Graded 2-Absorbing Submodules over Non-Commutative Rings

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Abstract: Let R be a non-commutative G-graded ring with unity and M be a G-graded left R-module. We define the concept of a graded 2-absorbing submodule and show that if the ring is commutative, then the concept is the same as the original definition of that of K. Al-Zoubi and R. Abu-Dawwas. We give an example to show that in general these two concepts are different. Many properties of graded 2-absorbing submodules are introduced which aresimilartotheresultsforcommutativerings.

Key–Words: Graded 2-absorbing submodules, graded 2-absorbing ideals, graded prime submodules, graded prime ideals.

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1 Introduction

The study of graded rings comes into being normally out of the study of affine schemes and admits them to establish and consolidate arguments by induction. However, this is not just an algebraic deception. The concept of grading in algebra, in particular graded modules is fundamental in the study of homological aspect of rings. Plenty of the contemporary growth of the commutative algebra give preference to graded rings. Graded rings play a principal role in algebraic geometry and commutative algebra. Gradings come into sight in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become progressively substantial and as a consequence, the graded analogues of various concepts are extensively studied.

In 2016, Naghani and Moghimi in [7] proposed the concept of graded 2-absorbing ideals of commutative rings with unity, which is a generalization of graded prime ideals, and investigated some properties. They defined a graded 2-absorbing ideal P of a commutative ring R with unity to be a proper graded ideal of R and if whenever $x, y, z \in h(R)$ with $xyz \in P$, then $xy \in P$ or $yz \in P$ or $xz \in P$. In 2019, Al-Zoubi, Abu-Dawwas and Çeken in [4] introduced several properties concerning graded 2-absorbing ideals over a commutative ring with unity. Al-Zoubi and Abu-Dawwas in [3] extended graded 2-absorbing ideals to graded 2-absorbing submodules over a commutative ring with unity. They defined a proper graded R-submodule N of M over a commutative ring R with unity to be graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then either $am \in N$ or $bm \in N$ or $ab \in (N :_R M) = \{r \in R : rM \subseteq N\}$. One can see that graded 2-absorbing submodules are generalization of graded prime submodules. Furthermore, it is recognizable that graded 2-absorbing ideals are special cases of graded 2-absorbing submodules.

In recent years, the study of the graded absorbing property of graded rings, graded modules and related concepts have been some of the topics of interest in the development of the graded ring and graded module theory. In this article, we study the concept of graded 2-absorbing submodules over non-commutative rings. Also, we introduce the concept of graded strong 2-absorbing submodules and show that in general if R is not a commutative ring, then the concepts of graded 2-absorbing and graded strong 2-absorbing submodules are not the same (Example 6). If R is commutative, then the concept of graded 2-absorbing submodules coincides with that of the original definition introduced by Al-Zoubi and Abu-Dawwas in [3].

Graded prime submodules and graded prime ideals over non-commutative ring have been studied by Abu-Dawwas, Bataineh and Al-Muanger in [2]. A proper graded R-submodule N of M is said to be graded prime if whenever $x \in h(R)$ and $m \in h(M)$ such that $xRm \subseteq N$, then $m \in N$ or $x \in (N :_R M)$. A graded R-module M is said to be graded prime if $\{0\}$ is a graded prime R-submodule of M. Also, in [2], a proper graded R-submodule N of a graded R-module M is said to be graded total prime if whenever $x \in h(R)$ and $m \in h(M)$ such that $xm \in N$, then either $m \in N$ or $x \in (N :_R M)$. Moreover, M is said to be graded total prime if $\{0\}$ is a graded total prime R-submodule of M. In general, a graded R-module M/N is a graded total prime R-module if and only if N is a graded total prime R-submodule of M.

