# The number of solutions to the boundary value problem with linear-quintic and linear-cubic-quintic nonlinearity

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*Abstract:* The nonlinear oscillators describing by differential equations of the form $x'' = -ax + cx^5$ and $x'' = -ax + bx^3 + cx^5$ are studied. Multiplicity results for both types of equations, given with the Neumann boundary conditions are obtained. It is shown that the number of solutions depend on the coefficien $a$ only. The exact estimates of the number of solutions are obtained. Practical issues, such as the representation of solutions in terms of Jacobian elliptic functions and calculation of the initial values for solutions of boundary value problems, are considered also. The illustrative examples are provided. Outlines of future research conclude the article.

*Key–Words:* Boundary value problem, quintic nonlinearity, cubic-quintic nonlinearity, phase trajectory, multiplicity of solutions, Jacobian elliptic function.

Received: June 4, 2020. Revised: October 21, 2020. Accepted: November 18, 2020. Published: December 8, 2020.

## 1 Introduction

Despite of the great number of papers devoted to nonlinear oscillators, there are still unsolved and not well-researched topics. An overview of results and issues can be found in [15].

In this article we will consider the second order differential equations of two types. The firs one is

$$x'' = -ax + cx^5, \quad a > 0,\ c > 0 \tag{1}$$

with the linear-quintic nonlinearity. The second equation is of the form

$$x'' = -ax + bx^3 + cx^5, \quad a > 0,\ b > 0,\ c > 0 \tag{2}$$

and it contains the linear-cubic-quintic nonlinearity. Both equations appear in applications. The list of areas where equations (1) and (2) arise can be found in the works [1], [2], [3], [16], [7], [6]. Equations of the form (1) and (2) appear in the theory of Schrödinger equation, dealing with wave propagation in fluids plasmas, and nonlinear optical media [19, subsection 5.1]. Making the traveling wave transformation in nonlinear reaction-diffusion equations also can lead to cubic-quintic equations of the form (2) [19, subsection 5.2].

We are motivated, on one hand, by a series of papers dealing with equations (1) and (2). In the work [1] formulas for a solution of the equation (1) were obtained using the Jacobi elliptic functions. On the other hand, we have obtained previously in the works [10], [11], [12], [13], [14] the results concerning the boundary value problems (BVP) for the equation

$$x'' = -ax + bx^3, \quad a > 0,\ b > 0. \tag{3}$$

In articles [10] and [12] we have obtained also, convenient for use, formulas for solutions of BVP (3)

$$x'(0) = 0, \quad x'(T) = 0. \tag{4}$$

Moreover, the equations for the initial conditions of solutions to the problem (3), (4), were derived. The initial conditions are needed for effective computation of solutions of BVP. More on the problems arising when findin periodic solutions can be found in [4]. The estimates of the number of solutions were obtained. The proof was conducted showing that these estimates are precise. These results were based on the monotonicity of periods of some solutions on the initial conditions. More details are given below.

In this paper we extend our results concerning the BVP (3), (4) to the similar problems for equations (1) and (2). Our aim is to get the estimates of the number of solutions to the problem (1), (4) and to the problem (2), (4). Our results are:

- the monotonicity of periods for solutions of the equations (1) and (2) is proved;
- the exact estimates of the number of solutions for the BVP (1), (4) and (2), (4) are obtained;

- the formulas for solutions of the initial value problem $x(0) = x_0, x'(0) = 0$ are given;
- the equations for the initial values of solutions of BVP (1), (4) and (2), (4) are derived;
- the examples are analyzed that show the validity of the above mentioned results and illustrate them.

A sample of the results follows.
Theorem 9: *Let $i$ be a positive integer such that $\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}$, where $T$ as in (4). The Neumann problem (1), (4) has exactly $2i$ nontrivial solutions.*

The structure of the paper is the following. In the next section (Section 2) we briefl describe previously obtained results on the Neumann problem for the cubic equation. In Section 3 and Section 4 we provide the main result on the number of solutions to the problem (1), (4) and to the problem (2), (4) accordingly. The formulas for solutions of (1), (4) and (2), (4) are obtained using theory of Jacobian elliptic functions ([17], [20], [8]) and using results in the papers [3], [1]. Then we are able to derive equations for determining of the values $x_\alpha$ which correspond to solutions of the BVP (1), (4) and (2), (4). Finally in Section 5 we demonstrate how all the developed technique and formulas work in a specifi example.

## 2 Review of the Neumann problem for cubic case

Consider the equation (3). There are three critical points of equation (3): $x_{1,3} = \pm\sqrt{\frac{a}{b}}$ both are saddle points, $x_2 = 0$ is a center. Two heteroclinic trajectories connect the two saddle points. $G_3$ is the open region bounded by the two heteroclinic trajectories connecting saddle points. The phase portrait of equation (3) is depicted in Fig. 1.

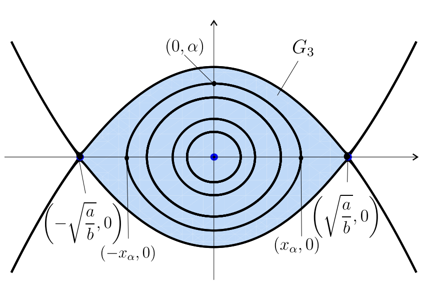


Figure 1: The phase portrait of equation (3), region $G_3$

Consider the Neumann boundary value problem (3), (4). In this section, we present the results about the exact number of solutions for this problem and using the theory of Jacobian elliptic functions, we recall expressions for solutions of the Cauchy problems (3),

$$x(0) = x_\alpha,\ x'(0) = 0,\ -\sqrt{\frac{a}{b}} < x_\alpha < \sqrt{\frac{a}{b}},\ x_\alpha \neq 0 \tag{5}$$

obtained in the articles [10], [12], [13]. We show how to fin the initial values $x_\alpha$ of solutions of the problem (3), (4). The following statement is true for the problem (3), (4).

