Moments and Identities Involving Inverted Wishart Distribution

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Abstract: - Moments of multivariate Wishart Distribution are known up to fourth order. But in many contexts, moments of some functions of Wishart distribution and Inverted Wishart Distribution have been found useful in risk theoretic estimation of covariance matrix and its characteristics. In this paper we review moments of some important functions of Wishart and Inverted Distributions.

Key-Words: - Wishert distribution, Inverted Wishert distribution, Covariance matrix, Wishert matrix, Jacobian, Moments

Received: November 13, 2019. Revised: February 24, 2020. Accepted: April 17, 2020. Published: April 29, 2020.

1 Introduction

The inverted Wishart distribution has got various applications in statistics. For instance, the distribution can be used as a natural conjugate prior when dealing with Bayesian estimation of covariance matrix under sampling from multivariate normal distribution (Anderson, 2003, Section 7.7). Moments of the inverted Wishart distribution have also been utilized in discriminant analysis (Das Gupta, 1968; Siskind, 1972 and Haff, 1982) while obtaining moments of the maximum likelihood estimators in the growth curve model (von Rosen, 1988 and Von Rosen, 1997). Useful results for the inverted Wishart distribution can also be found in Press (1982). Kaufman (1967) derived the moments by factorization theorem. Das Gupta (1968) utilized some invariance arguments. Both of them based their calculations on the inverse moments of chi-square variable. In a series of papers, Haff (1977, 1979, 1980, 1981, 1982) presented moment identities which are useful for deriving moments of inverted Wishart distribution. The identities were established by applying Stokes' theorem. Independently of Haff, von Rosen (1985) derived moments of inverted Wishart distribution with the help of matrix calculus. These moments have been found useful in risk theoretic estimation of covariance matrix and its characteristics. See for example Joarder (1997) and Joarder (1998). Let $X_1, X_2, \cdots X_N (N > p)$ be a p-dimensional independent normal random vector with mean vector $\bar{X}$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$.

Fisher (1915) derived the distribution of A for $p = 2$ in order to study the distribution of correlation coefficient from a normal sample. Wishart (1928) obtained the distribution for arbitrary p as the joint distribution of sample variances and covariances from multivariate normal population. Because of its important role in multivariate statistical analysis, various authors have derived it from different perspectives. See the references in Gupta and Nagar (2000, 87-88). The following theorem provides the density function of Wishart matrix.

Theorem 1.1 The random symmetric positive definite matrix A is said to have a Wishart distribution with parameters p, $m = N-1 > p$ and $\Sigma(p\times p) > 0$, written as $A \sim W_p(m,\Sigma)$ if its probability density function is given by

$$f_1(A) = c_{p,m}\,|\Sigma|^{-m/2}|A|^{(m-p-1)/2}\exp\left(-\frac{1}{2}tr\Sigma^{-1}A\right),$$
$$A > 0, m > p$$

where $c_{p,m}^{-1} = 2^{mp/2}\,\Gamma_p(m/2)$ and

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\alpha-(i-1)/2\right),$$
$$\alpha > (p-1)/2$$

(See e.g. Anderson, 2003, 252).

It is worth mentioning that $a_{11}/\sigma_{11} \sim \chi_m^2$. The first moment of A, trA, $\det(A)$, A^{-1} and similar beautiful quantities are known (Muirhead, 1982). Some moments of functions of $p\times p$ Wishart matrix are provided below:

$(i)\ E(A) = m\Sigma,$

$(ii)\ E\left(A^{*}\right) = N\Sigma,\ A^{*} = \sum_{j=1}^{N}(X_j-\mu)(X_j-\mu)',$

μ known,

$(iii)(m-p-1)E\left(A^{-1}\right) = \Sigma^{-1},$

$$(iv)E\left(|A|^k\right) = \frac{2^{kp}\prod_{i=1}^{p}\Gamma\left(\frac{m+2k}{2}-\frac{i-1}{2}\right)}{\prod_{i=1}^{p}\Gamma\left(\frac{m}{2}-\frac{i-1}{2}\right)}|\Sigma|^k$$

$(v)\ E(trA) = m\ tr\Sigma,$

$(vi)\ E(trA^{*}) = m\ tr\Sigma,\ \mu$ known,

$(vii)\ E\left[(trA)^2\right] = m^2(tr\Sigma)^2 + 2m\ tr\left(\Sigma^2\right),$

$(viii)\ E\left[tr\left(A^2\right)\right] = m\left[(tr\Sigma)^2 + (m+1)tr\left(\Sigma^2\right)\right]$

$(ix)\ E\left[(m+1)(trA)^2 - 2tr\left(A^2\right)\right] =$
$(m-1)m(m+2)tr\left(\Sigma^2\right),$

$(x)\ E\left[(m+2)(trA^{*})^2 - 2tr\left(A^{*2}\right)\right] =$
$m(m+1)(m+3)tr\left(\Sigma^2\right),\ \mu$ known,

$(xi)\ E\left[m\ tr\left(A^2\right) - (trA)^2\right] =$
$(m-1)m(m+2)tr\left(\Sigma^2\right),$

$(xii)\ E\left[(m+1)tr\left(A^{*2}\right) - (trA^{*})^2\right] =$
$m(m+1)(m+3)tr\left(\Sigma^2\right),\ \mu$ known.

