(Λ, sp)-closed sets and related topics in topological spaces

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Abstract: The purpose of the present paper is to introduce the concepts of Λ_{sp}-sets, (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of β-open sets and β-closed sets. Some characterizations of Λ_{sp}-submaximal spaces, Λ_{sp}-extremally disconnected spaces and Λ_{sp}-hyperconnected spaces are established. Moreover, several characterizations of upper and lower (Λ, sp)-continuous multifunctions are investigated.

Key–Words: Λ_{sp}-set, (Λ, sp)-closed set, (Λ, sp)-open set

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1 Introduction

The notions of open sets and continuous functions in topological spaces are extensively developed and used in many fields of applied sciences. Continuity is a basic concept for the study in topological spaces. This concept has been extended to the setting of multifunctions and has been generalized by weaker forms of open sets. Several different forms of continuous multifunctions have been introduced and studied over the years. Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. The first initiation of the notion of upper (lower) α-continuous multifunctions has been done by Neubrunn [15]. These multifunctions are further investigated by the present authors [21]. Popa and Noiri [22] introduced the notion of α-continuous multifunctions and investigated several characterizations of such multifunctions. Abd El-Monsef et al. [1] have introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets [3]. Noiri and Hatir [16] introduced the notions of Λ_{sp}-closed and spg-closed sets and investigated properties of these sets and introduced some related new separation axioms.

The concept of extremally disconnected topological spaces introduced by Gillman and Jerison [8]. Thompson [26] introduced the notion of S-closed spaces. Herrman [10, 11] showed that every S-closed weakly Hausdorff space is extremally disconnected. Cameron [6] proved that every maximally S-closed space is extremally disconnected. Niori [20] introduced the concept of locally S-closed spaces which is strictly weaker than that of S-closed spaces. Noiri [19] showed that every locally S-closed weakly Hausdorff space is extremally disconnected. Sivaraj [24] has obtained some characterizations of extremally disconnected spaces by utilizing semi-open sets due to Levine [14]. In [18], the present author obtained several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets. The concepts of maximality and submaximality of general topological spaces were introduced by Hewitt [12]. He discovered a general way of constructing maximal topologies. The existence of a maximal space that is Tychonoff is nontrivial and due to van Douwen [7]. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel'skiǐ and Collins [4]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is σ-discrete [4]. Every connected Hausdorff space which does not admit a larger connected topology is submaximal [9].

The notion of hyperconnected spaces was introduced by Steen and Seebach [25]. Several concepts which are equivalent to hyperconnectedness were defined and investigated in the literature. Levine [13] called a topological space (X, τ) a D-space if every nonempty open set of X is dense in X and showed that (X, τ) is a D-space if and only if it is hyperconnected. Sharma [23] indicated that a space is a D-space if it is a hyperconnected space due to Steen and Seebach. Ajmal and Kohli [2] have investigated the further properties of hyperconnected spaces. In [17], the present author obtained several characterizations of hyperconnected spaces by using semi-preopen sets and almost feebly continuous functions.

The paper is organized as follows. In section 3, we introduce (Λ, sp)-closed sets in topological spaces and investigate some of their fundamental properties. In section 4, we introduce the concept of Λ_{sp}-extremally disconnected spaces. Furthermore, several characterizations of Λ_{sp}-extremally disconnected spaces are discussed. In section 5 and 6, we investigate some characterizations of Λ_{sp}-submaximal spaces and Λ_{sp}-hyperconnected spaces, respectively. In the last section, we introduce the concepts of upper and lower (Λ, sp)-continuous multifunctions and investigate several characterizations of such multifunctions.

2 Preliminaries

Throughout the paper (X, τ) (or simply X) will always denote a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, the closure, interior and complement of A in (X, τ) are denoted by $\text{Cl}(A), \text{Int}(A)$ and $X - A$, respectively. By $\beta(X, \tau)$ and $\beta C(X, \tau)$ we denote the collection of all β-open sets and the collection of all β-closed sets of (X, τ), respectively. A subset A of a topological space (X, τ) is said to be β-open [1] if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$. The complement of a β-open set A is called β-closed and the β-closure of a set A, denoted by $\beta\text{Cl}(A)$, is the intersection of all β-closed sets containing A. The β-interior of a set A denoted by $\beta\text{Int}(A)$, is the union of all β-open sets contained in A.

Definition 1. *Let A be a subset of a topological space (X, τ). A subset $\Lambda_{sp}(A)$ [16] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$.*

Lemma 2. [16] *For subsets A, B and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) If $A \subseteq B$, then $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) If $U \in \beta(X, \tau)$, then $\Lambda_{sp}(U) = U$.

(5) $\Lambda_{sp}(\cap\{A_\alpha | \alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha | \alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

Definition 3. [16] *A subset A of a topological space (X, τ) is called Λ_{sp}-set if $A = \Lambda_{sp}(A)$. The family of all Λ_{sp}-sets of (X, τ) is denoted by $\Lambda_{sp}(X, \tau)$ (or simply Λ_{sp}).*

Lemma 4. [16] *For subsets A and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following hold:*

(1) $\Lambda_{sp}(A)$ is a Λ_{sp}-set.

(2) If A is β-open, then A is a Λ_{sp}-set.

(3) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cap_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

(4) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cup_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

By a multifunction, $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [5], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(Y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$ and for each $A \subseteq X$,

$$F(A) = \cup_{x\in A} F(x).$$

Then F is said to be surjection if $F(X) = Y$ and injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

3 On (Λ, sp)-closed sets

In this section, we introduce the concept of (Λ, sp)-closed sets in topological spaces. We also investigate some of their fundamental properties.

Definition 5. *A subset A of a topological space (X, τ) is called (Λ, sp)-closed if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The collection of all (Λ, sp)-closed sets in a topological space (X, τ) is denoted by $\Lambda_{sp}C(X, \tau)$.*

Theorem 6. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) A is (Λ, sp)-closed.

(2) $A = T \cap \beta Cl(A)$, where T is a Λ_{sp}-set.

(3) $A = \Lambda_{sp}(A) \cap \beta Cl(A)$.

