On the Diophantine Equation $(4^n)^x - p^y = z^2$

AMR ELSHAHED
Universiti Sains Malaysia
School of Mathematical Sciences
11800, USM, Pulau Pinang
MALAYSIA

HAILIZA KAMARULHAILI
Universiti Sains Malaysia
School of Mathematical Sciences
11800, USM, Pulau Pinang
MALAYSIA

Abstract: In this paper, the Diophantine equation $(4^n)^x - p^y = z^2$, where p is an odd prime, $n \in Z^+$ and x, y, z are non-negative integers, has been investigated to show that the solutions are given by

$$\{(x, y, z, p)\} = \{(k, 1, 2^{nk} - 1, 2^{nk+1} - 1)\} \cup \{(0, 0, 0, p)\}.$$

Key–Words: exponential Diophantine equation, Mersenne primes

Received: December 28, 2019. Revised: May 31, 2020. Accepted: June 15, 2020. Published: June 30, 2020.

1 Introduction

Diophantine Equation plays a significant role in the development of many technical, engineering areas and applied mathematical fields. Without its help, it would not be possible for large values of the argument to analyze the solution of various problems. In Cryptography, in particular, Diophantine Equations led to the emergence of useful applications. And also, diophantine equations are shown to apply to some problems in chemistry [11, 12, 13, 15]. In 1844, Catalan conjectured that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x, and y are integers such that $\min\{a, b, x, y\} > 1$ [2]. In 2004 the conjecture was proved by Mihailescu [3], and since that, it was used to solve many diophantine equations of the type $a^x + b^y = z^2$ [4, 5, 6, 9, 10, 14]. Rabago in 2018, by using Catalan's conjecture proved that the Diophantine equation $4^x - p^y = 3z^2$ has the two solutions $(0, 0, 0)$ and $(1, 0, 1)$[7]. In this paper, we will use the Catalan's conjecture to show that the Diophantine equation $16^x - 7^y = z^2$ has only two solutions (x, y, z), (the trivial solution) $(0, 0, 0)$, and $(1, 1, 3)$, where x, y, and z are non-negative integers, also we will solve the Diophantine equation $16^x - p^y = z^2$ to prove a generalization of the previous results, where $(4^n)^x - p^y = z^2$ and x, y, z are non-negative integers.

2 Preliminaries

Lemma 1 *[7] The Diophantine equation $4^x - p^y = z^2$ has the set of all solutions (x, y, z) given by $(x, y, z) = (0, 0, 0) \cup (q - 1, 1, 2^{q-1} - 1)$, for prime $p = 2^q - 1$ (with q also a prime). For $p \equiv 3 (mod\ 4)$ not of the form $2^q - 1$, the Diophantine equation $4^x - p^y = z^2$ has only the trivial solution $(x, y, z) = (0, 0, 0)$.*

3 Results

Theorem 2 *The Diophantine equation*

$$16^x - 7^y = z^2, \tag{1}$$

has only two solutions (x, y, z), (the trivial solution) $(0, 0, 0)$, and $(1, 1, 3)$.

Proof: Evidently, the case when $z = 0$ will give us $(x, y, z) = (0, 0, 0)$, so we may assume that $z > 0$. For $z > 0$, we consider three cases.
case 1. $x = 0$. This case is trivial.
case 2. $y = 0$. If $y = 0$, then we have $(2^x)^4 - z^2 = 1$ which is impossible according to Mihailescu's Theorem [3].
case 3. $x, y > 0$. For this case we have $((2^x)^4 - z^2) = ((2^x)^2 + z)((2^x)^2 - z) = 7^y$. It follows that $((2^x)^2 + z) + ((2^x)^2 - z)$ for some $\alpha < \beta$, where $\alpha + \beta = y$. Hence, $2^{2x+1} = 7^\alpha(7^{\beta-\alpha} + 1)$. Thus, $\alpha = 0$ and $2^{2x+1} - 7^\beta = 1$, which is true when $x = 1$ and $y = 1$. These give us the value $z = 3$. Therefore, $(1, 1, 3)$ is a solution of $16^x - 7^y = z^2$. Now,if we

assume $y > 1$, then we get $2^{2x+1} - 7^{\beta} = 1$ which has no solution according to Mihailescu's Theorem. Therefore, $(0, 0, 0)$, and $(1, 1, 3)$ are the only two solutions for Diophantine equation $16^x - 7^y = z^2$. □