(Voinov and Nikulin, 1996, 197-198). A nice update of moments of Wishart distribution is given in Gupta and Nagar (2000). By differentiating the moment generating function of $A \sim W_p(m,\Sigma)$, $m > p$, de Waal and Nel (1973) derived the following results:

$$(a)E(A^2) = m\left((m+1)\Sigma + (tr\Sigma)I_p\right)\Sigma$$

$$(b)E(A^3) = m\begin{pmatrix}(m^2+3m+4)\Sigma^2 + 2(m+1)(tr\Sigma)\Sigma \\ +(m+1)(tr\Sigma^2)I_p + (tr\Sigma)^2 I_p\end{pmatrix}\Sigma.$$

De Waal and Nel (1973) also derived the fourth moment of Wishart matrix. But in many contexts, moments of some functions of Wishart distribution have been found useful. In this paper we review some moments of functions of Wishart Distribution and also derive expectation of some functions of Inverted Wishart Distribution along Muirhead (1986).

2. Moments and Identities Involving Wishart Matrix

The following theorem and corollary are due to Muirhead (1986)

Theorem 2.1 Suppose that $A \sim W_p(m,\Sigma)$ defined in Theorem 1.1. Let $h(A)$ be a real-valued measurable function of A such that the function $f(t;A) = h(tA),\ t > 0,$ is differentiable at $t = 1$. Again let $f'(t;A) = \frac{\partial}{\partial t}f(t;A).$ Then

$$E\left[tr\left(\Sigma^{-1}A\right)h(A)\right]=mp\,E\left[h(A)\right]-2E\left[f'(1;A)\right] \quad (2.1)$$

Corollary 2.1 Let $h(xA)=x^{l}h(A)$ for some l.
Then $E\left[tr\left(\Sigma^{-1}A\right)h(A)\right]=(mp+2l)h(A).$

For any nonnegative integer k we will be using the notation:

$$a_{\{k\}}=a(a+1)(a+2)\cdots(a+k-1),$$
$$a^{\{k\}}=a(a-1)(a-2)\cdots(a-k+1).$$

That is $a_{\{k\}}=(a+k-1)^{\{k\}}$, $a^{\{k\}}=(a-k+1)^{\{k\}}$.

If $A\sim W_p(m,\Sigma)$, then the following results are known.

$$(i)\ E\left(|A|^k\right)=\frac{2^{pk}\,\Gamma_p(m/2+k)}{\Gamma_p(m/2)}\ |\Sigma|^k,$$

$$(ii)\ E\left(A^{-1}\right)=\frac{1}{m-p-1}\ \Sigma^{-1},\ m>p+1$$

The following moments and identities derived by Corollary 2.1 are due to Muirhead (1986).

1. $E\left[\left(tr\Sigma^{-1}A\right)^k\right]=2^k\,(mp/2)_{\{k\}}$.

2. $E\left[\left(tr\Sigma^{-1}A\right)^{-k}\right]=\dfrac{2^{-k}}{(mp/2-1)^{\{k\}}}$, $mp>2k$.

3. $E\left[\left(tr\Sigma^{-1}A\right)^k tr(A)\right]=2^k m(mp/2+1)_{\{k\}}\left(tr\Sigma\right)$

4. $E\left[\left(tr\Sigma^{-1}A\right)^k tr\left(A^{-1}\right)\right]=$
$\dfrac{2^k\left(mp/2-1\right)_{\{k\}}}{m-p-1}\ tr\Sigma^{-1},\ m>p+1$

5. $E\left[\left(tr\Sigma^{-1}A\right)^k tr\left(\Sigma A^{-1}\right)\right]=$
$2^k\ \dfrac{m(mp/2-1)_{\{k\}}}{m-p-1},\ m>p+1$

6. $E\left[\left(tr\Sigma^{-1}A\right)^k |A|^h\right]=$
$2^{hp+k}\ \left(mp/2+hp\right)_{\{k\}}\ \dfrac{\Gamma_p(m/2+h)}{\Gamma_p(m/2)}\ |\Sigma|^h$

7. $E\left[\left(tr\Sigma^{-1}A\right)(trA)\left(trA^{\alpha}\right)\right]=$
$[mp+2(\alpha+1)]\ E\left[(trA)\left(trA^{\alpha}\right)\right]$

Putting $\Sigma=I$ we have the identity

$$E\left[(trA)^2\left(trA^{\alpha}\right)\right]=$$
$$[mp+2(\alpha+1)]\ E\left[(trA)\left(trA^{\alpha}\right)\right]$$

which is the Sharma and Krishnamoorthy (1984) Identity.

3. Main Results

The probability distribution function of the inverted Wishart matrix $B=A^{-1}$ is given in the following theorem.

