Planar of special idealization rings

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Abstract: Let $\mathbf{R}(+)N$ be the idealization of the ring **R** by the **R**-module N. In this paper, we investigate when $\Gamma(\mathbf{R}(+)\mathbf{N})$ is a Planar graph where **R** is an integral domain and we investigate when $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ is a Planar graph.

Key-Words: The idealization rings **R**, Planar graph, Zero-divisor graph.

Received: July 27, 2020. Revised: November 23, 2020. Accepted: December 10, 2020. Published: December 23, 2020.

1 Introduction

I. Beck in [6] introduce the concept of zero-divisor graph also, D. D. Anderson and M. Naseer in [3] studied the context of coloring which is an interest concept of graph theory. Anderson and Livingston in [4, $Theorem$ 2.3] give the definition of the zero-divisor graph. For more information in zero-divisor graph see, [5].

/Let **R** be a commutative ring, the zero-divisor graph is the graph $\Gamma(\mathbf{R})$ which vertices are the non-zero zero divisors of **R**, with a and b adjacent if $a = b$ and $a.b = 0$.
For each ring **R**, the set of all zero-divisors of the ring **R** is $Z(\mathbf{R})$.

The idealization ring $\mathbf{R}(+)N$ is defined as **N** be an **R**-module and let $\mathbf{R}(+)\mathbf{N} = \{(a_1, h_1) : a_1 \in \mathbf{R},\ h_1 \in \mathbf{N}\}$ we have two operations $(a_1,\ h_1) + (a_2, h_2) = (a_1 + a_2,\ h_1 + h_2)$ and $(a_1, h_1)(a_2, h_2) = (a_1a_2,\ a1h_2 + a_2h_1)$.

Another concept of interest in the graph theory. The Planar graph is a graph isomorphic to a Plane graph. A Plane graph is graph that can be drawn on the plane without cross edging. If the graph has induced subgraph isomorphic to K_5 that is not a Planar graph, by Kuratoskies Theorem.

2 When $\Gamma(\mathbf{R}(+)\mathbf{N})$ is a Planar graph?

In this section, we investigate when $\Gamma(\mathbf{R}(+)\mathbf{N})$ is Planar graph where **R** is an integral domain and **N** be an **R**-module.

We begin with the following lemma when **R** is an integral domain for the idealization ring $\mathbf{R}(+)\mathbf{N}$.

Lemma 1:
[2] Suppose that **R** is an integral domain and **N** is an **R**−module. Then we have the following cases:

- **Case 1.** If **R** is an integral domain with $\mathbf{N} \cong \mathbf{Z}_2$ is an **R**−module and annihilator of $\mathbf{Z}_2$ is equal to zero, then the integral domain **R** is $\mathbf{R} \cong \mathbf{Z}_2$.
- **Case 2.** If **R** be an integral domain with $\mathbf{N} \cong \mathbf{Z}_3$ is an **R**−module and annihilator of $\mathbf{Z}_3$ is equal to zero, then the integral domain **R** is $\mathbf{R} \cong \mathbf{Z}_3$.

Theorem 1:
Suppose that **R** is an integral domain and $\mathbf{N} \cong \mathbf{Z}_2$ is an **R**−module. Then the graph $\Gamma(\mathbf{R}(+)\mathbf{Z}_2)$ is a Planar.
Proof:
To proof we have the following two cases to thoughtfulness:

- **Case 1**: If the annihilator of $\mathbf{Z}_2$ is equal to zero, then $\Gamma(\mathbf{Z}_2(+)\mathbf{Z}_2)$ is equal to $\{(0, 1)\}$ which is a Planar graph.
- **Case 2**: If the annihilator of $\mathbf{Z}_2$ is not equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{Z}_2) = \{(0, 1), (k_i, 0), (k_j, 1) : k_i,\ k_j \in ann(\mathbf{Z}_2)\}$. So, the graph $\Gamma(\mathbf{R}(+)\mathbf{Z}_2)$ is a star which is a Planar graph.

Theorem 2:
Suppose that **R** is an integral domain and $\mathbf{N} \cong \mathbf{Z}_3$ is an **R**−module. Then the graph $\Gamma(\mathbf{R}(+)\mathbf{Z}_3)$ is a

Planar.

Proof:

To proof we must note the following two cases to thoughtfulness:

- **Case 1**: If annihilator of $\mathbf{Z}_3$ is equal zero, then $\Gamma(\mathbf{R}(+)\mathbf{Z}_3)$ is equal to $\{(0,1),(0,2)\}$ that is a Planar graph.
- **Case 2**: If annihilator of $\mathbf{Z}_3$ is not equal zero, then graph $\Gamma(\mathbf{R}(+)\mathbf{Z}_3)$ is equal to $\{(0,1),(0,2),(r_i,0),(r_i,1),(r_i,2) \ : \ r_i \in ann(\mathbf{Z}_3)\}$. So, that is a Planar graph.