Proof. (1) $\Rightarrow$ (2): Let $A = T \cap C$,where T is a Λ_{sp}-set and C is a β-closed set. Since $A \subseteq C$, $\beta\text{Cl}(A) \subseteq C$ and hence $A = T \cap C \supseteq T \cap \beta\text{Cl}(A) \supseteq A$. Thus, $A = T \cap \beta\text{Cl}(A)$.

(2) $\Rightarrow$ (3): Let $A = T \cap \beta\text{Cl}(A)$, where T is a Λ_{sp}-set. Since $A \subseteq T$, we have $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(T) =$

T and hence $A \subseteq \Lambda_{sp}(A) \cap \beta\mathrm{Cl}(A) \subseteq T \cap \beta\mathrm{Cl}(A) = A$. Consequently, we obtain $A = \Lambda_{sp}(A) \cap \beta\mathrm{Cl}(A)$.

(3) $\Rightarrow$ (1): Since $\Lambda_{sp}(A)$ is a Λ_{sp}-set, $\beta\mathrm{Cl}(A)$ is a β-closed set and $A = \Lambda_{sp}(A) \cap \beta\mathrm{Cl}(A)$. This shows that A is (Λ, sp)-closed. □

Definition 7. *A subset A of a topological space (X, τ) is said to be (Λ, sp)-open if the complement of A is (Λ, sp)-closed. The collection of all (Λ, sp)-open sets in a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$.*

Theorem 8. *For subsets A and $A_\alpha (\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) If A is β-closed, then A is (Λ, sp)-closed.

(2) If A_α is (Λ, sp)-closed for each $\alpha \in \nabla$, then $\cap\{A_\alpha \mid \alpha \in \nabla\}$ is (Λ, sp)-closed.

(3) If A_α is (Λ, sp)-open for each $\alpha \in \nabla$, then $\cup\{A_\alpha \mid \alpha \in \nabla\}$ is (Λ, sp)-open.

Proof. (1) Suppose that A is a β-closed set. It is sufficient to observe that $A = X \cap A$, where the whole set X is a Λ_{sp}-set.

(2) Let A_α is (Λ, sp)-closed for each $\alpha \in \nabla$. Then, for each α, there exist a Λ_{sp}-set T_α and a β-closed set C_α such that $A_\alpha = T_\alpha \cap C_\alpha$. Thus,

$$\begin{aligned} \cap_{\alpha\in\nabla} A_\alpha &= \cap_{\alpha\in\nabla}(T_\alpha \cap C_\alpha) \\ &= (\cap_{\alpha\in\nabla} T_\alpha) \cap (\cap_{\alpha\in\nabla} C_\alpha). \end{aligned}$$

By Lemma 4, we have $\cap_{\alpha\in\nabla} T_\alpha$ is a Λ_{sp}-set and $\cap_{\alpha\in\nabla} C_\alpha$ is a β-closed set. This shows that $\cap_{\alpha\in\nabla} A_\alpha$ is (Λ, sp)-closed.

(3) Let A_α is (Λ, sp)-open for each $\alpha \in \nabla$. Then $X - A_\alpha$ is (Λ, sp)-closed for each $\alpha \in \nabla$. By (1), we have $X - \cup_{\alpha\in\nabla} A_\alpha = \cap_{\alpha\in\nabla}(X - A_\alpha)$ is (Λ, sp)-closed and hence $\cup_{\alpha\in\nabla} A_\alpha$ is (Λ, sp)-open. □

Definition 9. *Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, sp)-cluster point of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-closure of A and is denoted by $A^{(\Lambda,sp)}$.*

Lemma 10. *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ and $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ is (Λ, sp)-closed.

(4) A is (Λ, sp)-closed if and only if $A = A^{(\Lambda,sp)}$.

Theorem 11. *Let A be a subset of a topological space (X, τ). If A is (Λ, sp)-closed, then*

$$A = \Lambda_{sp}(A) \cap A^{(\Lambda,sp)}.$$

Proof. Let A be a (Λ, sp)-closed set. Then, there exist a Λ_{sp}-set T and a β-closed set C such that $A = T \cap C$. Since $A \subseteq T$, we have $A \subseteq \Lambda_{sp}(A) \subseteq \Lambda_{sp}(T) = T$. Since $A \subseteq C$, we have $A \subset A^{(\Lambda,sp)} \subseteq C^{(\Lambda,sp)} = C$. Thus, $A \subseteq \Lambda_{sp}(A) \cap A^{(\Lambda,sp)} \subseteq T \cap C = A$ and hence $A = \Lambda_{sp}(A) \cap A^{(\Lambda,sp)}$. □

Definition 12. *Let A be a subset of a topological space (X, τ). The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-interior of A and is denoted by $A_{(\Lambda,sp)}$.*

Lemma 13. *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ and $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, then $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ is (Λ, sp)-open.

(4) A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

Definition 14. *A subset A of a topological space (X, τ) is said to be:*

(i) $r(\Lambda, sp)$-open if $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(i) $s(\Lambda, sp)$-open if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$;

(iii) $p(\Lambda, sp)$-open if $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(iv) $\alpha(\Lambda, sp)$-open if $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(v) $\beta(\Lambda, sp)$-open if $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

The family of all $s(\Lambda, sp)$-open (resp. $r(\Lambda, sp)$-open, $p(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $r\Lambda_{sp}O(X, \tau)$, $p\Lambda_{sp}O(X, \tau)$, $\alpha\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$).

Theorem 15. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_{sp}O(X, \tau)$.

(2) $A^{(\Lambda,sp)} \in r\Lambda_{sp}C(X, \tau)$.

(3) $A^{(\Lambda,sp)} \in \beta\Lambda_{sp}O(X,\tau)$.

(4) $A^{(\Lambda,sp)} \in s\Lambda_{sp}O(X,\tau)$.

Proof. (1) $\Rightarrow$ (2): Let $A \in \beta\Lambda_{sp}O(X,\tau)$. Then

$$A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$$

and hence

$$A^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}.$$

This shows that $A^{(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Consequently, we obtain $A^{(\Lambda,sp)} \in r\Lambda_{sp}C(X,\tau)$.