In this article, we follow [6] to introduce the concept of graded strong 2-absorbing submodules. Among several results, we prove that if N is a graded prime (graded total prime) R-submodule of M, then N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M (Proposition 8), on the other hand, we prove that the converse is not true in general (Example 15). However, we prove that if N is a graded semi-commutative 2-absorbing R-submodule of M, then N is a graded strong 2-absorbing R-submodule of M (Proposition 11). Also, we show that if N_1 and N_2 are two graded prime (graded total prime) R-submodules of M, then $N_1 \bigcap N_2$ is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M (Proposition 14). We prove that if N and K are graded R-submodules of M such that $N \subseteq K$ and N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M, then N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of K (Proposition 16). We prove that if N and K are graded R-submodules of M such that $K \nsubseteq N$ and N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M, then $N \bigcap K$ is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of K (Proposition 21). Finally, we prove that if K is a graded R-submodule of M and N is an R-submodules of M such that $K \subseteq N$, then N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M if and only if N/K is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M/K (Proposition 23).

1.1 Preliminaries

All rings in this article are associative (not necessarily commutative) and all modules are left R-modules. Let G be a group with identity e and R be a ring. Then R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $supp(R,G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R and if R contains a unity 1, then $1 \in R_e$. Furthermore, $h(R) = \bigcup_{g\in G} R_g$.

Let I be an ideal of a graded ring R. Then I is said to be a graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. The following example shows that an ideal of a graded ring need not be graded.

Example 1. *Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbf{Z_4}$ (the group of integers modulo 4). Then R is G-graded by*

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \text{ and } R_1 = R_3 = \{0\}.$$

Consider the ideal $I = \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$ of R. Note that, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in I$ such that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\in R_0} + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\in R_2}$. If I is a graded ideal of R, then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I$ which is a contradiction. So, I is not graded ideal of R.

Let R be a G-graded ring and I be a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g+I)/I$ for all $g \in G$. If R and S are two G-graded rings, then $R \times S$ is G-graded by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M,G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$.

Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$, where $x_g \in N$ for all $g \in G$. Similarly, as in Example 1, an R-submodule of a graded R-module need not be graded.

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2 Graded 2-Absorbing Submodules and Graded strong 2-Absorbing Submodules

In this section, we study the concept of graded 2-absorbing submodules over non-commutative rings. Also, we introduce and study the concept of graded strong 2-absorbing submodules.

Definition 2. *Let R be a graded ring and P be a proper graded ideal of R. Then P is said to be a graded 2-absorbing ideal of R if whenever $x, y, z \in h(R)$ such that $xRyRz \subseteq P$, then either $xy \in P$ or $yz \in P$ or $xz \in P$.*

Definition 3. *Let M be a graded R-module and N be a proper graded R-submodule of M. Then N is said to be a graded 2-absorbing R-submodule of M if whenever $x, y \in h(R)$ and $m \in h(M)$ such that $xRyRm \subseteq N$, then either $xm \in N$ or $ym \in N$ or $xy \in (N :_R M)$. A graded R-module M is said to be graded 2-absorbing if $\{0\}$ is a graded 2-absorbing R-submodule of M.*

Remark 4. *If R is a commutative ring, then the concept of a graded 2-absorbing R-submodules coincides with that of Al-Zoubi and Abu-Dawwas in [3].*

Definition 5. *Let M be a graded R-module and N be a proper graded R-submodule of M. Then N is said to be a graded strong 2-absorbing R-submodule of M if whenever $x, y \in h(R)$ and $m \in h(M)$ such that $xym \in N$, then either $xm \in N$ or $ym \in N$ or $xy \in (N :_R M)$. A graded R-module M is said to be graded strong 2-absorbing if $\{0\}$ is a graded strong 2-absorbing R-submodule of M.*

The next example shows that if R is not commutative ring, then the concepts of graded 2-absorbing and graded strong 2-absorbing submodules are not the same.