Theorem 1.1 Suppose that the random symmetric positive definite matrix A has the Wishart distribution written as $A\sim W_p(m,\Sigma)$ with parameters p, $m=N-1>p$ and $\Sigma(p\times p)>0$. Then $B=A^{-1}$ has the probability density function

$$f(B)=$$
$$c_{p,m}\,|\Psi|^{m/2}\ \ |B|^{-(m+p+1)/2}\exp\left(-\frac{1}{2}tr\Psi B^{-1}\right),$$
$$B>0, m>p, \quad (3.1)$$

where

$$c_{p,m}^{-1}=2^{mp/2}\ \Gamma_p(m/2), \quad (3.2)$$

$$\Gamma_p(\alpha)=$$
$$\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\alpha-(i-1)/2\right),\ \alpha>(p-1)/2$$,and

$$\Psi=\Sigma^{-1}.$$

The density function in (1.1) will be denoted by $B\sim W_p^{-1}(\Psi,m)$. See e.g. Andeson , 2003, 272) or Anderson (1984, 268).

Proof. The Jacobian of the transformation $A = B^{-1}$ is given by $J(A \to B) = |B|^{-(p+1)}$ (See Gupta and Nagar, 200, 14). Then it follows from Theorem 1.1

$$f_1(A) = c_{p,m} |\Sigma|^{-m/2} |A|^{(m-p-1)/2} \exp\left(-\frac{1}{2} tr \Sigma^{-1} A\right)$$

That

$$f(B) = c_{p,m} |\Psi^{-1}|^{-m/2} |B^{-1}|^{(m-p-1)/2}$$
$$\exp\left(-\frac{1}{2} tr \Psi B^{-1}\right) |B|^{-(p+1)}$$

which simplifies to

$$f(B) =$$
$$c_{p,m} |\Psi|^{m/2} |B|^{-(m+p+1)/2} \exp\left(-\frac{1}{2} tr \Psi B^{-1}\right),$$

where $\quad c_{p,m}^{-1} = 2^{mp/2}\ \Gamma_p(m/2),$

$$\Gamma_p(\alpha) =$$
$$\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\alpha - (i-1)/2\right),\ \alpha > (p-1)/2$$

and $\Psi = \Sigma^{-1}$.

If $p = 1$, then the corresponding univariate density function would be

$$f_3(b_{11}) = \frac{1}{2^{m/2}\ \Gamma(m/2)}\ \psi_{11}^{m/2}\ b_{11}^{-(m+1+1)/2} \exp\left(-\frac{\psi_{11}}{2b_{11}}\right),$$

so that the density function of

$V = \psi_{11} / b_{11}$ is given by

$$f_4(v) =$$
$$\frac{1}{2^{m/2}\ \Gamma(m/2)}\ \psi_{11}^{m/2} \left(\frac{\psi_{11}}{v}\right)^{-(m+2)/2} e^{-v/2} \left(\frac{\psi_{11}}{v^2}\right),$$

$$f_4(v) = \frac{1}{2^{m/2}\ \Gamma(m/2)}\ v^{(m/2)-1}\ e^{-v/2},$$

It is worth mentioning that $U = b_{11} / \psi_{11} > 0$ has an inverted chi-square distribution with density function

$$f_5(u) = \frac{u^{-(m/2)-1}\ e^{-1/(2u)}}{2^{m/2}\ \Gamma(m/2)},\ u > 0$$

i.e. $1/U \sim \chi_m^2$.

Lemma 3.1

$$(i)\ E\left(|B|^h\right) = \frac{|\Psi|^h\ \Gamma_p(m/2-h)}{2^{hp}\ \Gamma_p(m/2)},\ m/2 > h$$

$$(ii)\ E(B) = \frac{1}{m-p-1}\Psi,\ m > p+1$$

Proof.

$$(i) E\left(|B|^h\right) =$$
$$\int_{B>0} |B|^h\ c_{p,m} |\Psi|^{m/2} \quad |B|^{-(m+p+1)/2}$$
$$\exp\left(-\frac{1}{2} tr \Psi B^{-1}\right) dB$$
$$= \left(c_{p,m} |\Psi|^{m/2}\right) \int_{B>0} |B|^h \quad |B|^{-(m-2h+p+1)/2}$$
$$\exp\left(-\frac{1}{2} tr \Psi B^{-1}\right) dB$$
$$= \left(c_{p,m} |\Psi|^{m/2}\right) \frac{1}{c_{p,m-2h} |\Psi|^{(m-2h)/2}}$$

The moment then follows by (1.1).

(ii) See Anderson (2003, 273) or Mardia, Kent and Bibby (1979, 85)

4. Main Results

Theorem 4.1 Suppose that $B \sim IW_p(\Psi, m)$ defined in Theorem 1.1. Let $h(B)$ be a real-valued measurable function of B such that the function $f(t;B) = h(tB),\ t > 0,$ is differentiable at $t = 1$. Again let $f'(t;B) = \frac{\partial}{\partial t} f(t:B)$.