Theorem 3 *The solutions of the Diophantine equation*

$$16^x - p^y = z^2, \tag{2}$$

are given by

$$\{(x, y, z, p)\} = \{(k, 1, 2^{2k}-1, 2^{2k+1}-1)\} \cup \{(0, 0, 0, p)\},$$

when p is an odd prime and k, x, y, z are non-negative integers.

Proof: By considering $y > 0$, we get

$$4^{2x} - z^2 = p^y$$

i.e

$$(4^x - z)(4^x + z) = p^y.$$

Where $4^x - z = p^v$ and $4^x + z = p^{y-v}$, $y > 2v$ and v is a non-negative integer. Then we get $p^{y-v} + p^v = 2^{2x+1}$ or $p^v(p^{y-2v} + 1) = 2^{2x+1}$. If $v = 0$, we obtain

$$p^y + 1 = 2^{2x+1}$$

i.e

$$2^{2x+1} - p^y = 1.$$

From Catalan's conjecture, $x = 0$, $x = 1$ or $y = 1$.

For $x = 0$, we get $2 - p^y = 1$, hence $p^y = 1$, this is impossible since $y > 0$.

For $x = 1$, we get $2^3 - p^y = 1$, hence $p^y = 7$, therefore the solution of the equation is $(p, y) = (7, 1)$. Hence the solution of the equation $16^x - p^y = z^2$ is $(x, y, z, p) = (1, 1, 3, 7)$.

For $y = 1$ we obtain $2^{2x+1} - p = 1$, hence $p = 2^{2x+1} - 1$, therefore the solution of the equation $16^x - p^y = z^2$ is $(x, y, z) = (x, 1, 2^{2x} - 1)$.

For $y = 0$ we get $4^{2x} - z^2 = 1$. From Catalan's conjecture, $x = 0$ or $x = 1$.

For $x = 1$, we get $4^2 - z^2 = 1$, hence $z^2 = 15$ which is impossible.

For $x = 0$, we get $1 - z^2 = 1$, hence $z^2 = 0$, hence $z = 0$, therefore the solution of the equation is $(x, z) = (0, 0)$, which implies that $1 - p^y = 0$, hence $p^y = 1$, and this is true for $(p, y) = (p, 0)$ therefore the solution of the equation $16^x - p^y = z^2$ is $(x, y, z, p) = (0, 0, 0, p)$.
Consequently, $\{(x, y, z, p)\} = \{(k, 1, 2^{2k} - 1, 2^{2k+1} - 1)\} \cup \{(0, 0, 0, p)\}$ are the solutions of the equation $16^x - p^y = z^2$ where k is a non-negative integer.

□

Theorem 4 *When p is an odd prime and k, x, y, z are non-negative integers, solutions of the Diophantine equation*

$$64^x - p^y = z^2 \tag{3}$$

is given by

$$\{(x, y, z, p)\} = \{(k, 1, 2^{3k}-1, 2^{3k+1}-1)\} \cup \{(0, 0, 0, p)\}.$$

Proof: By considering $y > 0$, we get

$$8^{2x} - z^2 = p^y$$

i.e

$$(8^x - z)(8^x + z) = p^y.$$

Where $8^x - z = p^v$ and $8^x + z = p^{y-v}$, $y > 2v$ and v is a non-negative integer. Then we get $p^{y-v} + p^v = 2^{3x+1}$ or $p^v(p^{y-2v} + 1) = 2^{3x+1}$. If $v = 0$, we obtain

$$p^y + 1 = 2^{3x+1}$$

i.e

$$2^{3x+1} - p^y = 1.$$

From Catalan's conjecture, $x = 0$, $x = 1$ or $y = 1$.