(2) $\Rightarrow$ (3): Let $A^{(\Lambda,sp)} \in r\Lambda_{sp}C(X,\tau)$. Then $A^{(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence

$$\begin{aligned} A^{(\Lambda,sp)} &= [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[[A^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}. \end{aligned}$$

Therefore, $A^{(\Lambda,sp)} \in \beta\Lambda_{sp}O(X,\tau)$.

(3) $\Rightarrow$ (4): Let $A^{(\Lambda,sp)} \in \beta\Lambda_{sp}O(X,\tau)$. Then $A^{(\Lambda,sp)} \subseteq [[[A^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence

$$\begin{aligned} A^{(\Lambda,sp)} &\subseteq [[[A^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}. \end{aligned}$$

Thus, $A^{(\Lambda,sp)} \in s\Lambda_{sp}O(X,\tau)$.

(4) $\Rightarrow$ (1): Let $A^{(\Lambda,sp)} \in s\Lambda_{sp}O(X,\tau)$. Then, we have $A \subseteq A^{(\Lambda,sp)} \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $A \in \beta\Lambda_{sp}O(X,\tau)$. $\square$

Corollary 16. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_{sp}C(X,\tau)$.

(2) $A_{(\Lambda,sp)} \in r\Lambda_{sp}O(X,\tau)$.

(3) $A_{(\Lambda,sp)} \in \beta\Lambda_{sp}C(X,\tau)$.

(4) $A_{(\Lambda,sp)} \in s\Lambda_{sp}C(X,\tau)$.

4 On characterizations of Λ_{sp}-extremally disconnected spaces

In this section, we introduce the notion of Λ_{sp}-extremally disconnected spaces and investigate some characterizations of such spaces.

Definition 17. *A topological space (X,τ) is called Λ_{sp}-extremally disconnected if $V^{(\Lambda,sp)}$ is (Λ,sp)-open in X for every (Λ,sp)-open set V of X.*

Theorem 18. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-extremally disconnected.

(2) $A_{(\Lambda,sp)}$ is (Λ,sp)-closed for every (Λ,sp)-closed set A of X.

(3) $[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every subset A of X.

(4) Every $s(\Lambda,sp)$-open set is $p(\Lambda,sp)$-open.

(5) The (Λ,sp)-closure of every $\beta(\Lambda,sp)$-open set of X is (Λ,sp)-open.

(6) Every $\beta(\Lambda,sp)$-open set is $p(\Lambda,sp)$-open.

(7) For every subset A of X, A is $\alpha(\Lambda,sp)$-open if and only if A is $s(\Lambda,sp)$-open.

Proof. (1) $\Rightarrow$ (2): Let A be a (Λ,sp)-closed set. Then $X-A$ is (Λ,sp)-open. By (1), we have $[X-A]^{(\Lambda,sp)} = X-A_{(\Lambda,sp)}$ is (Λ,sp)-open and hence $A_{(\Lambda,sp)}$ is (Λ,sp)-closed.

(2) $\Rightarrow$ (3): Let A be any subset of X. Then, we have $X-A_{(\Lambda,sp)}$ is (Λ,sp)-closed in X. By (2), $[X-A_{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ,sp)-closed and hence $[A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is (Λ,sp)-open. Thus, $[A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(3) $\Rightarrow$ (4): Let A be a $s(\Lambda,sp)$-open set. By (3), we have $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, A is $p(\Lambda,sp)$-open.

(4) $\Rightarrow$ (5): Let A be a $\beta(\Lambda,sp)$-open set. By Theorem 15, we have $A^{(\Lambda,sp)}$ is $s(\Lambda,sp)$-open and by (4), $A^{(\Lambda,sp)}$ is $p(\Lambda,sp)$-open. Thus,

$$A^{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$$

and hence $A^{(\Lambda,sp)}$ is (Λ,sp)-open.

(5) $\Rightarrow$ (6): Let A be a $\beta(\Lambda,sp)$-open set. By (5), $A^{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Therefore,

$$A \subseteq A^{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$$

and hence A is $p(\Lambda,sp)$-open.

(6) $\Rightarrow$ (7): Let A be a $s(\Lambda,sp)$-open set. Then A is $\beta(\Lambda,sp)$-open and by (6), A is $p(\Lambda,sp)$-open. Since A is $s(\Lambda,sp)$-open and $p(\Lambda,sp)$-open, we have A is $\alpha(\Lambda,sp)$-open.

(7) $\Rightarrow$ (1): Let A be a (Λ,sp)-open set. Then, we have $A^{(\Lambda,sp)}$ is $s(\Lambda,sp)$-open and by (7), $A^{(\Lambda,sp)}$ is $\alpha(\Lambda,sp)$-open. Therefore,

$$\begin{aligned} A^{(\Lambda,sp)} &\subseteq [[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &= [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \end{aligned}$$

and hence $A^{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that $A^{(\Lambda,sp)}$ is (Λ, sp)-open. Consequently, we obtain (X,τ) is Λ_{sp}-extremally disconnected. □

Theorem 19. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-extremally disconnected.

(2) For every (Λ, sp)-open sets U and V such that $U \cap V = \emptyset$, there exist disjoint (Λ, sp)-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$.

(3) $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$ for every (Λ, sp)-open sets U and V such that $U \cap V = \emptyset$.

(4) $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \cap U^{(\Lambda,sp)} = \emptyset$ for every subset A of X and every (Λ, sp)-open set U such that $A \cap U = \emptyset$.

Proof. (1) ⇒ (2): Suppose that (X,τ) is Λ_{sp}-extremally disconnected. Let U and V be (Λ, sp)-open sets such that $U \cap V = \emptyset$. Then, we have $U^{(\Lambda,sp)}$ and $X - U^{(\Lambda,sp)}$ are disjoint (Λ, sp)-closed sets containing U and V, respectively.

(2) ⇒ (3): Let U and V be (Λ, sp)-open sets such that $U \cap V = \emptyset$. By (2), there exist disjoint (Λ, sp)-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$. Thus, $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} \subseteq F \cap H = \emptyset$ and hence $U^{(\Lambda,sp)} \cap V^{(\Lambda,sp)} = \emptyset$.