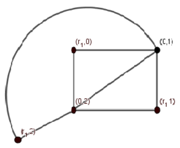

Figure 1: A graph which is a Planar graph.

We begin with the following lemma can be found in [7] to discus the case **N** of order 4.

Lemma 2:

If the graph **G** is a 3-connected planar, then there is a cycle through any five vertices of the graph **G**.

Theorem 3:

Suppose that **R** is an integral domain and $|\mathbf{N}| = 4$ is an $\mathbf{R}-$module. Then we have the following cases:

- **Case 1.** If the order of **N** is equal 4 and annihilator of N is equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is a Planar.
- **Case 2.** If the order of **N** is equal 4 and annihilator of N is not equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is not a Planar.

Proof:

To proof must note two cases to thoughtfulness:

- **Case 1.** If the order of **N** is equal 4 and annihilator of N is equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is equal to $\{(0,l_1),(0,l_2),(0,l_3) \ : \ l_i \in \mathbf{N}\}$. That is a Planar graph.
- **Case 2.** If the order of **N** is equal 4 and annihilator of N is not equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N}) = \{(r_i,l_i),(0,l_1),(0,l_2),(0,l_3) \ : \ l_i \in \mathbf{N}, \ r_i \in ann(\mathbf{N})\}$, by previous lemma then the graph is not a Planar graph.

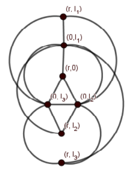

Figure 2: A graph which is not a Planar graph.

The next theorem will discuss when the order of **N** is greater than or equal 5.

Theorem 4:

Suppose that **R** is an integral domain and $|\mathbf{N}| \geq 5$ is an $\mathbf{R}-$module. Then we have the following cases:

- **Case 1.** If the order of **N** is equal to 5 and annihilator of N is equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is a Planar.
- **Case 2.** If the order of **N** is equal to 5 and annihilator of N is not equal to zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is not a Planar.
- **Case 3.** If the order of **N** is greater than 5 , then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is not a Planar.

Proof:

To proof must note two cases to thoughtfulness:

- **Case 1.** If the order of **N** is equal 5 and annihilator of N is equal zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is equal to $\{(0,l_1),(0,l_2),(0,l_3),(0,l_4) \ : \ l_i \in \mathbf{N}\}$. That is a Planar graph.
- **Case 2.** If the order of **N** is equal 5 and annihilator of N is not equal zero, then the graph $\Gamma(\mathbf{R}(+)\mathbf{N}) = \{(r_i,l_i),(0,l_1),(0,l_2),(0,l_3),(0,l_4) \ : \ l_i \in \mathbf{N}, \ r_i \in ann(\mathbf{N})\}$ has an induced subgraph isomorphic to K_5. That is not a Planar graph.

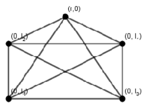

Figure 3: A graph which is not a Planar graph.

- **Case 3.** If the order of **N** is greater than 5, then graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is equal to $\{(0,l_1),(0,l_2),(0,l_3),(0,l_4),(0,l_5),...,(0,l_i) : l_i \in \mathbf{N}\}$. That has an induced subgraph isomorphic to K_5. So, the graph is not a Planar.

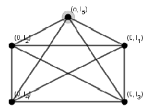

Figure 4: A graph which is not a Planar graph.

3 When $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ is Planar grah?

In this section, we consider the planar for the zero-divisor graph of the idealization ring $\mathbf{Z}_n(+)\mathbf{Z}_m$, $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ where $\mathbf{Z}_m$ be $\mathbf{Z}_n$-module.

Al-Labdi [1], she classified the zero-divisor graph of the idealization ring $\mathbf{Z}_n(+)\mathbf{Z}_m$.

We begin with the following lemma, when n is a prime number such that $n = p^\alpha$ and $m = p$.

Lemma 3:
Let $n = p^\alpha$ and $m = p$ where p is a prime number. Then the graph $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ have the following cases:

Case 1: If n is equal 4 and m is equal 2, then the graph $\Gamma(\mathbf{Z}_4(+)\mathbf{Z}_2)$ is a Planar.

Case 2: If n is equal p^α and m is equal p where p is a prime number, $\alpha \geq 3$, then the graph $\Gamma(\mathbf{Z}_{p^\alpha}(+)\mathbf{Z}_p)$ is not a Planar.