For $x = 0$, we get $2 - p^y = 1$, hence $p^y = 1$, this is impossible since $y > 0$.

For $x = 1$, we get $2^4 - p^y = 1$, hence $p^y = 15$, which has no solution.

For $y = 1$ we obtain $2^{3x+1} - p = 1$, hence $p = 2^{3x+1} - 1$, therefore the solution of the equation $64^x - p^y = z^2$ is $(x, y, z) = (x, 1, 2^{3x} - 1)$.

For $y = 0$ we get $8^{2x} - z^2 = 1$. From Catalan's conjecture, $x = 0$ or $x = 1$.

For $x = 1$, we get $8^2 - z^2 = 1$, hence $z^2 = 63$ which is impossible.

For $x = 0$, we get $1 - z^2 = 1$, hence $z^2 = 0$, hence $z = 0$, therefore the solution of the equation is $(x, z) = (0, 0)$, which implies that $1 - p^y = 0$, hence $p^y = 1$, and this is true for $(p, y) = (p, 0)$ therefore the solution of the equation $64^x - p^y = z^2$ is $(x, y, z, p) = (0, 0, 0, p)$.
Consequently, $\{(x, y, z, p)\} = \{(k, 1, 2^{3k} - 1, 2^{3k+1} - 1)\} \cup \{(0, 0, 0, p)\}$ are the solutions of the equation $64^x - p^y = z^2$ where k is a non-negative integer.

□

The generalization of the above Diophantine equations is given in the following theorem.

Theorem 5 *The solutions to*

$$(4^n)^x - p^y = z^2 \tag{4}$$

are given by

$$(x, y, z, p) = (k, 1, 2^{nk} - 1, 2^{nk+1} - 1)$$

where x,y,z are non-negative integers, k is a positive integer, and $2^{nk+1} - 1$ is a prime.

Proof: To solve this equation we will conside three cases where $y = 0$, $y = 1$, and $y \geq 2$.

For $y = 0$, the equation (3.4) becomes $4^{nx} - 1 = z^2$. If $x > 0$, then we have $-1 \equiv z^2 \pmod 4$, which is impossible, because squares are always $\equiv 0$ or $\equiv 1 \pmod 4$. Therefore $x = 0$ and the equation becomes $0 = z^2$, so $z = 0$. Also p is arbitrary. This is the trivial solution.

The cases $y = 1$ and $y \geq 2$ are similar to each other. We can use some divisibility observations here. From (3.4) we have

$$p^y = (2^{nx} - z)(2^{nx} + z).$$

The two factors on the right cannot both be divisible by p, because their sum is 2^{nx+1} which is not divisible by p. But they are both powers of p, so the smaller one is $2^{nx} - z = 1$ and the larger one is $2^{nx} + z = p^y$. Solving this system of two equations we obtain

$$p^y = 2^{nx+1} - 1, \tag{5}$$

$$z = 2^{nx} - 1.$$

We can also use a size (magnitude) observation: From (3.4) we have

$$4^{nx} = p^y + z^2 \geq p^y \geq 3$$

and therefore $nx \geq 1$. In particular $x \geq 1$.

If $y \geq 2$, we have that the equation (3.5) is impossible by Catalan's Conjecture, because both exponents are greater than or equal to 2 and it is not the one permissible case $3^2 - 2^3 = 1$.