**Theorem 1** *Let $i$ be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}} \tag{6}$$

*where $T$ is the end point of the interval $(0, T]$. The Neumann problem (3), (4) has exactly $2i$ nontrivial solutions such that $x(0) = x_\alpha \neq 0$, $x'(0) = 0$, $-\sqrt{\frac{a}{b}} < x_\alpha < \sqrt{\frac{a}{b}}$.*

The proof can be found in the articles [12], [13].
Consider the Cauchy problem (3), (5). Let $a$ and $T$ be given. We wish to fin $x_\alpha$ such that the respective solutions $x(t; x_\alpha)$ of the above problem satisfy the boundary condition $x'(T) = 0$, i.e. $x(t; x_\alpha)$ solve the Neumann problem (3), (4). The following assertion provides the explicit formula for the solution of (3), (4).

**Lemma 2** *[10], [13] The function*

$$x(t, a, b, x_\alpha) = x_\alpha cd\left(\sqrt{a - \frac{1}{2} b\, x_\alpha^2}\ t; k\right), \tag{7}$$

*where $k = \sqrt{\frac{b\,x_\alpha^2}{2a - b\,x_\alpha^2}}$ is a solution of the Cauchy problem (3), (5).*

The proof of Lemma 2 is given in article [13].

Denote $f(t, a, b, x_\alpha) = x'_t(t, a, b, x_\alpha)$. This derivative can be computed and the following formula

$$f(t, a, b, x_\alpha) = \frac{x_\alpha \sqrt{a - b\,x_\alpha^2}}{\sqrt{2}} (k^2 - 1) \times \\ \times nd\left(\sqrt{a - \tfrac{1}{2} b\,x_\alpha^2}\ t; k\right) sd\left(\sqrt{a - \tfrac{1}{2} b\,x_\alpha^2}\ t; k\right) \tag{8}$$

is valid.

**Lemma 3** *[13] The boundary value problem (3), (4) for the coefficient $a$ and $T$ given can be solved now by solving the equation with respect to $x_\alpha$*

$$f(T, a, b, x_\alpha) = 0. \tag{9}$$

**Theorem 4** *[13] A solution to the Neumann problem (3), (4) is given by (7), where $x_\alpha$ is a solution of (9).*

# 3 Main results for linear-quintic case

## 3.1 Multiplicity of solutions for linear-quintic case

Consider the equation (1). There are three critical points of equation (1): $x_{1,3} = \pm\sqrt[4]{\frac{a}{c}}$ both are saddle points, $x_2 = 0$ is a center. Two heteroclinic trajectories connect the two saddle points. $G5$ is an open region bounded by the two heteroclinic trajectories connecting saddle points. The phase portrait of equation (1) is depicted in Fig. 2.

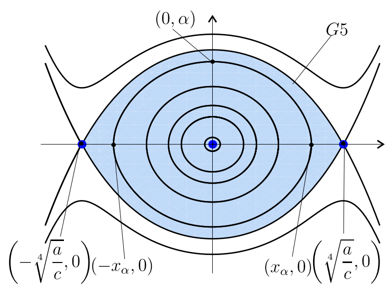


Figure 2: The phase portrait of equation (1), region $G5$

Consider trajectories (closed curves) that fil the region $G5$. The heteroclinic solution at infinit satisfie

$$\begin{aligned} 0 = x'^2(\infty) = -ax^2(\infty) + \tfrac{1}{3}\, c\, x^6(\infty) + C = \\ = -a\sqrt{\tfrac{a}{c}} + \tfrac{1}{3}\, c\left(\sqrt{\tfrac{a}{c}}\right)^3 + C = -\tfrac{2a\sqrt{a}}{3\sqrt{c}} + C, \end{aligned} \tag{10}$$

where $x^2(\infty) = \sqrt{\frac{a}{c}}$ and therefore the respective $C = \frac{2a}{3}\sqrt{\frac{a}{c}}$.

Then any trajectory located in the region $G5$ satisfie the relation (10), where $|C| < \frac{2a}{3}\sqrt{\frac{a}{c}}$.

It was proved in the papers [11], [13] that the period of a solution to the cubic equation (3) is increasing function. In a similar manner, we can prove that a solution to the quintic equation (1) also is increasing function.

Since any trajectory in the region $G5$ is closed it is convenient to consider the respective solutions in polar coordinates. Equation (1) written in polar coordinates

$$x(t) = \rho(t)\sin\phi(t), \quad x'(t) = \rho(t)\cos\phi(t) \tag{11}$$

turns to a system (12):

$$\begin{cases} \phi'(t) = \cos^2\phi(t) + a\sin^2\phi(t) - \rho^4(t)\; c\; \sin^6\phi(t), \\ \rho'(t) = \frac{1}{2}\,\rho(t)\sin 2\phi(t)\left(1 - a + \rho^4(t)\; c\; \sin^4\phi(t)\right). \end{cases} \tag{12}$$

Consider any solution of equation (1) with the initial conditions $(x(t_0), x'(t_0)) \in G5$. Let initial conditions be written as

$$\phi(t_0) = \phi_0,\ \rho(t_0) = \rho_0,\ (\phi_0, \rho_0) \in G5,\ \rho_0 > 0. \tag{13}$$

**Lemma 5** *The angular function of any solution of (12), (13) is monotonically increasing.*

**Proof:** Consider the firs equation of system (12) multiplied by $\rho^2(t)$

$$\begin{aligned} \rho^2(t)\phi'(t) = \rho^2(t)\cos^2\phi(t) + \\ + a\,\rho^2(t)\sin^2\phi(t) - c\,\rho^6(t)\; \sin^6\phi(t). \end{aligned} \tag{14}$$

Returning to $(x, x')$ coordinates

$$\begin{aligned} \rho^2(t)\phi'(t) = x'^2(t) + a\,x^2(t) - c\,x^6(t) = \\ = x'^2(t) + x^2(t)(a - c\,x^4(t)) > 0. \end{aligned} \tag{15}$$

Since $x^4(t) < \frac{a}{c}$ for any solution of (12), (13) the angular function $\phi(t)$ is increasing. □

**Corollary 6** *Let $x(t)$ be a solution of equation (1) with the initial conditions in $G5$. Then between any two consecutive zeros of $x(t)$ there is exactly one point of extremum.*

Therefore the following statement is valid (it is similar, for example, to Theorem 4 in the paper [12]).