Then

$$E\left[tr\left(\Psi B^{-1}\right) h(B)\right] = mp\ E\left[h(B)\right] - 2E\left[f'(1;B)\right]. \tag{4.1}$$

Proof. For $t > 0$, define the function $g(t)$ as

$$g(t) = c_{p,m}\left|\Psi\right|^{m/2} t^{mp/2} \int_{B>0} h(B)\ \left|B\right|^{-(m+p+1)} e^{-\frac{t}{2}tr\Psi B^{-1}} dB \tag{4.2}$$

where $c_{p,m} = 2^{mp/2}\,\Gamma_p(m/2)$ and note that $g(1) = E[h(B)]$. Rewrite

$$g(t) = c_{p,m}\left|\Psi\right|^{m/2} \int_{B>0} h(B)\ \left|B\right|^{-(m+p+1)} t^{mp/2} e^{-\frac{t}{2}tr\Psi B^{-1}} dB.$$

Differentiating with respect to t (justified by dominated convergence provided $E\left[tr\left(\Psi B^{-1}\right)h(B)\right]$ exist), we have

$$g'(t) = c_{p,m}\left|\Psi\right|^{m/2} \int_{B>0} h(B)\ \left|B\right|^{-(m+p+1)} \left[\frac{mp}{2}\, t^{(mp-2)/2}\, e^{-\frac{t}{2}tr\Psi B^{-1}} + t^{mp/2}\, e^{-\frac{t}{2}tr\Psi B^{-1}}\left(-\frac{1}{2}tr\Psi B^{-1}\right)\right] dB$$

or,

$$g'(t) = c_{p,m}\left|\Psi\right|^{m/2} \int_{B>0} h(B)\ \left|B\right|^{-(m+p+1)} \left[\frac{mp}{2}\, t^{(mp-2)/2} + t^{mp/2}\left(-\frac{1}{2}tr\Psi B^{-1}\right)\right] e^{-\frac{t}{2}tr\Psi B^{-1}} dB$$

Then

$$g'(1) = c_{p,m}\left|\Psi\right|^{m/2} \int_{B>0} h(B)\ \left|B\right|^{-(m+p+1)} \left[\frac{mp}{2} + \left(-\frac{1}{2}tr\Psi B^{-1}\right)\right] e^{-\frac{1}{2}tr\Psi B^{-1}} dB$$

or, $$g'(1) = \frac{mp}{2}E[h(B)] - \frac{1}{2}E\left[tr\left(\Psi B^{-1}\right)h(B)\right]. \tag{4.3}$$

Now make the transformation $B = tY^{-1}$ in (2.2) with Jacobian $J\left(B \to Y^{-1}\right) = t^{p(p+1)/2}$ since B is a symmetric matrix (equation 2.15.7 of Press, 1982) so that

$$g(t) = c_{p,m}\left|\Psi\right|^{m/2} \int_{Y^{-1}>0} h\left(tY^{-1}\right)\ \left|tY^{-1}\right|^{-(m+p+1)/2} t^{mp/2} e^{-\frac{t}{2}tr\Psi t^{-1}Y}\left[t^{p(p+1)/2}dY^{-1}\right]$$

$$= c_{p,m}\left|\Psi\right|^{m/2} \int_{Y^{-1}>0} f\left(t:Y^{-1}\right)\ \left|Y\right|^{(m+p+1)/2}\ e^{-\frac{t}{2}tr\Psi t^{-1}Y} dY^{-1}$$

and then differentiating we have

$$g'(t) = c_{p,m}\left|\Psi\right|^{m/2} \int_{Y^{-1}>0} f'\left(t:Y^{-1}\right)\ \left|Y\right|^{(m+p+1)/2}\ e^{-\frac{t}{2}tr\Psi t^{-1}Y} dY^{-1}.$$

Since $Y^{-1} = B$ for $t = 1$, we have

$$g'(1) = c_{p,m}\mid\Psi\mid^{m/2} \int_{Y^{-1}>0} \mid B\mid^{-(m+p+1)/2}\ f'\left(1:B\right))\ e^{-\frac{1}{2}tr\Psi Y} dY^{-1}$$

i.e. $g'(1) = E[f'(1;B)]$. (4.4)

The identity in (4.1) follows from (4.3) and (4.4).

In many applications, the function $h(.)$ has the property that, for $u > 0$, $h(uB) = u^l\ h(B)$ for some real l. Then $f(t:B) = h(tB) = t^l h(B)$, so that $f'(1:B) = l\ h(B)$. Then we have the following corollary.

Corollary 4.1: Let $h(xB) = x^l h(B)$ for some l. Then

$$E\left[tr\left(\Psi B^{-1}\right)h(B)\right] = (mp - 2l)h(B).$$

By the use of the above corollary and Lemma 1.1 we have derived the moments of some useful functions of inverted Wishart matrix B in the next section.

5 Some Special Moments

1. $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}\right]=2^{k}(mp/2)_{\{k\}}$.

Proof. Let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}\right]=E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]$ where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}$. Then $h_1(xB)=\left(tr\left(\Psi (xB)^{-1}\right)\right)^{k-1}=x^{-k+1}\,h_1(B)$ so that by Corollary 2.1, we have $l=-(k-1)$ and $E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]=(mp-2l)E\left[h_1(B)\right]$, i.e.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}\right]=$$
$$[mp+2(k-1)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}\right].$$

Next , let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}\right]=E\left[tr\left(\Psi B^{-1}\right)h_2(B)\right]$ where $h_2(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}$. Then $h_2(xB)=x^{-k+2}\,h_2(B)$ so that by Corollary 2.1, we have $l=-(k-2)$ and $E\left[tr\left(\Psi B^{-1}\right)h_2(B)\right]=(mp-2l)E\left[h_2(B)\right]$, i.e