Example 6. *Let $R = M_2(\mathbf{Z})$ (the ring of all 2×2 matrices with integer entries) and $M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ where the entries of the matrices in M are from the ring $\mathbf{Z_2}$. Then M is an R-module. Suppose that $G = \mathbf{Z_4}$. Then R is G-graded by $R_0 = \begin{pmatrix} \mathbf{Z} & 0 \\ 0 & \mathbf{Z} \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \mathbf{Z} \\ \mathbf{Z} & 0 \end{pmatrix}$ and $R_1 = R_3 = \{0\}$. Also, M is G-graded by $M_0 = M$ and $M_1 = M_2 = M_3 = \{0\}$. The possible graded nontrivial proper R-submodules of M are $N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ and $L = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ that are not closed under multiplication by R; for N: $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in R$ such that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin N$, for K: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in R$ such that*

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \notin K$$

and for L: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin L.$$

So, M has no graded nontrivial proper R-submodules, i.e., M is a graded simple and hence M is a graded prime R-module and as such graded 2-absorbing. On the other hand, $x = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix} \in h(R)$ and $m = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in h(M)$ such that $xym = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $xm = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $ym = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $xyM \neq \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M$ such that $xy\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, M is not graded strong 2-absorbing R-module.

Lemma 7. *Let M be a G-graded R-module and N be a graded prime R-submodule of M. If $x, y \in h(R)$ and $m \in h(M)$ such that $xRyRm \subseteq N$ and $xm \notin N$, then $y \in (N :_R M)$.*

Proof: Firstly, we show that $yRm \subseteq N$. Let $r \in R$. Then $r = \sum_{g \in G} r_g$ where $r_g \in R_g$ for all $g \in G$, and then for $g \in G$, $xR(yr_g m) \subseteq xR(yRm) \subseteq N$. Since N is graded prime, either $yr_g m \in N$ or $x \in (N :_R M)$. Then $yr_g m \in N$ because $xm \notin N$. That is $yR_g m \subseteq N$ for all $g \in G$, which implies that $yRm \subseteq N$. Since N is graded prime, either $m \in N$ or $y \in (N :_R M)$. Since $xm \notin N$, we have $m \notin N$, so that $y \in (N :_R M)$. □

Proposition 8. *Let N be a graded prime (graded total prime) R-submodule of M. Then N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M.*

Proof: Suppose that N is a graded prime R-submodule of M. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xRyRm \subseteq N$. Assume that $xm \notin N$. Then by Lemma 7, we have $y \in (N :_R M)$. Then $ym \in N$ and $xyM \subseteq xN \subseteq N$. Hence, N is a graded 2-absorbing R-submodule of M. Now, suppose that N is a graded total prime R-submodule of M. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xym \in N$. Assume that $ym \notin N$. Since N is graded total prime, we have $x \in (N :_R M)$. Hence, $xyM \subseteq xM \subseteq N$, and we have N is a graded strong 2-absorbing R-submodule of M. □

Remark 9. *We will prove in Example 15 that the converse of Proposition 8 is not true in general, even if the ring R is commutative.*

Definition 10. *Let M be a graded R-module and N be a graded R-submodule of M. Then N is said to be a graded semi-commutative R-submodule of M if whenever $x \in h(R)$ and $m \in h(M)$ such that $xm \in N$, then $xRm \subseteq N$.*

Proposition 11. *Let M be a graded R-module and N be a proper graded semi-commutative R-submodule of M. If N is a graded 2-absorbing R-submodule of M, then N is a graded strong 2-absorbing R-submodule of M.*

Proof: Let $x, y \in h(R)$ and $m \in h(M)$ such that $xym \in N$. Since N is graded semi-commutative, we have $xRyRm \subseteq N$. Since N is graded 2-absorbing, we have $xm \in N$ or $ym \in N$ or $xyM \subseteq N$. Hence, N is a graded strong 2-absorbing R-submodule of M. □

Compare the next Proposition with Theorem 2.5(ii) in [3].