**Lemma 7** *Suppose $T_{x_\alpha}$ is the time needed to move on a phase plane from $(x_\alpha, 0)$ to the vertical axis $x = 0$ (a quarter of a period). The function $T_{x_\alpha}$ monotonically increases from $\frac{\pi}{2\sqrt{a}}$ to $+\infty$ as $x_\alpha$ changes from zero to $\sqrt[4]{\frac{a}{c}}$.*

**Proof:** Equation (1) has an integral

$$x'^2(t) = -a\,x^2(t) + \frac{1}{3}\,c\,x^6(t) + C, \tag{16}$$

where $C$ is an arbitrary constant. It is clear from the analysis of the phase portrait that solutions (trajectories) of the Neumann problem can exist only in the region $G5$ between the heteroclinic trajectories. Solutions $x(t; x_\alpha)$ of the Cauchy problem (1), (17)

$$x(0) = x_\alpha,\quad x'(0) = 0,\quad 0 < x_\alpha < \sqrt[4]{\frac{a}{c}} \tag{17}$$

satisfy the relation (16), where $C = a\,x_\alpha^2 - \dfrac{1}{3}\,c\,x_\alpha^6$. It follows

$$x'^2(t;x_\alpha) = -ax^2(t;x_\alpha) + \tfrac{1}{3}\,c\,x^6(t;x_\alpha) + ax_\alpha^2 - \tfrac{1}{3}\,c\,x_\alpha^6,$$

$$\frac{dx}{dt} = \pm\sqrt{-ax^2(t;x_\alpha) + \tfrac{1}{3}\,c\,x^6(t;x_\alpha) + ax_\alpha^2 - \tfrac{1}{3}\,c\,x_\alpha^6} \tag{18}$$

(notice that $x'(t;x_\alpha) < 0$) and therefore

$$-\frac{dx}{\sqrt{-ax^2(t;x_\alpha) + \frac{1}{3}\,c\,x^6(t;x_\alpha) + ax_\alpha^2 - \frac{1}{3}\,c\,x_\alpha^6}} = dt. \tag{19}$$

$$\begin{aligned} &-\int_{x_\alpha}^{0} \frac{dx}{\sqrt{-ax^2+\frac{1}{3}\,c\,x^6+ax_\alpha^2-\frac{1}{3}\,c\,x_\alpha^6}} = \\ &= \int_{0}^{x_\alpha} \frac{dx}{\sqrt{-ax^2+\frac{1}{3}\,c\,x^6+ax_\alpha^2-\frac{1}{3}\,c\,x_\alpha^6}} = \int_0^t dt = t. \end{aligned} \tag{20}$$

One has that

$$\begin{aligned} &\int_0^{x_\alpha} \frac{dx}{\sqrt{-ax^2 + \frac{1}{3}\,c\,x^6 - \frac{1}{3}\,c\,x_\alpha^4 + ax_\alpha^2}} = \\ &= \int_0^{x_\alpha} \frac{dx}{\sqrt{(ax_\alpha^2 - \frac{1}{3}\,c\,x_\alpha^6) - (ax^2 - \frac{1}{3}\,c\,x^6)}} = \\ &= \left|\xi = \frac{x}{x_\alpha}\right| = \int_0^1 \frac{d\xi}{\sqrt{a(1-\xi^2) - \frac{1}{3}\,c\,x_\alpha^4(1-\xi^6)}} = \int_0^{T_{x_\alpha}} dt = T_{x_\alpha}, \end{aligned} \tag{21}$$

where $T_{x_\alpha}$ is the time needed to move on a phase plane from $(x_\alpha, 0)$ to the vertical axis $x = 0$ (a quarter of a period). It follows then that $T_{x_\alpha}$ is increasing function of $x_\alpha$.

Compare $T_{\alpha_1} = \int_0^1 \frac{d\xi}{\sqrt{a\,(1-\xi^2) - 1/3\,c\,x_{\alpha_1}^4(1-\xi^6)}}$ and $T_{\alpha_2} = \int_0^1 \frac{d\xi}{\sqrt{a(1-\xi^2) - 1/3\,c\,x_{\alpha_2}^4(1-\xi^6)]}}$. If $x_{\alpha_1} < x_{\alpha_2}$, then $T_{\alpha_1} < T_{\alpha_2}$. □

**Remark 8** *The results on the monotonicity of periods of solution of equations similar to (1) and (2) were proved in [5], but they are not applicable to equations (1) and (2). Therefore we prove Lemma 7 and Lemma 14.*

For the problem (1), (4) the following statement is true.

**Theorem 9** *Let $i$ be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}, \tag{22}$$

*where $T$ is as in (4). The Neumann problem (1), (4) has exactly $2i$ nontrivial solutions such that $x(0) = x_\alpha \neq 0$, $x'(0) = 0$, $-\sqrt[4]{\dfrac{a}{c}} < x_\alpha < \sqrt[4]{\dfrac{a}{c}}$.*