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}\right]=$$
$$[mp+2(k-2)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}\right].$$

Next , let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}\right]=E\left[tr\left(\Psi B^{-1}\right)h_3(B)\right]$ where $h_3(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-3}$. Then $h_3(xB)=x^{-k+3}\,h_3(B)$ so that by Corollary 2.1, we have $l=-(k-3)$ and $E\left[tr\left(\Psi B^{-1}\right)h_3(B)\right]=(mp-2l)E\left[h_3(B)\right]$, i.e

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}\right]=$$
$$[mp+2(k-3)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-3}\right].$$

Similarly, we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-2)}\right]=$$
$$[mp+2\{k-(k-1)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)}\right]$$

and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)}\right]=$$
$$[mp+2(k-k)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k}\right].$$

Finally, we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}\right]=$$
$$[mp-2(-k+1)][mp-2(-k+2)][mp-2(-k+3)]$$
$$\cdots[mp-2(-k+k)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k}\right].$$
$$=2^{k}\left[\frac{mp}{2}+(k-1)\right]\left[\frac{mp}{2}+(k-2)\right]\left[\frac{mp}{2}+(k-3)\right]$$
$$\cdots\left[\frac{mp}{2}+(k-k)\right]=2^{k}\ \left(mp/2\right)_{\{k\}}$$

2. $E\left[tr\left(\Psi B^{-1}\right)^{-k}\right]=\dfrac{2^{-k}}{(mp/2-1)^{\{k\}}}$, $mp>2k$.

Proof. Let $E\left[tr\left(\Psi B^{-1}\right)^{0}\right]=E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]$ where $h_1(B)=\left[tr\left(\Psi B^{-1}\right)\right]^{-1}$. Then

$h_1(xB)=xh_1(B)$ so that $l=1$ and $E\left[tr\left(\Psi B^{-1}\right)^{0}\right]=(mp-2)h_1(B)$, i.e.,

$$E\left[tr\left(\Psi B^{-1}\right)^{0}\right]=(mp-2)E\left[tr\left(\Psi B^{-1}\right)^{-1}\right].$$

Next, let $E\left[tr\left(\Psi B^{-1}\right)^{-1}\right]=E\left[tr\left(\Psi B^{-1}\right)h_2(B)\right]$ where $h_2(B)=\left[tr\left(\Psi B^{-1}\right)\right]^{-2}$. Then

$h_2(xB)=x^{2}\,h_2(B)$ so that $l=2$ and $E\left[tr\left(\Psi B^{-1}\right)^{-1}\right]=[mp-2(2)]h_1(B),$, i.e.,

$$E\left[tr\left(\Psi B^{-1}\right)^{-1}\right]=[mp-2(2)]E\left[tr\left(\Psi B^{-1}\right)^{-2}\right].$$

Next, let $E\left[tr\left(\Psi B^{-1}\right)^{-2}\right]=E\left[tr\left(\Psi B^{-1}\right)h_3(B)\right]$ where $h_3(B)=\left[tr\left(\Psi B^{-1}\right)\right]^{-3}$. Then

$h_3(xB)=x^3\ h_3(B)$ so that $l=3$ and $E\left[tr\left(\Psi B^{-1}\right)^{-2}\right]=[mp-2(3)]h_2(B),$, i.e.,

$$E\left[tr\left(\Psi B^{-1}\right)^{-2}\right]=[mp-2(3)]E\left[tr\left(\Psi B^{-1}\right)^{-3}\right].$$

Similarly we have

$$E\left[tr\left(\Psi B^{-1}\right)^{-3}\right]=[mp-2(4)]E\left[tr\left(\Psi B^{-1}\right)^{-4}\right],$$

.−.−.−.

$$E\left[tr\left(\Psi B^{-1}\right)^{-(k-2)}\right]=$$
$$[mp-2(k-1)]E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}\right] \text{ and}$$

$$E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}\right]=[mp-2(k)]E\left[tr\left(\Psi B^{-1}\right)^{-k}\right].$$

That is

$$\frac{1}{(mp-2)}E\left[tr\left(\Psi B^{-1}\right)^{0}\right]=E\left[tr\left(\Psi B^{-1}\right)^{-1}\right],$$

$$\frac{1}{[mp-2(2)]}E\left[tr\left(\Psi B^{-1}\right)^{-1}\right]=E\left[tr\left(\Psi B^{-1}\right)^{-2}\right],$$

$$\frac{1}{[mp-2(3)]}E\left[tr\left(\Psi B^{-1}\right)^{-2}\right]=E\left[tr\left(\Psi B^{-1}\right)^{-3}\right],$$

.−.−.−.

$$\frac{1}{[mp-2(k-1)]}E\left[tr\left(\Psi B^{-1}\right)^{-(k-2)}\right]=$$
$$E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}\right]$$

and

$$\frac{1}{[mp-2(k)]}E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}\right]=E\left[tr\left(\Psi B^{-1}\right)^{-k}\right].$$

In general for any $k>0$, we have

$$E\left[tr\left(\Psi B^{-1}\right)^{-k}\right]=$$
$$\frac{1}{mp-2(k)}\times\frac{1}{mp-2(2)}\times\frac{1}{mp-2}E\left[tr\left(\Psi B^{-1}\right)^{0}\right]$$
$$=\frac{2^{-k}}{(mp/2-1)^{\{k\}}},mp>2k.$$

3. $E\left[\left(tr\left(\Psi^{-1}B\right)\right)^{k}tr(B)\right]=$
$2^{k-1}(mp/2)_{\{k-1\}}E\left[\left(tr\left(\Psi B^{-1}\right)\right)trB\right]$.