(3) ⇒ (4): Let A be any subset of X and U be a (Λ, sp)-open set such that $A \cap U = \emptyset$. Since $[A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ, sp)-open and

$$[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \cap U = \emptyset,$$

by (3), $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \cap U^{(\Lambda,sp)} = \emptyset$.

(4) ⇒ (1): Let U be a (Λ, sp)-open set. Then $(X - U^{(\Lambda,sp)}) \cap U = \emptyset$. Since $X - U^{(\Lambda,sp)}$ is (Λ, sp)-open and by (4),

$$[[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \cap [X - U^{(\Lambda,sp)}]^{(\Lambda,sp)} = \emptyset.$$

Since U is (Λ, sp)-open, we have

$$U^{(\Lambda,sp)} \cap [X - [U^{(\Lambda,sp)}]_{(\Lambda,sp)}] = \emptyset$$

and hence $U^{(\Lambda,sp)} \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Thus, $U^{(\Lambda,sp)}$ is (Λ, sp)-open. This shows that (X,τ) is Λ_{sp}-extremally disconnected. □

Theorem 20. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-extremally disconnected.

(2) For every $r(\Lambda, sp)$-open set of X is (Λ, sp)-closed.

(3) For every $r(\Lambda, sp)$-closed set of X is (Λ, sp)-open.

Proof. (1) ⇒ (2): Suppose that (X,τ) is Λ_{sp}-extremally disconnected. Let U be a $r(\Lambda, sp)$-open set. Then $U = [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Since U is (Λ, sp)-open, we have $U^{(\Lambda,sp)}$ is (Λ, sp)-open. Thus,

$$U = [U^{(\Lambda,sp)}]_{(\Lambda,sp)} = U^{(\Lambda,sp)}$$

and hence U is (Λ, sp)-closed.

(2) ⇒ (1): Suppose that every $r(\Lambda, sp)$-open set of X is (Λ, sp)-closed. Let U be a (Λ, sp)-open set. Since $[U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open, we have $[U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ, sp)-closed and hence

$$U^{(\Lambda,sp)} \subseteq [[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [U^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Thus, $U^{(\Lambda,sp)}$ is (Λ, sp)-open. This shows that (X,τ) is Λ_{sp}-extremally disconnected.

(2) ⇔ (3): The proof is obvious. □

Theorem 21. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-extremally disconnected.

(2) The (Λ, sp)-closure of every $s(\Lambda, sp)$-open set of X is (Λ, sp)-open.

(3) The (Λ, sp)-closure of every $p(\Lambda, sp)$-open set of X is (Λ, sp)-open.

(4) The (Λ, sp)-closure of every $r(\Lambda, sp)$-open set of X is (Λ, sp)-open.

Proof. (1) ⇒ (2): Let U be a $s(\Lambda, sp)$-open set. Then U is $\beta(\Lambda, sp)$-open and by Theorem 18, $U^{(\Lambda,sp)}$ is (Λ, sp)-open.

(1) ⇒ (3): Let U be a $p(\Lambda, sp)$-open set. Then U is $\beta(\Lambda, sp)$-open. By Theorem 18, we have $U^{(\Lambda,sp)}$ is (Λ, sp)-open.

(2) ⇒ (4): Let U be a $r(\Lambda, sp)$-open set. Then U is $s(\Lambda, sp)$-open. By (2), we have $U^{(\Lambda,sp)}$ is (Λ, sp)-open.

(3) ⇒ (4): Let U be a $r(\Lambda, sp)$-open set. Then U is $p(\Lambda, sp)$-open and by (3), $U^{(\Lambda,sp)}$ is (Λ, sp)-open.

(4) ⇒ (1) Suppose that the (Λ, sp)-closure of every $r(\Lambda, sp)$-open set of X is (Λ, sp)-open. Let U be a (Λ, sp)-open set. Then, we have $[U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $r(\Lambda, sp)$-open and hence $[[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ is (Λ, sp)-open. Therefore,

$$\begin{aligned} U^{(\Lambda,sp)} &\subseteq [[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \\ &= [[[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &= [U^{(\Lambda,sp)}]_{(\Lambda,sp)}. \end{aligned}$$

Thus, $U^{(\Lambda,sp)}$ is (Λ, sp)-open. This shows that (X,τ) is Λ_{sp}-extremally disconnected. □

5 On characterizations of Λ_{sp}-submaximal spaces

In this section, we introduce the notion of Λ_{sp}-extremally disconnected spaces. Moreover, several characterizations of Λ_{sp}-extremally disconnected spaces are investigated.

Definition 22. *A subset A of a topological space (X,τ) is said to be locally (Λ, sp)-closed if $A = U \cap F$, where $U \in \Lambda_{sp}O(X,\tau)$ and F is a (Λ, sp)-closed set.*

Lemma 23. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is locally (Λ, sp)-closed.

(2) $A = U \cap A^{(\Lambda,sp)}$ for some $U \in \Lambda_{sp}O(X,\tau)$.

(3) $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed.

(4) $A \cup [X - A^{(\Lambda,sp)}] \in \Lambda_{sp}O(X,\tau)$.

(5) $A \subseteq [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)}$.

Proof. (1) $\Rightarrow$ (2): Let A be a locally (Λ, sp)-closed set. Then, there exist a (Λ, sp)-open set U and a (Λ, sp)-closed set F such that $A = U \cap F$. Since $A \subseteq F$, we have $A^{(\Lambda,sp)} \subseteq F^{(\Lambda,sp)} = F$. Since $A \subseteq U$, $A \subseteq U \cap A^{(\Lambda,sp)} \subseteq U \cap F = A$ and hence $A = U \cap A^{(\Lambda,sp)}$.

(2) $\Rightarrow$ (3): Suppose that $A = U \cap A^{(\Lambda,sp)}$ for some $U \in \Lambda_{sp}O(X,\tau)$. Then, we have

$$\begin{aligned} A^{(\Lambda,sp)} - A &= [X - [U \cap A^{(\Lambda,sp)}]] \cap A^{(\Lambda,sp)} \\ &= (X - U) \cap A^{(\Lambda,sp)} \end{aligned}$$

and hence $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed.