**Proof:** Consider solutions of the Cauchy problem (1), $x(0) = x_\alpha,\ x'(0) = 0$, where $0 < x_\alpha < \sqrt[4]{\frac{a}{c}}$. These solutions, denoted $x(t;x_\alpha)$, are continuously dependent on the initial data, due to a simple form of the equation. Solutions for $x_\alpha$ small enough are well approximated by solutions $y(t)$ of the equation of variations $y'' = -a\,y$ around the trivial solution. Solutions of the linearized equation which satisfy the initial conditions $y(0) = x_0, y'(0) = 0$, are given by the formula

$$y(t) = x_\alpha \cos\sqrt{a}t. \tag{23}$$

Due to the assumption $\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}$ solutions $y(t)$ along with solutions $x(t;x_\alpha)$ (for small enough $x_\alpha$) have exactly $i$ extrema in the interval $(0, T)$ and $t = T$ is not an extremum point. Notice that zeros and extrema of solutions $x(t;x_\alpha)$ alternate, and there is exactly one point of extremum between two consecutive zeros. These extrema due to Lemma 5 and Lemma 7 move monotonically to the right as $x_\alpha$ increases. Solutions $x(t;x_\alpha)$ with $0 < x_\alpha < \sqrt[4]{\frac{a}{c}}$ and close enough to $\sqrt[4]{\frac{a}{c}}$ have not extrema in $(0, T]$ since the respective trajectories are close to the upper heteroclinic (and the "period" of a heteroclinic solution is infinite) Therefore there are exactly $i$ solutions of the problem (1), (4). The additional $i$ solutions are obtained considering solutions with $x_\alpha \in \left(-\sqrt[4]{\frac{a}{c}}\,, 0\right)$ due to symmetry arguments. □

## 3.2 Formulae of solutions for linear-quintic case

Consider the problem

$$\begin{aligned} &x'' = -ax + cx^5, \\ &x(0) = x_\alpha, \quad x'(0) = 0, \quad 0 < x_\alpha < \sqrt[4]{\frac{a}{c}}. \end{aligned} \tag{24}$$

For $a$ and $T$ given we wish to fin $x_\alpha$ such that the respective solutions $x(t;x_\alpha)$ of the above problem satisfy the boundary condition $x'(T) = 0$, i.e. $x(t;x_\alpha)$ solve the Neumann problem (1), (4). The following assertion provides the explicit formula for a solution of (24).

**Lemma 10** *The function*

$$x(t,a,c,x_\alpha) = \frac{x_\alpha\,cn\left(\sqrt{\frac{1}{3}\,c\,\xi_1\,(x_\alpha^2 - \xi_2)}\,t; k_1\right)}{\sqrt{1 - \frac{x_\alpha^2}{\xi_1}\,sn^2\left(\sqrt{\frac{1}{3}\,c\,\xi_1\,(x_\alpha^2 - \xi_2)}\,t; k_1\right)}}, \tag{25}$$

*where* $k_1 = \sqrt{\frac{x_\alpha^2\ (\xi_1 - \xi_2)}{\xi_1\ (x_\alpha^2 - \xi_2)}}$,

$$\xi_1 := \xi_1(x_\alpha) = \frac{1}{2\sqrt{c}} \left( -x_\alpha^2 \sqrt{c} + \sqrt{3(4a - x_\alpha^4\, c)} \right),$$

$$\xi_2 := \xi_2(x_\alpha) = \frac{1}{2\sqrt{c}} \left( -x_\alpha^2 \sqrt{c} - \sqrt{3(4a - x_\alpha^4\, c)} \right), \tag{26}$$

*is a solution of the Cauchy problem (24).*

In equation (25) we introduce a new variable

$$A := A(x_\alpha) = \sqrt{\frac{1}{3}\, c\, \xi_1\ (x_\alpha^2 - \xi_2)} \tag{27}$$

and obtain the formula for a solution of the Cauchy problem (24)

$$x(t, a, c, x_\alpha) = \frac{x_\alpha\, cn(A\, t; k_1)}{\sqrt{1 - \dfrac{x_\alpha^2}{\xi_1}\, sn^2(A\, t; k_1)}}, \tag{28}$$

where $k_1 = \sqrt{\frac{x_\alpha^2\ (\xi_1 - \xi_2)}{\xi_1\ (x_\alpha^2 - \xi_2)}}$, but $\xi_1$ and $\xi_2$ as in (26). We can compute $x'_t(t, a, c, x_\alpha)$ and obtain

$$x'_t(t, a, c, x_\alpha) = x_\alpha\, sn(A\, t; k_1)\, dn(A\, t; k_1) \times \\ \times \frac{x_\alpha^2\, cn(A\, t; k_1) - \xi_1 \sqrt{1 - \frac{x_\alpha^2}{\xi_1}\, sn^2(A\, t; k_1)}}{\xi_1 \left( \sqrt{1 - \frac{x_\alpha^2}{\xi_1}\, sn^2(A\, t; k_1)} \right)^3}, \tag{29}$$

where $\xi_1$, $\xi_2$ and $A$ are given by the formulas (26), (27). Denote $g(T, a, c, x_\alpha) = x'_t(t, a, c, x_\alpha)$.

**Lemma 11** *The problem (1), (4) for $a$ and $T$ given can be obtained as a solution of the equation*

$$g(T, a, c, x_\alpha) = 0 \tag{30}$$

*with respect to $x_\alpha$, where $x_\alpha$ is the initial value for solution.*

**Theorem 12** *A solution to the Neumann problem (1), (4) is given by (25), where $x_\alpha$ is a solution of (30).*

# 4 Main results for linear-cubic-quintic case

## 4.1 Multiplicity of solutions for linear-cubic-quintic case

In this section we consider the linear-cubic-quintic equation (2).

There are three critical points of equation (2): $x_2 = 0$ is a center, $x_{1,3} = \pm\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 + 4ac}}$ both are saddle points. Two heteroclinic trajectories connect the two saddle points. $G$ is an open region bounded by the two heteroclinic trajectories connecting saddle points. The phase portrait of equation (2) is depicted in Fig. 3.