Proof. Let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}trB\right]=E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]$ where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}trB.$ Then $h_1(xB)=x^{-k+2}\ h_1(B)$ so that by Corollary 2.1, we have $l=-k+2$ and $E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]=[mp-2(-k+2)]E\left[h_1(B)\right]$, i.e.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}trB\right]=$$
$$[mp+2(k-2)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}trB\right].$$

Similarly,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}trB\right]=$$
$$[mp+2(k-3)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}trB\right],$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}trB\right]=$$
$$[mp+2(k-4)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-3}trB\right],$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-3}trB\right]=$$
$$[mp+2(k-5)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-4}trB\right],\ \ldots$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-3)} trB\right]=$$
$$[mp+2(k-(k-1))]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-2)} trB\right]$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-2)} trB\right]=$$
$$[mp+2(k-k)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} trB\right]$$ and

Proceeding thus,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} trB\right]$$
$$=[mp+2(k-2)][mp+2(k-3)]\cdots$$
$$\times\cdots[mp+2(k-(k-1))][mp+2(k-k)]E$$
$$\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} trB\right],$$

which can be written as

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} trB\right]$$
$$=2^{k-1}\times\left[\frac{mp}{2}+(k-2)\right]\left[\frac{mp}{2}+(k-3)\right]\cdots$$
$$\times\cdots\left[\frac{mp}{2}+2\{k-(k-1)\}\right]\left[\frac{mp}{2}+2(k-k)\right]$$
$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} trB\right],$$

which can be written as

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} trB\right]=$$
$$2^{k-1}\left(\frac{mp}{2}\right)_{\{k-1\}}E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} trB\right].$$

4. $E\left[\left(tr\Psi^{-1}B\right)^{-k} trB\right]=$
$$\frac{1}{2^{k}\ (m-p-1)(mp/2)_{\{k\}}}\ tr\Psi,\quad m>p+1.$$

Proof. Let $E\left[tr\left(\Psi^{-1}B\right)^{0} trB\right]=E\left[\left(tr\Psi^{-1}B\right)h_1(B)\right]$ where $h_1(B)=\left(tr\Psi^{-1}B\right)^{-1} trB$. Then

$h_1(xB)=x^0h_1(B)$ so that $l=0$ and $E\left[\left(tr\Psi^{-1}B\right)h_1(B)\right]=(mp-2\times 0)E[h_1(B)]$, i.e.,

$E\left[tr\left(\Psi^{-1}B\right)^{0} trB\right]=mp\ E\left[\left(tr\Psi^{-1}B\right)^{-1} trB\right]$.Next let $E\left[tr\left(\Psi^{-1}B\right)^{-1} trB\right]=E\left[\left(tr\Psi^{-1}B\right)^{1} h_2(B)\right]$ where $h_2(B)=\left(tr\Psi^{-1}B\right)^{-2} trB$. Then

$h_2(xB)=\left(tr\Psi^{-1}xB\right)^{-2} tr(xB)=x^{-1}h_2(B)$ so that $l=-1$ and

$E\left[\left(tr\Psi^{-1}B\right)h_2(B)\right]=[mp-2(-1)]E[h_2(B)]$, i.e.,

$$E\left[\left(tr\Psi^{-1}B\right)^{-1} trB\right]=$$
$$[mp\ \ -2(-1)]E\left[\left(tr\Psi^{-1}B\right)^{-2} trB\right].$$

Next let $E\left[\left(tr\Psi^{-1}B\right)^{-2} trB\right]=E\left[\left(tr\Psi^{-1}B\right)h_3(B)\right]$ where $h_3(B)=\left(tr\Psi^{-1}B\right)^{-3} trB$. Then

$h_3(xB)=\left(tr\Psi^{-1}xB\right)^{-3} tr(xB)=x^{-2}h_3(B)$ so that $l=-2$ and

$E\left[\left(tr\Psi^{-1}B\right)h_3(B)\right]=[mp-2(-2)]E[h_3(B)]$, i.e.,

$$E\left[\left(tr\Psi^{-1}B\right)^{-2} trB\right]=$$
$$[mp\ \ -2(-2)]E\left[\left(tr\Psi^{-1}B\right)^{-3} trB\right].$$

Similarly,

$$E\left[\left(tr\Psi^{-1}B\right)^{-(k-2)} trB\right]=$$
$$[mp-2\{-(k-2)\}]E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)} trB\right]$$ and

$$E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)} trB\right]=$$
$$[mp-2\{-(k-1)\}]E\left[tr\left(\Psi^{-1}B\right)^{-k} trB\right]$$