Proposition 12. *Let M be a G-graded R-module and N be a graded R-submodule of M. If N_g is a 2-absorbing R_e-submodule of M_g for some $g \in G$, then $(N_g :_{R_e} M_g)$ is a 2-absorbing ideal of R_e.*

Proof: Let $x, y, z \in R_e$ such that $xR_eyR_ez \subseteq (N_g :_{R_e} M_g)$ and suppose that $xz \notin (N_g :_{R_e} M_g)$ and $yz \notin (N_g :_{R_e} M_g)$. We prove that $xy \in (N_g :_{R_e} M_g)$. Since $xz, yz \notin (N_g :_{R_e} M_g)$, there exist $m, s \in M_g$ such that $xzm \notin N_g$ and $yzs \notin N_g$. Now, $xR_eyR_ez(m + s) \subseteq N_g$. Since N_g is 2-absorbing, we have $xy \in (N_g :_{R_e} M_g)$ or $xz(m + s) \in N_g$ or $yz(m + s) \in N_g$. If $xz(m + s) \in N_g$, then $xzs \notin N_g$ since $xzm \notin N_g$. Since $xR_eyR_ezs \subseteq N_g$ and $yzs \notin N_g$ and $xzs \notin N_g$, we have $xy \in (N_g :_{R_e} M_g)$. Similar to the case $yz(m + s) \in N_g$, we obtain $xy \in (N_g :_{R_e} M_g)$. Hence, $(N_g :_{R_e} M_g)$ is a 2-absorbing ideal of R_e. □

Lemma 13. *([5], Lemma 2.1) Let M be a graded R-module. If N and K are graded R-submodules of M, then $N \bigcap K$ is a graded R-submodule of M.*

Proof: Clearly, $N \bigcap K$ is an R-submodule of M. Let $m \in N \bigcap K$. Then $m \in N$ and $m \in K$. Since N and K are graded R-submodules of M, we have $m_g \in N$ and $m_g \in K$ for all $g \in G$, which implies that $m_g \in N \bigcap K$ for all $g \in G$. Hence, $N \bigcap K$ is a graded R-submodule of M. □

Compare the next Proposition with Theorem 2.4 in [3].

Proposition 14. *If N_1 and N_2 are two graded prime (graded total prime) R-submodules of a graded R-module M, then $N_1 \bigcap N_2$ is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M.*

Proof: By Lemma 13, we have $N_1 \bigcap N_2$ is a graded R-submodule of M. If $N_1 = N_2$, then $N_1 \bigcap N_2$ is a graded prime (graded total prime) R-submodule of M, so that $N_1 \bigcap N_2$ is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M. Assume that N_1 and N_2 are distinct. Since N_1 and N_2 are proper graded R-submodules of M, it follows that $N_1 \bigcap N_2$ is a proper graded R-submodule of M. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xRyRm \subseteq N_1 \bigcap N_2$ ($xym \in N_1 \bigcap N_2$) with $xm \notin N_1 \bigcap N_2$ and $xyM \nsubseteq N_1 \bigcap N_2$. Then we have the following two situations:

1. $xm \notin N_1$ or $xm \notin N_2$.
2. $xyM \nsubseteq N_1$ or $xyM \nsubseteq N_2$.

These two situations give the following four cases:

1. $xm \notin N_1$ and $xyM \nsubseteq N_1$.
2. $xm \notin N_1$ and $xyM \nsubseteq N_2$.
3. $xm \notin N_2$ and $xyM \nsubseteq N_1$.
4. $xm \notin N_2$ and $xyM \nsubseteq N_2$.

Let N_1 and N_2 be two graded prime R-submodules of a graded R-module M. Firstly, we consider Case (1). Since $xRyRm \subseteq N_1 \bigcap N_2 \subseteq N_1$ and $xm \notin N_1$, it follows from Lemma 7 that $yM \subseteq N_1$, a contradiction since $xyM \nsubseteq N_1$. So, Case (1) is impossible. Similarly, Case (4) does not occur. Now, we consider Case (2). Again, we get that $yM \subseteq N_1$ and then $ym \in N_1$. Let $r \in R$. Then $r = \sum_{g \in G} r_g$ where $r_g \in R_g$ for all $g \in G$. Since $xRyRm \subseteq N_1 \bigcap N_2 \subseteq N_2$, we have for $g \in G$, $xRyr_gm \subseteq N_2$, which implies that $xM \subseteq N_2$ or $yr_gm \in N_2$ since N_2 is graded prime. If $xM \subseteq N_2$, then $xyM \subseteq xM \subseteq N_2$ contradicts $xyM \nsubseteq N_2$. Thus $yr_gm \in N_2$ for all $g \in G$ and then $yrm \in N_2$, that is $yRm \subseteq N_2$. Since N_2 is graded prime, we have $yM \subseteq N_2$ or $m \in N_2$. If $yM \subseteq N_2$, then $xyM \subseteq N_2$ leading to the same contradiction. Hence, $m \in N_2$ and then $ym \in N_2$. Therefore, $ym \in N_1 \bigcap N_2$. The proof of Case (3) is similar to that of Case (2).