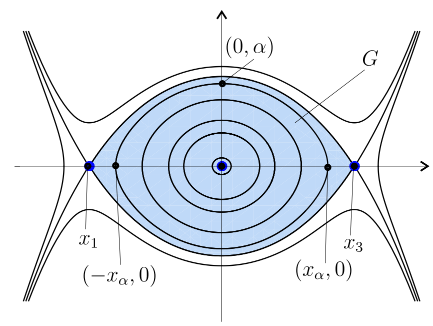


Figure 3: The phase portrait of equation (2), region $G$

Consider trajectories (closed curves) that fil the region $G$. The heteroclinic solution at infinit satisfie

$$0 = x'^2(\infty) = -ax^2(\infty) + \frac{1}{2} b\, x^4(\infty) + \frac{1}{3} c\, x^6(\infty) + C, \tag{31}$$

where $x^2(\infty) = \frac{-b + \sqrt{b^2 + 4ac}}{2c}$. Then any trajectory located in the region $G$ satisfie the relation (31), where $|C| < \frac{-b + \sqrt{b^2 + 4ac}}{2c} \left( \frac{2a}{3} - \frac{b}{6} \left( \frac{-b + \sqrt{b^2 + 4ac}}{2c} \right) \right)$.
Since any trajectory in the region $G$ is closed it is convenient to consider the respective solutions in polar coordinates. Equation (2) written in polar coordinates (11) turns to a system (32):

$$\begin{cases} \phi'(t) = \cos^2 \phi(t) + a \sin^2 \phi(t) - \rho^2(t)\ b\ \sin^4 \phi(t) - \\ -\rho^4(t)\ c\ \sin^6 \phi(t), \\ \rho'(t) = \frac{1}{2}\, \rho(t) \sin 2\phi(t) \times \\ \times \left( 1 - a + \rho^2(t)\ b\ \sin^2 \phi(t) + \rho^4(t)\ c\ \sin^4 \phi(t) \right). \end{cases} \tag{32}$$

Consider any solution of equation (2) with the initial conditions $(x(t_0), x'(t_0)) \in G$. Let the initial conditions be written as

$$\phi(t_0) = \phi_0,\ \rho(t_0) = \rho_0,\ (\phi_0, \rho_0) \in G,\ \rho_0 > 0. \tag{33}$$

**Lemma 13** *The angular function of any solution of (32), (33) is monotonically increasing.*

**Proof:** Consider the firs equation of system (32) multiplied by $\rho^2(t)$

$$\rho^2\,(t)\phi'(t) = \rho^2(t) \cos^2 \phi(t) + a\, \rho^2(t) \sin^2 \phi(t) - \\ -b\, \rho^4(t)\ \sin^4 \phi(t) - c\, \rho^6(t)\ \sin^6 \phi(t). \tag{34}$$

Returning to $(x, x')$ coordinates

$$\rho^2(t)\phi'(t) = x'^2(t) + a\,x^2(t) - b\,x^4(t) - c\,x^6(t) = \\ = x'^2(t) + x^2(t)(a - b\,x^2(t) - c\,x^4(t)) > 0. \tag{35}$$

Since $\frac{-b+\sqrt{b^2+4ac}}{2c} < x^2(t) < \frac{-b+\sqrt{b^2+4ac}}{2c}$ for any solution of (32), (33) the angular function $\phi(t)$ is increasing. □

**Lemma 14** *Suppose $T_{x_\alpha}$ is the time needed to move on a phase plane from $(x_\alpha, 0)$ to the vertical axis $x = 0$ (a quarter of a period). The function $T_{x_\alpha}$ monotonically increases from $\dfrac{\pi}{2\sqrt{a}}$ to $+\infty$ as $x_\alpha$ changes from zero to $\sqrt{-\dfrac{b}{2c} + \dfrac{1}{2c}\sqrt{b^2+4ac}}$.*

**Proof:** Equation (2) has an integral

$$x'^2(t) = -a\,x^2(t) + \frac{1}{2}\,b\,x^4(t) + \frac{1}{3}\,c\,x^6(t) + C, \tag{36}$$

where $C$ is an arbitrary constant. The solutions (trajectories) of the Neumann problem can exist only in the region $G$ between the heteroclinic trajectories. Solutions $x(t; x_\alpha)$ of the Cauchy problem (2), (37)

$$x(0) = x_\alpha,\ x'(0) = 0, \\ 0 < x_\alpha < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}} \tag{37}$$

satisfy the relation (36), $C = a\,x_\alpha^2 - \frac{1}{2}\,b\,x_\alpha^4 - \frac{1}{3}\,c\,x_\alpha^6$. Performing the transformations

$$x'^2(t; x_\alpha) = -ax^2(t; x_\alpha) + \tfrac{1}{2}\,b\,x^4(t; x_\alpha) + \\ + \tfrac{1}{3}\,c\,x^6(t; x_\alpha) + ax_\alpha^2 - \tfrac{1}{2}\,b\,x_\alpha^4 - \tfrac{1}{3}\,c\,x_\alpha^6,$$

$$\frac{dx}{dt} = \\ = \pm\sqrt{-ax^2 + \tfrac{1}{2}\,b\,x^4 + \tfrac{1}{3}\,c\,x^6 + ax_\alpha^2 - \tfrac{1}{2}\,b\,x_\alpha^4 - \tfrac{1}{3}\,c\,x_\alpha^6} \tag{38}$$

(notice that $x'(t; x_\alpha) < 0$) one obtains

$$-\frac{dx}{\sqrt{-ax^2 + \frac{1}{2}\,b\,x^4 + \frac{1}{3}\,c\,x^6 + ax_\alpha^2 - \frac{1}{2}\,b\,x_\alpha^4 - \frac{1}{3}\,c\,x_\alpha^6}} = dt. \tag{39}$$

$$-\int_{x_\alpha}^{0} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}\,b\,x^4+\frac{1}{3}\,c\,x^6+ax_\alpha^2-\frac{1}{2}\,b\,x_\alpha^4-\frac{1}{3}\,c\,x_\alpha^6}} = \\ = \int_{0}^{x_\alpha} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}\,b\,x^4+\frac{1}{3}\,c\,x^6+ax_\alpha^2-\frac{1}{2}\,b\,x_\alpha^4-\frac{1}{3}\,c\,x_\alpha^6}} = \int_0^t dt = t. \tag{40}$$