Proceeding thus, we have

$$mp\ E\left[tr\left(\Psi^{-1}B\right)^{0} trB\right]= E\left[\left(tr\Psi^{-1}B\right)^{-1} trB\right],$$

$$\frac{1}{[mp\ -2(-1)]}E\left[\left(tr\Psi^{-1}B\right)^{-1} trB\right]= E\left[\left(tr\Psi^{-1}B\right)^{-2} trB\right],$$

$$\frac{1}{[mp-2\{-(k-2)\}]}E\left[\left(tr\Psi^{-1}B\right)^{-(k-2)} trB\right]= E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)} trB\right]$$ and

$$\frac{1}{[mp-2\{-(k-1)\}]}E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)} trB\right]= E\left[tr\left(\Psi^{-1}B\right)^{-k} trB\right].$$

Finally, we have

$$E\left[\left(tr\Psi^{-1}B\right)^{-k} trB\right] = \frac{1}{2^k\left[\frac{mp}{2}+(k-1)\right]}\times\frac{1}{\left[\frac{mp}{2}+(k-2)\right]}\times\cdots\times \frac{1}{\left[\frac{mp}{2}+2\right]}\times\frac{1}{\left[\frac{mp}{2}+1\right]}\times\frac{1}{\left[\frac{mp}{2}\right]}E\left[tr\left(\Psi^{-1}B\right)^{0} trB\right],$$

which can be written as

$$E\left[\left(tr\Psi^{-1}B\right)^{-k} trB\right]= \frac{1}{2^k\ (mp/2)_{\{k\}}}\ E(trB), m>p+1.$$

5. $E\left[\left(tr\left(\Psi^{-1}B\right)\right)^{k} tr\left(B^{-1}\right)\right]= 2^k\ m\left((mp/2)+1\right)_{\{k\}} tr\Psi^{-1}.$

Proof. Let

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(B^{-1}\right)\right] = E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]$$

where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(B^{-1}\right).$

Then $h_1(xB)=x^{-k}\ h_1(B)$

so that by Corollary 2.1, we have $l=-k$ and $E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]= [mp-2(-k+2)]E\left[h_1(B)\right],$ i.e.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(B^{-1}\right)\right]= [mp+2k]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(B^{-1}\right)\right].$$

Similarly,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(B^{-1}\right)\right]= [mp+2(k-1)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2} tr\left(B^{-1}\right)\right]$$

and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} tr\left(B^{-1}\right)\right]= [mp+2\{k-(k-1)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k} tr\left(B^{-1}\right)\right].$$

Proceeding thus,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(B^{-1}\right)\right] =[mp+2k][mp+2(k-1)]\cdots \ldots[mp+2\{k-(k+1)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k} tr\left(B^{-1}\right)\right],$$

which simplifies to

$$E\left[\left(tr\left(\Psi^{-1}B\right)\right)^{k} tr\left(B^{-1}\right)\right]=2^k\ m\left(\frac{mp}{2}+1\right)_{\{k\}}\ EtrB^{-1}.$$

Since, $EtrB^{-1}=trE(B^{-1})=trE(A)=mtr\Sigma=mtr\Psi^{-1}$, we have

$$E\left[\left(tr\left(\Psi^{-1}B\right)\right)^{k} tr\left(B^{-1}\right)\right]= 2^k\ m\left((mp/2)+1\right)_{\{k\}} tr\Psi^{-1}.$$

6. $E\left[\left(tr\Psi^{-1}B\right)^{k} tr\left(B^{-1}\right)\right]=\dfrac{mtr\Psi^{-1}}{2^k\ \left(\frac{mp}{2}+2\right)_{\{k\}}}.$

Proof. Let

$$E\left[\left(tr\Psi^{-1}B\right)^{0}tr\left(B^{-1}\right)\right]=E\left[\left(tr\Psi^{-1}B\right)h_1(B)\right]$$

where $h_1(B)=\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)$.

Then $h_1(xB)=\left(tr\Psi^{-1}xB\right)^{-1}tr\left(\left(xB\right)^{-1}\right)=x^{-2}h_1(B)$

so that $l=-2$ and $E\left[\left(tr\Psi^{-1}B\right)h_1(B)\right]=\left[mp-2\times(-2)\right]E[h_1(B)]$, i.e.,

$$E\left[tr\left(B^{-1}\right)\right]=\left[mp+2(2)\right]E\left[\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)\right].$$

Let $E\left[\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)\right]=E\left[\left(tr\Psi^{-1}B\right)h_2(B)\right]$ where $h_2(B)=\left(tr\Psi^{-1}B\right)^{-2}tr\left(B^{-1}\right)$. Then $h_2(xB)=\left(tr\Psi^{-1}xB\right)^{-2}tr\left(\left(xB\right)^{-1}\right)=x^{-3}h_2(B)$ so that $l=-3$ and $E\left[\left(tr\Psi^{-1}B\right)h_2(B)\right]=\left[mp-2\times(-3)\right]E[h_2(B)]$, i.e.,

$$E\left[\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)\right]=\left[mp+2(3)\right]E\left[\left(tr\Psi^{-1}B\right)^{-2}tr\left(B^{-1}\right)\right].$$

