344e-3	1.3440e-3	1.6689E-06
512	6.754e-4	6.7549e-4	2.3842E-07	6.754e-4	6.7547e-4	2.3842E-07

Table 5: Computational results in terms of Maximum absolute errors for different values of N and ε for example problem- 2

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-6}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-8}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-10}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-15}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-25}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07
10^{-30}	9.033E-04	2.707E-04	7.409E-05	1.949E-05	5.066E-06	1.669E-06	2.384E-07

Table 6: Comparison of computational results (MAE) with existing results for various values of ε and N for example problem-3.

	$\varepsilon=10^{-4}$			$\varepsilon=10^{-8}$		
$N\downarrow$	Mohapatra and Mahalik[16]	Mohapatra and Reddy[17]	Our Result	Mohapatra and Mahalik[16]	Mohapatra and Reddy[17]	Our Result
16	1.1143e-2	1.1143e-2	1.5593E-04	1.1141e-2	1.1141e-2	1.1960E-04
32	5.6345e-3	5.6345e-2	6.6429E-05	5.6343e-3	5.6343e-2	2.9892E-05
64	2.8197e-3	2.8197e-3	4.5240E-05	2.8192e-3	2.8192e-3	7.5400E-06
128	1.3958e-3	1.3958e-3	3.8445E-05	1.3955e-3	1.3955e-3	1.6689E-06
256	6.8346e-4	6.8346e-4	3.9727E-05	6.8342e-4	6.8342e-4	5.3644E-07
512	3.2758e-4	3.2758e-4	4.4465E-05	3.2754e-4	3.2754e-4	3.2783E-07

Table 7: Comparison of computational results(MAE) with existing results [with fitting factor(w.f.f.) and without fitting factor(w.o.f.f.)] for various values of ε and N for example problem-3.

	$\varepsilon=10^{-3}$				$\varepsilon=10^{-5}$			
	SOUJANYA et.al.[18]		Our Result		SOUJANYA et.al.[18]		Our Result	
$N\downarrow$	with f.f	w.o. f.f	with f.f	w.o. f.f	with f.f	w.o. f.f	with f.f	without f.f
8	2.02e-02	1.23e+00	8.41E-04	6.97E-03	2.06e-02	1.39e+00	4.79E-04	4.62E-04
16	1.06e-02	1.04e+00	4.87E-04	1.51E-02	1.10e-02	1.42e+00	1.23E-04	1.19E-04
32	5.27e-03	8.67e-01	3.97E-04	3.05E-02	5.63e-03	1.42e+00	3.35E-05	2.85E-04
64	2.48e-03	7.61e-01	3.74E-04	5.97E-02	2.84e-03	1.36e+00	1.17E-05	6.27E-04
128	1.06e-03	5.89e-01	3.68E-04	1.13E-01	1.42e-03	1.20e+00	4.71E-06	1.27E-03

Table 8: Computational results in terms of Maximum absolute errors for different values of N and ε for example problem- 3

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	5.122E-04	1.559E-04	6.643E-05	4.524E-05	3.845E-05	3.973E-05	4.447E-05
10^{-6}	4.763E-04	1.200E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07
10^{-8}	4.758E-04	1.196E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07
10^{-10}	4.758E-04	1.196E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07
10^{-15}	4.758E-04	1.196E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07
10^{-25}	4.758E-04	1.196E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07
10^{-30}	4.758E-04	1.196E-04	2.989E-05	7.540E-06	1.669E-06	5.364E-07	3.278E-07

results. Comparisons show that the proposed scheme is better than the schemes presented in the articles[16,17,29,18]. One can easily observed from these Tables:2,3,6 and 9 that the presented fitted scheme is capable of producing highly accurate uniformly convergent solution for any fixed value of step size $N = 1/h > \varepsilon$, when perturbation parameter $\varepsilon \to 0$. The main feature of the proposed fitted scheme is that it does not depends on the very fine mess size.

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