(3) $\Rightarrow$ (4): Since

$$X - [A^{(\Lambda,sp)} - A] = [X - A^{(\Lambda,sp)}] \cup A$$

and by (3), $A \cup [X - A^{(\Lambda,sp)}] \in \Lambda_{sp}O(X,\tau)$.

(4) $\Rightarrow$ (5): By (4), we have

$$A \subseteq A \cup [X - A^{(\Lambda,sp)}] = [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)}.$$

(5) $\Rightarrow$ (1): We put $U = [A \cup [X - A^{(\Lambda,sp)}]]_{(\Lambda,sp)}$. Then $U \in \Lambda_{sp}O(X,\tau)$ and

$$\begin{aligned} A = A \cap U &\subseteq U \cap A^{(\Lambda,sp)} \\ &\subseteq [A \cup [X - A^{(\Lambda,sp)}]] \cap A^{(\Lambda,sp)} \\ &= A \cap A^{(\Lambda,sp)} = A. \end{aligned}$$

Thus, $A = U \cap A^{(\Lambda,sp)}$. Consequently, we obtain A is a locally (Λ, sp)-closed set. □

Definition 24. *A subset A of a topological space (X,τ) is said to be:*

(i) Λ_{sp}-dense if $A^{(\Lambda,sp)} = X$;

(ii) Λ_{sp}-codense if its complement is Λ_{sp}-dense.

Definition 25. *A topological space (X,τ) is said to be Λ_{sp}-submaximal if for each Λ_{sp}-dense subset of X is (Λ, sp)-open.*

Theorem 26. *A topological space (X,τ) is Λ_{sp}-submaximal if and only if for each Λ_{sp}-codense subset of X is (Λ, sp)-closed.*

Proof. Let A be a Λ_{sp}-codense subset of X. Then $X - A$ is Λ_{sp}-dense. Since (X,τ) is Λ_{sp}-submaximal, we have $X - A$ is (Λ, sp)-open and hence A is (Λ, sp)-closed.

Conversely, suppose that every Λ_{sp}-codense subset of X is (Λ, sp)-closed. Let A be a Λ_{sp}-dense subset of X. Then $X - A$ is Λ_{sp}-codense and hence $X - A$ is (Λ, sp)-closed. This shows that A is (Λ, sp)-open. Thus, (X,τ) is Λ_{sp}-submaximal. □

Theorem 27. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) is Λ_{sp}-submaximal.

(2) Every subset of X is a locally (Λ, sp)-closed set.

(3) Every subset of X is the union of a (Λ, sp)-open set and a (Λ, sp)-closed set.

(4) Every Λ_{sp}-dense set of X is the intersection of a (Λ, sp)-closed set and a (Λ, sp)-open set.

(5) Every Λ_{sp}-codense set of X is the union of a (Λ, sp)-open set and a (Λ, sp)-closed set.

Proof. (1) $\Rightarrow$ (2): Suppose that (X,τ) is Λ_{sp}-submaximal. Let A be any subset of X. Then

$$\begin{aligned} &[X - [A^{(\Lambda,sp)} - A]]^{(\Lambda,sp)} \\ &= [A \cup [X - A^{(\Lambda,sp)}]]^{(\Lambda,sp)} = X. \end{aligned}$$

Thus, $X - [A^{(\Lambda,sp)} - A]$ is Λ_{sp}-dense and hence

$$X - [A^{(\Lambda,sp)} - A]$$

is (Λ, sp)-open. Since

$$X - [A^{(\Lambda,sp)} - A] = A \cup [X - A^{(\Lambda,sp)}],$$

we have $A \cup [X - A^{(\Lambda,sp)}]$ is (Λ, sp)-open. Therefore, $A = [A \cup [X - A^{(\Lambda,sp)}]] \cap A^{(\Lambda,sp)}$ is a locally (Λ, sp)-closed set.

(2) $\Leftrightarrow$ (3): Suppose that every subset of X is a locally (Λ, sp)-closed set. Let A be any subset of X. By (2), we have $X - A = U \cap F$, where U is a (Λ, sp)-open set and F is a (Λ, sp)-closed set. This implies that $A = (X - U) \cup (X - K)$, where $X - U$ is a (Λ, sp)-closed set and $X - F$ is a (Λ, sp)-open set. The converse is similar.

(2) $\Rightarrow$ (4): This is obvious.

(4) $\Leftrightarrow$ (5): The proof is obvious.

(4) $\Rightarrow$ (1): Let A be a Λ_{sp}-dense set. By (4), there exist a (Λ, sp)-open set U and a (Λ, sp)-closed set F such that $A = U \cap F$. Since $A \subseteq F$ and A is a Λ_{sp}-dense set, $X \subseteq F$ and hence $F = X$. This shows that $A = U$ is (Λ, sp)-open. Consequently, we obtain (X, τ) is Λ_{sp}-submaximal. □

Definition 28. *A subset A of a topological space (X, τ) is said to be:*

(i) $t(\Lambda, sp)$-set if $A_{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(ii) $s(\Lambda, sp)$-regular if A is a $t(\Lambda, sp)$-set and $s(\Lambda, sp)$-open;

(iii) $\mathcal{B}(\Lambda, sp)$-set if $A = U \cap V$, where

$$U \in \Lambda_{sp}O(X, \tau)$$

and V is a $t(\Lambda, sp)$-set;

(iv) $\mathcal{AB}(\Lambda, sp)$-set if $A = U \cap V$, where

$$U \in \Lambda_{sp}O(X, \tau)$$

and V is a $s(\Lambda, sp)$-regular set.

Theorem 29. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-submaximal.

(2) $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed for every subset A of X.

(3) Every subset of X is locally (Λ, sp)-closed.

(4) Every subset of X is a $\mathcal{B}(\Lambda, sp)$-set.

(5) Every Λ_{sp}-dense set of X is a $\mathcal{B}(\Lambda, sp)$-set.

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ) is Λ_{sp}-submaximal. Let A be any subset of X. Then, we have

$$\begin{aligned}&[X - [A^{(\Lambda,sp)} - A]]^{(\Lambda,sp)} \\ &= [A \cup [X - A^{(\Lambda,sp)}]]^{(\Lambda,sp)} = X\end{aligned}$$

and hence $X - [A^{(\Lambda,sp)} - A]$ is Λ_{sp}-dense. By the hypothesis, $X - [A^{(\Lambda,sp)} - A]$ is (Λ, sp)-open. Thus, $A^{(\Lambda,sp)} - A$ is (Λ, sp)-closed.