One has that

$$\int_{0}^{x_\alpha} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}\,b\,x^4+\frac{1}{3}\,c\,x^6+ax_\alpha^2-\frac{1}{2}\,b\,x_\alpha^4-\frac{1}{3}\,c\,x_\alpha^6}} = \\ = \left|\xi = \frac{x}{x_\alpha}\right| = \int_0^1 \frac{d\xi}{\sqrt{a(1-\xi^2)-\frac{1}{2}\,b\,x_\alpha^2(1-\xi^4)-\frac{1}{3}\,c\,x_\alpha^4(1-\xi^6)}} = \\ = \int_0^{T_{x_\alpha}} dt = T_{x_\alpha}, \tag{41}$$

where $T_{x_\alpha}$ is the time needed to move on a phase plane from $(x_\alpha, 0)$ to the vertical axis $x = 0$ (a quarter of a period). It follows then that $T_{x_\alpha}$ is increasing function of $x_\alpha$. Compare

$T_{\alpha_1} = \int\limits_0^1 \frac{d\xi}{\sqrt{a\ (1-\xi^2)-1/2\,b\,x_{\alpha_1}^2(1-\xi^4)-1/3\,c\,x_{\alpha_1}^4(1-\xi^6)}}$ and

$T_{\alpha_2} = \int\limits_0^1 \frac{d\xi}{\sqrt{a(1-\xi^2)-1/2\,b\,x_{\alpha_2}^2(1-\xi^4)-1/3\,c\,x_{\alpha_2}^4(1-\xi^6)]}}$.

If $x_{\alpha_1} < x_{\alpha_2}$ then $T_{\alpha_1} < T_{\alpha_2}$. □

For the problem (2), (4) the following statement is true.

**Theorem 15** *Let $i$ be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}} \tag{42}$$

*where $T$ is as in (4). The Neumann problem (2), (4) has exactly $2i$ nontrivial solutions such that $x(0) = x_\alpha \neq 0$, $x'(0) = 0$, $-\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}} < x_\alpha < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}}$.*

**Proof:** Consider solutions of the Cauchy problem (1), $x(0) = x_\alpha$, $x'(0) = 0$, where $0 < x_\alpha < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}}$. Solutions for $x_\alpha$ small enough behave like solutions of the equation of variations $y'' = -a\,y$ around the trivial solution. Solution of the linearized equation is $y(t) = x_\alpha \cos\sqrt{a}t$. Due to the assumption $\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}$ solutions $y(t)$ along with solutions $x(t; x_\alpha)$ (for small enough $x_\alpha$) have exactly $i$ extrema in the interval $(0, T)$ and $t = T$ is not an extremum point. These extrema due to Lemma 13 and Lemma 14 move monotonically to the right as $x_\alpha$ increases. Solutions $x(t; x_\alpha)$ with $0 < x_\alpha < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}}$ and close enough to $\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}}$ have not extrema in $(0, T]$ since the respective trajectories are close to the upper heteroclinic (and the "period" of a heteroclinic solution is infinite) Therefore there are exactly $i$ solutions of the problem (1), (2). The additional $i$ solutions are obtained considering solutions with $x_\alpha \in \left(-\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2+4ac}}, 0\right)$ due to symmetry arguments. □

## 4.2 Formulae of solutions for linear-cubic-quintic case

Consider the problem

$$x'' = -ax + bx^3 + cx^5, \\ x(0) = x_\alpha,\ x'(0) = 0, \\ 0 < x_\alpha < \sqrt{-\tfrac{b}{2c} + \tfrac{1}{2c}\sqrt{b^2+4ac}}. \tag{43}$$

For $a$ and $T$ given we wish to fin $x_\alpha$ such that the respective solutions $x(t;x_\alpha)$ of the above problem satisfy the boundary condition $x'(T)=0$, i.e. $x(t;x_\alpha)$ solve the Neumann problem (2), (4). The following assertion provides the explicit formula for a solution of (43).

**Lemma 16** *The function*

$$x(t,a,b,c,x_\alpha)=\frac{x_\alpha\, cn\left(\sqrt{\frac{1}{3}\,c\,\psi_1\,(x_\alpha^2-\psi_2)}\;t;k_2\right)}{\sqrt{1-\frac{x_\alpha^2}{\psi_1}\,sn^2\left(\sqrt{\frac{1}{3}\,c\,\psi_1\,(x_\alpha^2-\psi_2)}\;t;k_2\right)}},$$
$$k_2=\sqrt{\frac{x_\alpha^2\,(\psi_1-\psi_2)}{\psi_1\,(x_\alpha^2-\psi_2)}}, \tag{44}$$

$$\psi_1:=\psi_1(x_\alpha)=\frac{1}{4c}(-3\,b-2\,c\,x_\alpha^2-$$
$$-\sqrt{3\,(3\,b^2-4\,b\,c\,x_\alpha^2-4\,c^2\,x_\alpha^4+16\,a\,c)}), \tag{45}$$
$$\psi_2:=\psi_2(x_\alpha)=\frac{1}{4c}(-3\,b-2\,c\,x_\alpha^2+$$
$$+\sqrt{3\,(3\,b^2-4\,b\,c\,x_\alpha^2-4\,c^2\,x_\alpha^4+16\,a\,c)}),$$

*is a solution of the Cauchy problem (43).*

In equation (44) we introduce a new variable

$$B:=B(x_\alpha)=\sqrt{\frac{1}{3}\,c\,\psi_1\,(x_\alpha^2-\psi_2)} \tag{46}$$

and obtain the formula for a solution of the Cauchy problem (43)

$$x(t,a,b,c,x_\alpha)=\frac{x_\alpha\, cn(B\,t;k_2)}{\sqrt{1-\frac{x_\alpha^2}{\psi_1}\,sn^2(B\,t;k_2)}}, \tag{47}$$

where $k_2=\sqrt{\frac{x_\alpha^2\,(\psi_1-\psi_2)}{\psi_1\,(x_\alpha^2-\psi_2)}}$, but $\psi_1$ and $\psi_2$ as in (45).
We can compute $x'_t(t,a,b,c,x_\alpha)$ and obtain

$$x'_t(t,a,b,c,x_\alpha)=x_\alpha\,sn(B\,t;k_2)\,dn(B\,t;k_2)\times$$
$$\times\frac{x_\alpha^2\,cn(B\,t;k_2)-\psi_1\sqrt{1-\frac{x_\alpha^2}{\psi_1}\,sn^2(B\,t;k_2)}}{\psi_1\left(\sqrt{1-\frac{x_\alpha^2}{\psi_1}\,sn^2(B\,t;k_2)}\right)^3}, \tag{48}$$

where $\psi_1$, $\psi_2$ and $B$ are given by formulas (45), (46). Denote $s(T,a,b,c,x_\alpha)=x'_t(t,a,b,c,x_\alpha)$.