Let N_1 and N_2 be graded total prime R-submodules of M, $x, y \in h(R)$ and $m \in h(M)$ such that $xym \in N_1 \bigcap N_2 \subseteq N_1$. Consider Case (1), since N_1 is graded total prime, we have $xM \subseteq N_1$ or $ym \in N_1$. If $xM \subseteq N_1$, then $xyM \subseteq xM \subseteq N_1$ which is impossible. So, $ym \in N_1$ and then $yM \subseteq N_1$ or $m \in N_1$ which is impossible. Therefore, Case (1) does not happen. Similarly, Case (4) does not occur. Consider Case (2), we have $xym \in N_1 \bigcap N_2 \subseteq N_2$ and since N_2 is graded total prime, we have $xM \subseteq N_2$ or $ym \in N_2$. If $xM \subseteq N_2$, then $xyM \subseteq xM \subseteq N_2$ that is a contradiction. So, $ym \in N_2$. Since $xym \subseteq N_1 \bigcap N_2 \subseteq N_1$, we have $xM \subseteq N_1$ or $ym \in N_1$. Since $xm \notin N_1$, $xM \subseteq N_1$ is impossible. Hence, $ym \in N_1 \bigcap N_2$. The proof of Case (3) is similar to that of Case (2). □

The next example shows that the converse of Proposition 8 is not true in general, even if the ring R is commutative.

Example 15. *Let $R = \mathbf{Z}$, $M = \mathbf{Z_6}[\mathbf{i}]$ and $G = \mathbf{Z_2}$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbf{Z_6}$ and $M_1 = i\mathbf{Z_6}$. Now, $N = \{0\}$ is a graded R-submodule of M. Note that $2 \in h(R)$ and $3 \in h(M)$ such that $2.3 \in N$, but $3 \notin N$ and $2 \notin (N :_R M) = 6\mathbf{Z}$. Therefore, N is not a graded prime (graded total prime) R-submodule of M. On the other hand, $2, 3 \in h(M)$, $N_1 = \langle 2 \rangle$ and $N_2 = \langle 3 \rangle$ are graded prime (graded total prime) R-submodules of M, and by Proposition 14, $N = N_1 \bigcap N_2$ is a graded 2-absorbing (strong 2-absorbing) R-submodule of M.*

Proposition 16. *Let N and K be two graded R-submodules of a graded R-module M such that $N \subseteq K$. If N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M, then N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of K.*

Proof: If $K = M$, then we are done. Suppose that $K \neq M$. Let $x, y \in h(R)$ and $m \in h(K)$ such that $xRyRm \subseteq N$ ($xym \in N$). Since $m \in h(K)$, $m \in K_g = K \bigcap M_g$ for some $g \in G$, so $m \in h(M)$. As N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M, we have $xm \in N$ or $ym \in N$ or $xy \in (N :_R M)$. Since $N \subseteq K$, we have $(N :_R M) \subseteq (N :_R K)$, and then $xm \in N$ or $ym \in N$ or $xy \in (N :_R K)$. Hence, N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of K. □

Compare the next Lemma with Theorem 2.5(i) in [3].