Proof:
We consider two cases to proof:

Case 1: If n is equal 4 and m is equal 2, then graph $\Gamma(\mathbf{Z}_4(+)\mathbf{Z}_2)$ is equal to $\{(0,1),(2,0),(2,1)\}$. So, that the graph is a Planar.
Case 2: If n is equal p^α and m is equal p where p is a prime number greater than 2, $\alpha \geq 3$, then the graph $\Gamma(\mathbf{Z}_{p^\alpha}(+)\mathbf{Z}_p)$ is equal $\{(0,1),(0,2),...,(0,p-1),(kp,0),...,(kp,p-1) : k \in\}$. So, it has an induced subgraph K_5 that is not a Planar graph.

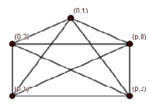

Figure 5: A graph which is not a Planar graph.

Theorem 5:
Let m is a product of powers of prime numbers $m = {p_1}^{k_1} \times {p_2}^{k_2} \times ... \times {p_l}^{k_l}$ and n is product power of primes $n = {p_1}^{s_1} \times {p_2}^{s_2} \times ... \times {p_r}^{s_r}$ where p_i is a prime number and $l \leq r$. Then the graph $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ is not a Planar graph.
Proof
We consider two cases to proof:
If m is product power of primes $m = {p_1}^{k_1} \times {p_2}^{k_2} \times ... \times {p_l}^{k_l}$ and n is product power of primes $n = {p_1}^{s_1} \times {p_2}^{s_2} \times ... \times {p_r}^{s_r}$ where p_i is a prime number and $l \leq r$. Then the graph $\Gamma(Z_{{p_1}^{s_1} \times {p_2}^{s_2} \times ... \times {p_r}^{s_r}} \; (+)Z_{{p_1}^{k_1} \times {p_2}^{k_2} \times ... \times {p_l}^{k_l}})$ is equal to $\{(0,h_i),(b_i,h_i) : b_i \in n \; h_i \in m\}$ such that $gcd(b_i,n) \neq 1$ or $gcd(b_i,m) \neq 1$. So, it has an induced subgraph K_5 that is not a Planar graph.

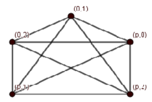

Figure 6: A graph which is not a Planar graph.

4 Outcome and questions

In this article, we classify the planarity for the graph of idealization $\Gamma(\mathbf{R}(+)\mathbf{N})$, we conclude in the following theorem.

Theorem 6:
Let $\mathbf{R}(+)\mathbf{N}$ be an idealization ring. Then the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ is a Planar graph if the ring $\mathbf{R}$ is an integral domain and the order of $\mathbf{N}$ is less than or equal 4 with $ann(\mathbf{N}) = 0$, or the order of $\mathbf{N}$ is equal to 5 with $ann(\mathbf{N}) = 0$ and the graph $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$ is a Planar when $n = 4$, $m = 2$.

One can ask the following questions:

(1) When the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$ are Eulerian graph?
(2) When the complement graph of idealization ring $\Gamma(\mathbf{R}(+)\mathbf{N})$ are Planar graph?
(3) What is the matching number of the graph $\Gamma(\mathbf{R}(+)\mathbf{N})$?

Possible engineering applications of this study can be found in problems of [8] and [9].

References:

[1] M. Allabadi M, Futher results on the diameter of zero-divisor graphs of some special idealizations, *International Journal of Algebra*, Vol. 12 (2010), pp. 609-614.

[2] M. Allabadi, On the Diameter of Zero-Divisor Graphs of Idealizations with Respect to Integral Domain, *Jordan Journal of Mathematics and Statistics*, Vol. 3 (2010), pp. 127-131.

[3] DD. Anderson, M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* Vol.159 (1993), pp. 500-514.

[4] DF. Anderson, PS. Livingston, The zero-divisor graph of a commutative, *J. Algebra*, Vol.217 (1999), pp. 434-447.

[5] M. Axtell, J. Stickle, The zero-divisor graph of a commutative rings, textitJornal of Pure and Applied Algebra, Vol.204 (2006), pp. 235-243.

[6] I.Beck, Coloring of a commutative ring, *J. Algebra*, Vol. 116 (1988), pp. 208-226.

[7] B. Jackson, Longest cycles in 3-connected cubic, *J. Combin. Theory Ser B*, Vol. 41 (1986), pp. 17-26.

[8] N. Boonsim, Racing Bib Number Localization on Complex Backgrounds, *WSEAS Transactions on Systems and Control*, Vol.13 (2018), pp. 226-231.

[9] T. Ashkan Tashk, H.Jurgen, Esmaeil Nadimi, Automatic Segmentation of Colorectal Polyps based on a Novel and Innovative Convolutional Neural Network Approach, WSEAS Transactions on Systems and Control, Vpl.14 (2019), pp. 384-391.