We have arrived at the statement.

**Lemma 17** *A solution to the problem (2), (4) for $a$ and $T$ given has the initial value $x(0)=x_\alpha$ that is a root of the equation*

$$s(T,a,b,c,x_\alpha)=0. \tag{49}$$

**Theorem 18** *A solution to the Neumann problem (2), (4) is given by (44) where $x_\alpha$ is a solution of (49).*

# 5 Example

**Example 1**
Consider equation (1) with $a=50$, $b=25$:

$$x''=-50x+25x^5 \tag{50}$$

with the initial conditions $x(0)=x_\alpha,\ x'(0)=0$, $0<x_\alpha<\sqrt[4]{2}$. Then the number of solutions satisfying the boundary conditions (4), where $T=1$, is two and, symmetrically, for initial conditions $x(0)=x_\alpha,\ x'(0)=0,\ -\sqrt[4]{2}<x_\alpha<0$ there are two additional solutions to the problem (50), (4), totally four solutions. By Theorem 9, this is the case for $T=1$ and $i=2$ (namely $\frac{2\pi}{\sqrt{50}}<T=1<\frac{3\pi}{\sqrt{50}}$) in the inequality (22).
Consider equation (30) with $a=50$, $b=25$ and T=1. We have

$$g(1,50,25,x_\alpha)=x'_t(1,50,25,x_\alpha)=0. \tag{51}$$

The next Figure shows that the equation (51) provides four nonzero initial values $x_\alpha$ for solutions of the problem (50), (4).

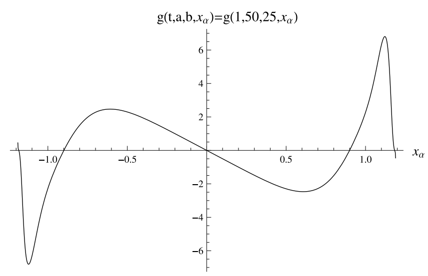


Figure 4: The graph of $g(1,50,25,x_\alpha)$ for equation (50)

In Fig. 5 and Fig. 6 the graphs of solutions $x(t)$ and the derivatives $x'(t)$ of the problem (50), (4) are depicted.

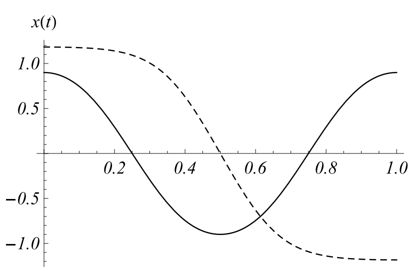


Figure 5: Graphs $x(t)$ for solutions of the problem (50), (4), $x_\alpha\approx 0.8995$ (solid), $x_\alpha\approx 1.1831$ (dashed).

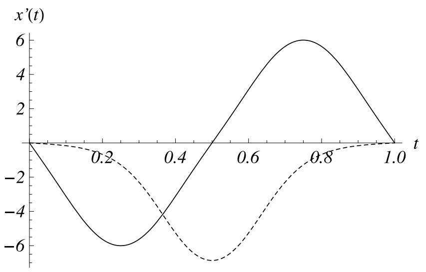


Figure 6: Graphs $x'(t)$ for solutions of the problem (50), (4), $x_\alpha \approx 0.8995$ (solid), $x_\alpha \approx 1.1831$ (dashed).

**Example 2**
Equation (2) is of the form:

$$x'' = -50x + 25x^3 + 25x^5 \tag{52}$$

and the initial conditions are $x(0) = x_\alpha,\ x'(0) = 0$, $0 < x_\alpha < 1$, then the number of solutions satisfying the boundary conditions (4), where $T = 1$, is two and, symmetrically, for initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, $-1 < x_\alpha < 0$ there are also two solutions to the problem (52), (4), totally four solutions. By Theorem 15, this is the case for $T = 1$ and $i = 2$ (namely $\frac{2\pi}{\sqrt{50}} < T = 1 < \frac{3\pi}{\sqrt{50}}$) in the inequality (42). Consider equation (49) with $a = 50$, $b = 25$, $c = 25$ and T=1, we have the equation

$$s(1, 50, 25, 25, x_\alpha) = x'_t(1, 50, 25, 25, x_\alpha) = 0. \tag{53}$$

In Fig. 7 four nonzero initial values $x_\alpha$ are indicated for solutions of the problem (50), (4).

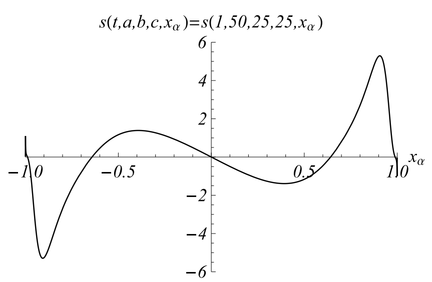


Figure 7: The graph of $s(1, 50, 25, 25, x_\alpha)$ for equation (52)

In Fig. 8 and Fig. 9 the graphs of solutions $x(t)$ and the derivatives $x'(t)$ of the problem (52), (4) are depicted.

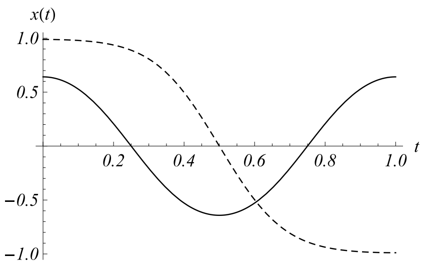


Figure 8: Graphs $x(t)$ for solutions of the problem (52), (4), $x_\alpha \approx 0.6417$ (solid), $x_\alpha \approx 0,9886$ (dashed).