If $y = 1$, all variables of this form are solutions. That is, if k is a positive integer and $2^{nk+1}-1$ is prime, then

$$(x, y, z, p) = (k, 1, 2^{nk} - 1, 2^{nk+1} - 1)$$

is a solution of (3.4), because we have

$$z^2 = (2^{nk} - 1)^2 = 2^{2nk} - 2^{nk+1} + 1 = (4^n)^k - p^1.$$

□

4 Conclusion

In this paper, Catalan's conjecture has been used to show that the Diophantine equation $16^x - 7^y = z^2$ has only two solutions (x, y, z), (the trivial solution) $(0, 0, 0)$, and $(1, 1, 3)$, where x, y, and z are non-negative integers. Also the Diophantine equation $16^x - p^y = z^2$ has been solved to prove a generalization of the previous results, where $(4^n)^x - p^y = z^2$ and x, y, z are non-negative integers have the solutions given by

$$\{(x, y, z, p)\} = \{(k, 1, 2^{nk}-1, 2^{nk+1}-1)\} \cup \{(0, 0, 0, p)\}.$$

And we noticed that some of the solutions contain parts that similar to Mersenne prime, which is a prime number of the form $2^p - 1$, p is a prime.

Acknowledgements: This work is supported by the Research University (RU) funding, account number 1001 /PMATHS/ 8011121, Universiti Sains Malaysia.

References:

[1] Sroysang, Banyat. "On the Diophantine equation $7^x + 8^y = z^2$." International Journal of Pure and Applied Mathematics 84.1 (2013): 111-114.

[2] Catalan, Eugene. "Note extraite d'une lettre adressée à l'éditeur par Mr. E. Catalan, Répétiteur à l'école polytechnique de Paris." Journal für die reine und angewandte Mathematik 27 (1844): 192-192.

[3] Mihailescu, Preda. "Primary cyclotomic units and a proof of Catalan's conjecture." Journal Fur die reine und angewandte Mathematik 572 (2004): 167-196.

[4] Sroysang, Banyat. "On the Diophantine equation $7^x + 8^y = z^2$." International Journal of Pure and Applied Mathematics 84.1 (2013): 111-114.

[5] Sroysang, Banyat. "More on the Diophantine equation $3^x + 85^y = z^2$." International Journal of Pure and Applied Mathematics 91.1 (2014): 131-134.

[6] Sroysang, B. "On the Diophantine equation $5^x + 7^y = z^2$." International Journal of Pure and Applied Mathematics 89.1 (2013): 115-118.

[7] Rabago, Julius Fergy Tiongson. "On the Diophantine Equation $4^x - p^y = 3z^2$ where p is a Prime." Thai Journal of Mathematics 16.3 (2018): 643-650.

[8] Peker, Bilge, and Selin Inag Cenberci. "On The Solutions of The Equation $(4^n)^x + p^y = z^2$." arXiv preprint arXiv:1202.2267 (2012).

[9] Chotchaisthit, Somchit. "On the Diophantine equation $2^x + 11^y = z^2$" Maejo International Journal of Science and Technology 7.2 (2013): 291.

[10] Qi, Lan, and Xiaoxue Li. ”The Diophantine equation 8x+ py= z2.” The Scientific World Journal 2015 (2015).

[11] Okumura, Shinya. ”A public key cryptosystem based on diophantine equations of degree increasing type.” Pacific Journal of Mathematics for Industry 7, no. 1 (2015): 4.

[12] Ding, Jintai, Momonari Kudo, Shinya Okumura, Tsuyoshi Takagi, and Chengdong Tao. ”Cryptanalysis of a public key cryptosystem based on Diophantine equations via weighted LLL reduction.” Japan Journal of Industrial and Applied Mathematics 35, no. 3 (2018): 1123-1152.

[13] Crocker, Roger. ”Application of Diophantine equations to problems in chemistry.” Journal of Chemical Education 45, no. 11 (1968): 731.

[14] Ddamulira, Mahadi, Carlos A. Gómez, and Florian Luca. ”On a problem of Pillai with k–generalized Fibonacci numbers and powers of 2.” Monatshefte für Mathematik 187, no. 4 (2018): 635-664.

[15] KLAŠKA, JIRÍ. ”Real-world Applications of Number Theory.” South Bohemia Mathematical Letters (2017).