(2) $\Rightarrow$ (3): It is obvious by Lemma 23.

(3) $\Rightarrow$ (4): It follows from the fact that every locally (Λ, sp)-closed set is a $\mathcal{B}(\Lambda, sp)$-set.

(4) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): Let A be a Λ_{sp}-dense subset of X. By (5), we have A is a $\mathcal{B}(\Lambda, sp)$-set and hence $A = U \cap F$, where U is (Λ, sp)-open and

$$F_{(\Lambda,sp)} = [F^{(\Lambda,sp)}]_{(\Lambda,sp)}.$$

Since $A \subseteq F$, $A^{(\Lambda,sp)} \subseteq F^{(\Lambda,sp)}$ and $X = F^{(\Lambda,sp)}$. Thus, $X = [F^{(\Lambda,sp)}]_{(\Lambda,sp)} = F_{(\Lambda,sp)}$ which implies that $F = X$. Consequently, we obtain

$$A = U \cap F = U \cap X = U$$

and hence A is (Λ, sp)-open. This shows that (X, τ) is Λ_{sp}-submaximal. □

Theorem 30. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-submaximal.

(2) Every subset of X is a $\mathcal{B}(\Lambda, sp)$-set.

(3) Every $\beta(\Lambda, sp)$-open set is a $\mathcal{B}(\Lambda, sp)$-set.

(4) Every Λ_{sp}-dense set is a $\mathcal{B}(\Lambda, sp)$-set.

Proof. (1) $\Rightarrow$ (2): It follows from Theorem 29.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): It follows from the fact that every Λ_{sp}-dense set is a $\beta(\Lambda, sp)$-open set.

(4) $\Rightarrow$ (1): It follows from Theorem 29. □

6 On characterizations of Λ_{sp}-hyperconnected spaces

We begin this section by introducing the notion of Λ_{sp}-hyperconnected spaces.

Definition 31. *A topological space (X, τ) is called Λ_{sp}-hyperconnected if U is Λ_{sp}-dense for every nonempty (Λ, sp)-open set U of X.*

Definition 32. *A subset N of a topological space (X, τ) is said to be Λ_{sp}-nowhere dense if $[N^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$.*

Lemma 33. *A subset A of a topological space (X, τ) is $s(\Lambda, sp)$-open if and only if there exists a (Λ, sp)-open set U such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$.*

Proof. Suppose that A is $s(\Lambda, sp)$-open. Then, we have $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Put $U = A_{(\Lambda,sp)}$. Then U is a $s(\Lambda, sp)$-open set such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$.

Conversely, suppose that there exists a (Λ, sp)-open set U such that $U \subseteq A \subseteq U^{(\Lambda,sp)}$. Then $U \subseteq A_{(\Lambda,sp)}$ and hence $U^{(\Lambda,sp)} \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $A \subseteq U^{(\Lambda,sp)}$, we have $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$. This shows that A is $s(\Lambda, sp)$-open. □

Theorem 34. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-hyperconnected.

(2) A is Λ_{sp}-dense or Λ_{sp}-nowhere dense for every subset A of X.

(3) $U \cap V \neq \emptyset$ for every nonempty (Λ, sp)-open sets U and V of X.

(4) $U \cap V \neq \emptyset$ for every nonempty $s(\Lambda, sp)$-open sets U and V of X.

Proof. (1) ⇒ (2): Suppose that A is not Λ_{sp}-nowhere dense. Then $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$. Since (X, τ) is Λ_{sp}-hyperconnected, $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = X$ and hence $X \subseteq A^{(\Lambda,sp)}$. Consequently, we obtain $A^{(\Lambda,sp)} = X$. This shows that A is Λ_{sp}-dense.

(2) ⇒ (3): Suppose that $U \cap V = \emptyset$ for some nonempty (Λ, sp)-open sets U and V of X. Then, we have $U^{(\Lambda,sp)} \cap V = \emptyset$ and hence U is not Λ_{sp}-dense. Since U is (Λ, sp)-open, we have $U \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and hence U is not Λ_{sp}-nowhere dense.

(3) ⇒ (4): Suppose that $U \cap V = \emptyset$ for some nonempty $s(\Lambda, sp)$-open sets U and V of X. By Lemma 33, there exist $G, W \in \Lambda_{sp}O(X, \tau)$ such that $G \subseteq U \subseteq G^{(\Lambda,sp)}$ and $W \subseteq V \subseteq W^{(\Lambda,sp)}$. Since U and V are nonempty, G and W are nonempty. Moreover, we have $G \cap W \subseteq U \cap V = \emptyset$.

(4) ⇒ (1): Suppose that (X, τ) is not Λ_{sp}-hyperconnected. There exists a nonempty (Λ, sp)-open set G of X such that $G^{(\Lambda,sp)} \neq X$. Thus, $X - G^{(\Lambda,sp)} \neq \emptyset$ and hence $[X - G^{(\Lambda,sp)}] \cap G = \emptyset$. This is a contradiction. □

Theorem 35. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-hyperconnected.

(2) V is Λ_{sp}-dense for every nonempty $\beta(\Lambda, sp)$-open set V of X.

(3) $V \cup [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = X$ for every nonempty $\beta(\Lambda, sp)$-open set V of X.

Proof. (1) ⇒ (2): Suppose that (X, τ) is Λ_{sp}-hyperconnected. Let V be a nonempty $\beta(\Lambda, sp)$-open set. It follows that $[V^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$ and hence $X = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = V^{(\Lambda,sp)}$. This shows that V is Λ_{sp}-dense.

(2) ⇒ (3): Let V be a nonempty $\beta(\Lambda, sp)$-open set. By (2), we have

$$V \cup [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = V \cup X_{(\Lambda,sp)} = X.$$

(3) ⇒ (1): Let V be a nonempty (Λ, sp)-open set. It follows (3) that

$$V^{(\Lambda,sp)} \supseteq V \cup [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = X$$

and hence $V^{(\Lambda,sp)} = X$. Consequently, we obtain (X, τ) is Λ_{sp}-hyperconnected. □

Theorem 36. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_{sp}-hyperconnected.