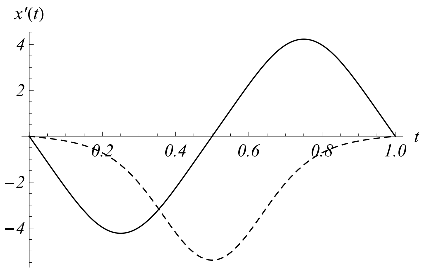


Figure 9: Graphs $x'(t)$ for solutions of the problem (52), (4), $x_\alpha \approx 0.6417$ (solid), $x_\alpha \approx 0,9886$ (dashed).

# Concluding remarks

In this article, we have considered an undamped, unforced cubic-quintic oscillator. We have studied boundary value problems with the Neumann type boundary conditions. Our conclusion is that the number of solutions to the problem under consideration depends only on the coefficien $a$ in a linear part. Besides, the formulas for the number of solutions was obtained and, moreover, the proof was given that these estimates are exact. The complicated analytical expressions for solutions of the cubic-quintic equations were analyzed and as a result, equations for the initial conditions for solutions of the boundary value problems were obtained. This information makes it much easier to do the calculations.

In the work [12] some remarks were made concerning nonlinear oscillators with higher degree polynomial nonlinearities. In the work [11] cubic oscillators with non-constant (piece-wise linear) coefficient were considered, following the results and ideas in [18], [9]. Studies in both directions and combinations of both ones are possible as future research.

# Acknowledgements

The author would like to sincerely thank Professor Felix Sadyrbaev for his useful comments.

*References:*

[1] A. Beléndez, E. Arribas, T. Beléndez, C. Pascual, E. Gimeno, M. L. Álvarez. Closed-Form Exact Solutions for the Unforced Quintic Nonlinear Oscillator. *Advances in Mathematical Physics*, Hindawi, 2017.

[2] A. Beléndez, G. Bernabeu, J. Francés, D. I. Méndez, S. Marini. An accurate closed-form approximate solution for the quintic Duffin oscillator equation. *Mathematical and Computer Modelling*, vol. 52(3-4), pp. 637–641, 2010.

[3] A. Beléndez, T. Beléndez, F. J. Martinez, C. Pascual, M. L. Alvarez and E. Arribas. Exact solution for the unforced Duffin oscillator with cubic and quintic nonlinearities. *Nonlinear Dynamics*, vol. 86, pp. 1687–1700, 2016.

[4] C. Bota, B. Căruntu, and O. Bundău. Approximate Periodic Solutions for Oscillatory Phenomena Modelled by Nonlinear Differential Equations. *Mathematical Problems in Engineering*, vol. 2014 (2014).

[5] C. Chicone. Geometric Methods for Two-Point Nonlinear Boundary Value Problems. *Journal of differential equation*, vol. 72, pp. 360–407, 1988.

[6] V. Chua. Cubic-quintic Duffin oscillators. *College of Engineering, School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, Georgia*, 2003.

[7] A. Elías-Zúñiga. Exact solution of the cubic-quintic Duffin oscillator, *Applied Mathematical Modelling*, vol. 37(4), pp.2574-2579, 2013.

[8] I. S. Gradshteyn and I. M. Ryzhik. Table of Integrals. *Series and Products, Academic Press, San Diego, Calif, USA, 6th edition*, 2000.

[9] A. Gritsans, F. Sadyrbaev. Extension of the example by Moore-Nehari. *Tatra Mountains Mathematical Publications*, vol. 63, pp. 115–127, 2015.

[10] A. Kirichuka. The number of solutions to the boundary value problem for the second order differential equation with cubic nonlinearity, *WSEAS Transactions on Mathematics*, vol. 18(31), pp. 230–236, 2019.

[11] A. Kirichuka, F. Sadyrbaev. On boundary value problem for equations with cubic nonlinearity and step-wise coefficient *Differential Equations and Applications*, vol. 10(4), pp. 433–447, 2018.

[12] A. Kirichuka, F. Sadyrbaev. Remark on boundary value problems arising in Ginzburg-Landau theory, *WSEAS Transactions on Mathematics*, Vol. 17, pp. 290–295, 2018.

[13] A. Kirichuka. The number of solutions to the Neumann problem for the second order differential equation with cubic nonlinearity. *Proceedings of IMCS of University of Latvia*, vol. 17(0), pp. 44–51, 2017.

[14] A. Kirichuka. The number of solutions to the Dirichlet and mixed problem for the second order differential equation with cubic nonlinearity. *Proceedings of IMCS of University of Latvia*, vol. 18(0), pp. 63–72, 2018.

[15] I. Kovacic. Nonlinear Oscillations. *Springer International Publishing*, ISBN: 9783030531720, August 2020.

[16] S. K. Lai, C. W. Lim, B. S. Wu, C. Wang, Q. C. Zeng and X. F. He. Newton - harmonic balancing approach for accurate solutions to nonlinear cubic - quintic Duffin oscillators. *Applied Mathematical Modelling*, vol. 33, no. 2, pp. 852–866, 2009.

[17] L. M. Milne-Thomson. Handbook of Mathematical Functions, Chapter 16. Jacobian Elliptic Functions and Theta Functions. *Dover Publications, New York, NY, USA, Edited by: M. Abramowitz and I. A. Stegun*, 1972.

[18] R. Moore and Z. Nehari. Nonoscillation theorems for a class of nonlinear differential equations. *Trans. Amer. Math. Soc.*, vol. 93, pp. 3052, 1959).

[19] A.H. Salas and S.C. Trujillo. A New Approach for Solving the Complex Cubic-Quintic Duffin Oscillator Equation for Given Arbitrary Initial Conditions. *Mathematical Problems in Engineering*, vol. 2020, 8 pages, 2020).

[20] E. T. Whittaker and G. N. Watson. A Course of Modern Analysis. *Cambridge University Press*, 1940, 1996.