Lemma 17. *Let M be a G-graded R-module and N be a graded R-submodule of M. Assume that $g \in G$ such that N_g is a 2-absorbing (strong 2-absorbing) R_e-submodule of M_g. Then $xR_eyK \subseteq N_g$ ($xyK \subseteq N_g$) implies that $xy \in (N_g :_{R_e} M_g)$ or $xK \subseteq N_g$ or $yK \subseteq N_g$ for every $x, y \in R_e$ and R_e-submodule K of M_g.*

Proof: Suppose that $xy \notin (N_g :_{R_e} M_g)$, $xK \nsubseteq N_g$ and $yK \nsubseteq N_g$. Then there exist $m, s \in K$ such that $xm \notin N_g$ and $ys \notin N_g$. Since $xR_eyR_em \subseteq xR_eyK \subseteq N_g$ ($xym \in xyK \subseteq N_g$) and $xy \notin (N_g :_{R_e} M_g)$ and $xm \notin N_g$, we have $ym \in N_g$. Also, since $xR_eyR_es \subseteq xR_eyK \subseteq N_g$ ($xys \in xyK \subseteq N_g$) and $xy \notin (N_g :_{R_e} M_g)$ and $ys \notin N_g$, we have $xs \in N_g$. Now, since $xR_eyR_e(m+s) \subseteq xR_eyK \subseteq N_g$ ($xy(m+s) \in xyK \subseteq N_g$) and $xy \notin (N_g :_{R_e} M_g)$, we have $x(m+s) \in N_g$ or $y(m+s) \in N_g$. If $x(m+s) \in N_g$, then since $xs \in N_g$, we have $xm \in N_g$ which is a contradiction. If $y(m+s) \in N_g$, then since $ym \in N_g$, we have $ys \in N_g$ which is a contradiction. Thus, $xy \in (N_g :_{R_e} M_g)$ or $xK \subseteq N_g$ or $yK \subseteq N_g$. □

Proposition 18. *Let M be a G-graded R-module and N be a graded R-submodule of M. Assume that $g \in G$ such that N_g is a 2-absorbing R_e-submodule of M_g. Then $(N_g :_{R_e} R_em)$ is a 2-absorbing ideal of R_e for every $m \in M_g - N_g$.*

Proof: Let $x, y, z \in R_e$ and $m \in M_g - N_g$ such that $xR_eyR_ez \subseteq (N_g :_{R_e} R_em)$. Then $xR_ey(R_ezR_e)m \subseteq N_g$. Since R_ezR_e is an ideal of R_e, we have $K = (R_ezR_e)m$ is an R_e-submodule of M_g. Now, by Lemma 17, $xK \subseteq N_g$ or $yK \subseteq N_g$ or $xyM_g \subseteq N_g$. Hence, $xzR_em \subseteq N_g$ or $yzR_em \subseteq N_g$ or $xyM_g \subseteq N_g$. Thus, $xz \in (N_g :_{R_e} R_em)$ or $yz \in (N_g :_{R_e} R_em)$ or $xyR_em \subseteq xyM_g \subseteq N_g$ that is $xy \in (N_g :_{R_e} R_em)$. □

Lemma 19. *Let M be a G-graded R-module, N be a graded R-submodule of M and I be an ideal of R_e. Assume that $g \in G$ such that N_g is a 2-absorbing (strong 2-absorbing) R_e-submodule of M_g. If $x \in R_e$ and $m \in M_g$ such that $IR_exR_em \subseteq N_g$ ($Ixm \subseteq N_g$), then $xm \in N_g$ or $Im \subseteq N_g$ or $Ix \subseteq (N_g :_{R_e} M_g)$.*

Proof: Suppose that $xm \notin N_g$ and $Ix \nsubseteq (N_g :_{R_e} M_g)$. Then there exists $y \in I$ such that $yx \notin (N_g :_{R_e} M_g)$. Now, $yR_exR_em \subseteq N_g$ ($yxm \in N_g$) implies that $ym \in N_g$ as N_g is a 2-absorbing (strong 2-absorbing) R_e-submodule of M_g. We show that $Ix \subseteq N_g$. Let $z \in I$. Then $(y+z)R_exR_em \subseteq IR_exR_em \subseteq$