(2) V is Λ_{sp}-dense for every nonempty $s(\Lambda, sp)$-open set V of X.

(3) $V \cup [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = X$ for every nonempty $s(\Lambda, sp)$-open set V of X.

Proof. The proof follows from Theorem 35. □

7 On upper and lower (Λ, sp)-continuous multifunctions

In this section, we introduce the notions of upper and lower (Λ, sp)-continuous multifunctions. Moreover, some characterizations of upper and lower (Λ, sp)-continuous multifunctions are discussed.

Definition 37. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) upper (Λ, sp)-continuous if for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V$;

(ii) lower (Λ, sp)-continuous if for each $x \in X$ and each (Λ, sp)-open set V of Y such that

$$F(x) \cap V \neq \emptyset,$$

there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

Theorem 38. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper (Λ, sp)-continuous.

(2) $F^+(V)$ is (Λ, sp)-open in X for every (Λ, sp)-open set V of Y.

(3) $F^-(K)$ is (Λ, sp)-closed in X for every (Λ, sp)-closed set K of Y.

(4) $[F^-(B)]^{(\Lambda,sp)} \subseteq F^-[B^{(\Lambda,sp)}]$ for every subset B of Y.

(5) $F^+[B_{(\Lambda,sp)}] \subseteq [F^+(B)]_{(\Lambda,sp)}$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let $V \in \Lambda_{sp}O(Y, \sigma)$ and $x \in F^+(V)$. Then $F(x) \subseteq V$. There exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in [F^+(V)]_{(\Lambda,sp)}$. This shows that $F^+(V) \subseteq [F^+(V)]_{(\Lambda,sp)}$. Consequently, we obtain $F^+(V)$ is (Λ, sp)-open in X.

(2) $\Rightarrow$ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By Lemma 10, we have $B^{(\Lambda,sp)}$ is (Λ, sp)-closed and by (3), $[F^-(B)]^{(\Lambda,sp)} \subseteq [F^-[B^{(\Lambda,sp)}]]^{(\Lambda,sp)} = F^-[B^{(\Lambda,sp)}]$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - [F^+(B)]_{(\Lambda,sp)} &= [X - F^+(B)]^{(\Lambda,sp)} \\ &= [F^-(Y - B)]^{(\Lambda,sp)} \\ &\subseteq F^-[[Y - B]^{(\Lambda,sp)}] \\ &= F^-[Y - B_{(\Lambda,sp)}] \\ &= X - F^+[B_{(\Lambda,sp)}] \end{aligned}$$

and hence $F^+[B_{(\Lambda,sp)}] \subseteq [F^+(B)]_{(\Lambda,sp)}$.

(5) $\Rightarrow$ (1): Let $x \in X$ and $V \in \Lambda_{sp}O(X, \tau)$ such that $F(x) \subseteq V$. Then $x \in F^+(V) = F^+[V_{(\Lambda,sp)}] \subseteq [F^+(V)]_{(\Lambda,sp)}$. There exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper (Λ, sp)-continuous. □

Theorem 39. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous.

(2) $F^-(V)$ is (Λ, sp)-open in X for every (Λ, sp)-open set V of Y.

(3) $F^+(K)$ is (Λ, sp)-closed in X for every (Λ, sp)-closed set K of Y.

(4) $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+[B^{(\Lambda,sp)}]$ for every subset B of Y.

(5) $F[A^{(\Lambda,sp)}] \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X.

(6) $F^-[B_{(\Lambda,sp)}] \subseteq [F^-(B)]_{(\Lambda,sp)}$ for every subset B of Y.

Proof. We prove only the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) being the proofs of the other similar to those of Theorem 38.

(4) $\Rightarrow$ (5): Let A be any subset of X. By (4), $A^{(\Lambda,sp)} \subseteq [F^+[F(A)]]^{(\Lambda,sp)} \subseteq F^+[[F(A)]^{(\Lambda,sp)}]$ and hence $F[A^{(\Lambda,sp)}] \subseteq [F(A)]^{(\Lambda,sp)}$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. By (5),

$$\begin{aligned} F[[F^+(Y - B)]^{(\Lambda,sp)}] &\subseteq [F[F^+(Y - B)]]^{(\Lambda,sp)} \\ &\subseteq [Y - B]^{(\Lambda,sp)} \\ &= Y - B_{(\Lambda,sp)}. \end{aligned}$$

Since

$$\begin{aligned} F[[F^+(Y - B)]^{(\Lambda,sp)}] &= F[[X - F^-(B)]^{(\Lambda,sp)}] \\ &= F[X - [F^-(B)]_{(\Lambda,sp)}], \end{aligned}$$

we have

$$\begin{aligned} X - [F^-(B)]_{(\Lambda,sp)} &\subseteq F^+[Y - B_{(\Lambda,sp)}] \\ &= X - F^-[B_{(\Lambda,sp)}] \end{aligned}$$

and hence $F^-[B_{(\Lambda,sp)}] \subseteq [F^-(B)]_{(\Lambda,sp)}$. □

Definition 40. *Let A be a subset of a topological space (X, τ). The $\theta(\Lambda, sp)$-closure of A, $A^{\theta(\Lambda,sp)}$, is defined as follows: $A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap U^{(\Lambda,sp)} \neq \emptyset$ for each $U \in \Lambda_{sp}O(X, \tau)$ containing $x\}$.*

Definition 41. *A subset A of a topological space (X, τ) is called $\theta(\Lambda, sp)$-closed if $A = A^{\theta(\Lambda,sp)}$. The complement of a $\theta(\Lambda, sp)$-closed set is said to be $\theta(\Lambda, sp)$-open.*

Lemma 42. *Let A be a subset of a topological space (X, τ). Then, $x \in A^{(\Lambda,sp)}$ if and only if $U \cap A \neq \emptyset$ for every $U \in \Lambda_{sp}O(X, \tau)$ containing x.*

Lemma 43. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is (Λ, sp)-open in X, then $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$.

(2) $A^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed.