N_g ($(y+z)xm \in Ixm \subseteq N_g$). Hence, $(y+z)m \in N_g$ or $(y+z)x \in (N_g :_{R_e} M_g)$. If $(y+z)m \in N_g$, then since $ym \in N_g$, we have that $zm \in N_g$. If $(y+z)x \in (N_g :_{R_e} M_g)$, then $zx \notin (N_g :_{R_e} M_g)$, on the other hand, $zR_exR_em \subseteq N_g$ ($zxm \in N_g$), so $zm \in N_g$. Therefore, $Im \subseteq N_g$. □

Proposition 20. *Let M be a G-graded R-module and N be a graded R-submodule of M. Assume that $g \in G$ such that N_g is a 2-absorbing (strong 2-absorbing) R_e-submodule of M_g. Then $(N_g :_{M_g} I) = \{m \in M_g : Im \subseteq N_g\}$ is a 2-absorbing (strong 2-absorbing) R_e-submodule of M_g for every ideal I of R_e.*

Proof: Let I be an ideal of R_e. Suppose that $x, y \in R_e$ and $m \in M_g$ such that $xR_eyR_em \subseteq (N_g :_{M_g} I)$ ($xym \subseteq (N_g :_{M_g} I)$). So, $IxR_eyR_em \subseteq N_g$ ($Ixym \subseteq N_g$), and then $(IxR_e)R_eyR_em \subseteq IxR_eyR_em \subseteq N_g$. Thus by Lemma 19, we have $(IxR_e)m \subseteq N_g$ or $(IxR_ey) \subseteq (N_g :_{R_e} M_g)$ or $ym \in N_g$. If $ym \in N_g$, then $Iym \subseteq N_g$ and then $ym \in (N_g :_{M_g} I)$ as required. If $IxR_ey \subseteq (N_g :_{R_e} M_g)$, then $xy \in xR_ey \subseteq ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$. If $(IxR_e)m \subseteq N_g$, then $xm \in xR_em \subseteq (N_g :_{M_g} I)$. Hence, $ym \in (N_g :_{M_g} I)$ or $xm \in (N_g :_{M_g} I)$ or $xy \in ((N_g :_{M_g} I) :_{R_e} M_g)$ that ends the proof for 2-absorbing. For $Ixym \subseteq N_g$ and N_g is strong 2-absorbing, it follows from Lemma 19 that $xym \in N_g$ or $Im \subseteq N_g$ or $Ixy \subseteq (N_g :_{R_e} M_g)$. If $xym \in N_g$, then $xm \in N_g$ or $ym \in N_g$ or $xy \in (N_g :_{R_e} M_g)$. So, for $xm \in N_g$ it follows that $Ixm \subseteq IN_g \subseteq N_g$ and we have $xm \in (N_g :_{M_g} I)$. For $ym \in N_g$ it follows that $Iym \subseteq IN_g \subseteq N_g$ and we have $ym \in (N_g :_{M_g} I)$. For $xy \in (N_g :_{R_e} M_g)$, we have $xy \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$. For $Im \subseteq N_g$, we have $m \in (N_g :_{M_g} I)$ and hence $xm \in (N_g :_{M_g} I)$. For $Ixy \subseteq (N_g :_{R_e} M_g)$, we have $xy \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$ and $(N_g :_{M_g} I)$ is a strong 2-absorbing R_e-submodule of M_g. □

Proposition 21. *Let M be a graded R-module, N and K be graded R-submodules of M such that $K \nsubseteq N$. If N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M, then $N \bigcap K$ is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of K.*

Proof: By Lemma 13 and since $K \nsubseteq N$, $N \bigcap K$ is a proper graded R-submodule of K. Let $x, y \in h(R)$ and $k \in h(K)$ such that $xRyRk \subseteq N \bigcap K$ ($xyk \in N \bigcap K$). Clearly, $xyK \subseteq K$ and $xk, yk \in K$. Furthermore, since $xRyRk \subseteq N \bigcap K \subseteq N$ ($xyk \in N \bigcap K \subseteq N$) and N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M, $xyM \subseteq N$ or $xk \in N$ or $yk \in N$. So, $xyK \subseteq xyK \bigcap xyM \subseteq N \bigcap K$ or $xk \in N \bigcap K$ or $yk \in N \bigcap K$. Hence, $N \bigcap K$ is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of K. □

Let M be a G-graded R-module and N be an R-submodule of M. Then M/N may be made into a graded module by putting $(M/N)_g = (M_g + N)/N$ for all $g \in G$ (see [8]). Moreover, we have the following.

Lemma 22. *([1], Lemma 1.2) Let M be a graded R-module and K be a graded R-submodule of M. Suppose that N is an R-submodules of M such that $K \subseteq N$. Then N is a graded R-submodule of M if and only if N/K is a graded R-submodule of M/K*

Proof: Suppose that N is a graded R-submodule of M. Clearly, N/K is an R-submodule of M/K. Let $x + K \in N/K$. Then $x \in N$ and since N is graded, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$ and then $(x + K)_g = x_g + K \in N/K$ for all $g \in G$. Hence, N/K is a graded R-submodule of M/K. Conversely, let $x \in N$. Then $x = \sum_{g \in G} x_g$ where $x_g \in M_g$ for all $g \in G$ and then $(x_g + K) \in (M_g + K)/K = (M/K)_g$ for all $g \in G$ such that

$$\sum_{g \in G}(x+K)_g = \sum_{g \in G}(x_g + K) = \left(\sum_{g \in G} x_g\right) + K = x + K \in N/K.$$

Since N/K is graded, $x_g + K \in N/K$ for all $g \in G$ which implies that $x_g \in N$ for all $g \in G$. Hence, N is a graded R-submodule of M. □

Proposition 23. *Let M be a graded R-module and K be a graded R-submodule of M. Suppose that N is an R-submodules of M such that $K \subseteq N$. Then N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M if and only if N/K is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M/K.*

Proof: Suppose that N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M. Then by Lemma 22, N/K is a proper graded R-submodule of M/K. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xRyR(m + K) \subseteq N/K$ ($xy(m + K) \subseteq N/K$). Assume that $a, b \in R$. Then $xaybm + K = xayb(m + K) \subseteq xRyR(m + K) \subseteq N/K$, and then there exists $n \in N$ such that $xaybm + K = n + K$ which implies that $-n + xaybm \in K \subseteq N$, and hence $xaybm \in N$. So, $xRyRm \subseteq N$. (For the graded strong 2-absorbing case, $xym + K = xy(m + K) \subseteq N/K$ and then there exists $n \in N$ such that $xym + K = n + K$, so $-n + xym \in K \subseteq N$, and then $xym \in N$). Thus, $xm \in N$ or $ym \in N$ or $xyM \subseteq N$ as N is a graded 2-absorbing (graded strong 2-absorbing) R-submodule of M. Hence, $x(m + K) \in N/K$ or $y(m + K) \in N/K$ or $xy(M/K) \subseteq N/K$. Therefore, N/K is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M/K. Conversely, N is a proper graded R-submodule of M. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xRyRm \subseteq N$ ($xym \in N$). Then $m+K \in h(M/K)$ such that $xRyR(m+K) \subseteq N/K$ ($xy(m + K) \in N/K$), and then $x(m + K) \in N/K$ or $y(m+K) \in N/K$ or $xy(M/K) \subseteq N/K$ as N/K is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M/K. So, $xm \in N$ or $ym \in N$ or $xyM \subseteq N$, and hence N is a graded 2-absorbing (graded strongly 2-absorbing) R-submodule of M. □

3 Conclusion

In this article, we study the concept of graded 2-absorbing submodules over non-commutative rings. Also, we introduce the concept of graded strong 2-absorbing submodules and show that in general if R is not a commutative ring, then the concepts of graded 2-absorbing and graded strong 2-absorbing submodules are not the same (Example 6). If R is commutative, then the concept of graded 2-absorbing submodules coincides with that of the original definition introduced by Al-Zoubi and Abu-Dawwas in [3].

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