Proof. (1) In general, the holds that

$$A^{(\Lambda,sp)} \subseteq A^{\theta(\Lambda,sp)}.$$

Suppose that $x \notin A^{(\Lambda,sp)}$. By Lemma 42, there exists $U \in \Lambda_{sp}O(X,\tau)$ containing x such that $U \cap A = \emptyset$; hence $A \cap U^{(\Lambda,sp)} = \emptyset$, since A is (Λ, sp)-open. This shows that $x \notin A^{\theta(\Lambda,sp)}$. Consequently, we obtain $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$.

(2) Let $x \in X - A^{\theta(\Lambda,sp)}$. Then $x \notin A^{\theta(\Lambda,sp)}$. There exists $U_x \in \Lambda_{sp}O(X,\tau)$ containing x such that $A \cap U_x^{(\Lambda,sp)} = \emptyset$ and hence $U_x \cap A^{\theta(\Lambda,sp)} = \emptyset$. Thus, $x \in U_x \subseteq X - A^{\theta(\Lambda,sp)}$. Therefore,

$$X - A^{\theta(\Lambda,sp)} = \cup U_x \in \Lambda_{sp}O(X,\tau).$$

This shows that $A^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed. □

Definition 44. *A topological space (X,τ) is said to be Λ_{sp}-regular if for each (Λ, sp)-closed set F and each $x \notin F$, there exist disjoint (Λ, sp)-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 45. *A topological space (X,τ) is Λ_{sp}-regular if and only if for each $x \in X$ and each (Λ, sp)-open set U containing x, there exists a (Λ, sp)-open set V such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq U$.*

Proof. Let $x \in X$ and U be a (Λ, sp)-open set containing x. Then, we have $x \notin X - U$ and $X - U$ is (Λ, sp)-closed. There exist disjoint (Λ, sp)-open sets V and W such that $x \in U$ and $X - U \subseteq W$. Thus, $V \subseteq X - W \subseteq U$. Since $X - W$ is (Λ, sp)-closed, $V^{(\Lambda,sp)} \subseteq X - W \subseteq U$ and hence

$$x \in V \subseteq V^{(\Lambda,sp)} \subseteq U.$$

Conversely, let F be a (Λ, sp)-closed set and let $x \notin F$. Then $x \in X - F \in \Lambda_{sp}O(X,\tau)$ and there exists a (Λ, sp)-open set V such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq X - F$. By Lemma 13, we have $F \subseteq X - V^{(\Lambda,sp)} = [X - V]_{(\Lambda,sp)} \in \Lambda_{sp}O(X,\tau)$. This shows that (X,τ) is Λ_{sp}-regular. □

Lemma 46. *Let (X,τ) be a Λ_{sp}-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$ for every subset A of X.

(2) Every (Λ, sp)-open set is $\theta(\Lambda, sp)$-open.

Proof. (1) In general, we have $A^{(\Lambda,sp)} \subseteq A^{\theta(\Lambda,sp)}$ for every subset A of X. Next, we show that $A^{\theta(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}$. Let $x \in A^{\theta(\Lambda,sp)}$ and U be any (Λ, sp)-open set containing x. By Lemma 45, there exists a (Λ, sp)-open set V such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq U$. Since $x \in A^{\theta(\Lambda,sp)}$, it follows that $A \cap V^{(\Lambda,sp)} \neq \emptyset$ and hence $U \cap A \neq \emptyset$. By Lemma 42, we have $x \in A^{(\Lambda,sp)}$. Thus, $A^{\theta(\Lambda,sp)} \subseteq A^{(\Lambda,sp)}$.

(2) Let $V \in \Lambda_{sp}O(X,\tau)$. By (1), we have

$$X - V = [X - V]^{(\Lambda,sp)} = [X - V]^{\theta(\Lambda,sp)}$$

and hence $X - V$ is $\theta(\Lambda, sp)$-closed. Therefore, V is $\theta(\Lambda, sp)$-open. □

Theorem 47. *Let (Y,σ) be a Λ_{sp}-regular space. For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is upper (Λ, sp)-continuous.

(2) $F^-[B^{\theta(\Lambda,sp)}]$ is (Λ, sp)-closed in X for every subset B of Y.

(3) $F^-(K)$ is (Λ, sp)-closed in X for every $\theta(\Lambda, sp)$-closed set K of Y.

(4) $F^+(V)$ is (Λ, sp)-open in X for every $\theta(\Lambda, sp)$-open set V of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 43, we have $B^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed in Y. Since F is upper (Λ, sp)-continuous, by Theorem 38, $F^-[B^{\theta(\Lambda,sp)}]$ is (Λ, sp)-closed in X.

(2) ⇒ (3): Let K be any $\theta(\Lambda, sp)$-closed set of Y. Then $K^{\theta(\Lambda,sp)} = K$ and by (2), we have $F^-(K)$ is (Λ, sp)-closed in X.

(3) ⇒ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) ⇒ (1): Let V be any (Λ, sp)-open set of Y. Since (Y,σ) is Λ_{sp}-regular, we have V is $\theta(\Lambda, sp)$-open in Y and by (4), $F^+(V)$ is (Λ, sp)-open in X. Thus, F is upper (Λ, sp)-continuous by Theorem 38. □

Theorem 48. *Let (Y,σ) be a Λ_{sp}-regular space. For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous.

(2) $F^+[B^{\theta(\Lambda,sp)}]$ is (Λ, sp)-closed in X for every subset B of Y.

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\theta(\Lambda, sp)$-closed set K of Y.

(4) $F^-(V)$ is (Λ, sp)-open in X for every $\theta(\Lambda, sp)$-open set V of Y.

Proof. The proof follows from Theorem 47. □

8 Conclusion

This paper deals with the concept of (Λ, sp)-closed sets which is defined as the intersection of a Λ_{sp}-set and a β-closed set. Some basic properties of (Λ, sp)-closed sets and (Λ, sp)-open sets are considered. Especially, several characterizations of Λ_{sp}-submaximal spaces, Λ_{sp}-extremally disconnected spaces and Λ_{sp}-hyperconnected spaces are obtained. Moreover, some characterizations of upper and lower (Λ, sp)-continuous multifunctions are explored. The ideas and results of this paper may motivate further research.

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