Similarly

$$E\left[\left(tr\Psi^{-1}B\right)^{-(k-2)}tr\left(B^{-1}\right)\right]=\left[mp+2(k)\right]E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)}tr\left(B^{-1}\right)\right],$$

$$E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)}tr\left(B^{-1}\right)\right]=\left[mp+2(k+1)\right]E\left[\left(tr\Psi^{-1}B\right)^{-k}tr\left(B^{-1}\right)\right]$$

That is

$$E\left[\left(tr\Psi^{-1}B\right)^{-k}tr\left(B^{-1}\right)\right]=\frac{1}{\left[mp+2(k+1)\right]}E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)}tr\left(B^{-1}\right)\right],$$

$$E\left[\left(tr\Psi^{-1}B\right)^{-(k-1)}tr\left(B^{-1}\right)\right]=\frac{1}{\left[mp+2(k)\right]}E\left[\left(tr\Psi^{-1}B\right)^{-(k-2)}tr\left(B^{-1}\right)\right]$$

.−.−.−.

$$E\left[\left(tr\Psi^{-1}B\right)^{-2}tr\left(B^{-1}\right)\right]=\frac{1}{\left[mp+2(3)\right]}E\left[\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)\right]$$

$$E\left[\left(tr\Psi^{-1}B\right)^{-1}tr\left(B^{-1}\right)\right]=\frac{1}{\left[mp+2(2)\right]}E\left[tr\left(B^{-1}\right)\right]$$

Finally, we have

$$E\left[\left(tr\Psi^{-1}B\right)^{-k}tr\left(B^{-1}\right)\right]=\frac{1}{\left[mp+2(k+1)\right]}\frac{1}{\left[mp+2(k)\right]}\cdots\frac{1}{\left[mp+2(3)\right]}\frac{1}{\left[mp+2(2)\right]}E\left[tr\left(B^{-1}\right)\right]$$

$$=\frac{1}{2^k\times\frac{mp}{2}+(k+1)}\times\frac{1}{\frac{mp}{2}+k}\times\cdots\times\frac{1}{\frac{mp}{2}+3}\times\frac{1}{\frac{mp}{2}+2}tr\left[E\left(B^{-1}\right)\right]$$

which simplifies to

$$E\left[\left(tr\Psi^{-1}B\right)^{-k}tr\left(B^{-1}\right)\right]=\frac{1}{2^k}\times\frac{1}{\left(\frac{mp}{2}+2\right)_{\{k\}}}tr\left[E\left(A\right)\right],$$

which can be evaluated to

$$E\left[\left(tr\Psi^{-1}B\right)^{k}tr\left(B^{-1}\right)\right]=\frac{mtr\Sigma}{2^k\left(\frac{mp}{2}+2\right)_{\{k\}}}.$$

7. $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k}tr\left(\Psi^{-1}B\right)\right]=\frac{2^{k-2}p}{m-p-1}\left((mp/2)+1\right)_{\{k-2\}},\ m>p+1.$

Proof. Let

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(\Psi^{-1}B\right)\right]=E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]$$

where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(\Psi^{-1}B\right)$. Then $h_1(xB)=x^{-k+2}\, h_1(B)$ so that by Corollary 2.1, we have $l=-k+2$ and $E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]=[mp-2(-k+2)]E\left[h_1(B)\right]$, i.e.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(\Psi^{-1}B\right)\right]=$$

$$[mp+2(k-2)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(\Psi^{-1}B\right)\right].$$

Similarly, we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} tr\left(\Psi^{-1}B\right)\right]=$$

$$[mp+2(k-3)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2} tr\left(\Psi^{-1}B\right)\right]$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-2)} tr\left(\Psi^{-1}B\right)\right]=$$

$$[mp+2\{k-(k)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} tr\left(\Psi^{-1}B\right)\right]$$

and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)} tr\left(\Psi^{-1}B\right)\right]=$$

$$[mp+2\{k-(k+1)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k} tr\left(\Psi^{-1}B\right)\right].$$

Finally we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(\Psi^{-1}B\right)\right]$$

$$=[mp+2(k-2)][mp+2(k-3)]\cdots$$

$$...[mp+2\{k-(k+1)\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k} tr\left(\Psi^{-1}B\right)\right].$$

which can be simplified to be

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(\Psi^{-1}B\right)\right]=$$

$$2^{k-2}\ \frac{\left((mp/2)+1\right)_{\{k-2\}} tr\Psi^{-1}E(B)}{m-p-1},\ \ m>p+1,$$

Since

$$tr\Psi^{-1}E(B)=tr\Psi^{-1}\left(\frac{1}{m-p-1}\Psi\right)=$$

$$\frac{1}{m-p-1}trI_p=\frac{p}{m-p-1},$$

we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} tr\left(\Psi^{-1}B\right)\right]=$$

$$\frac{2^{k-2}p}{m-p-1}\ \left((mp/2)+1\right)_{\{k-2\}},\ \ m>p+1.$$

8. $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right]=$

$$\frac{2^{-k}p}{(m-p-1)\left((mp/2)-k-1\right)_{\{k\}}},$$

$m>\max\{p+1,\ 4/p\}$

Proof. Let

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{0} tr\left(\Psi^{-1}B\right)\right]=E\left[\left(tr\left(\Psi B^{-1}\right)\right)h_1(B)\right]$$

where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)$. Then $h_1(xB)=x^2h_1(B)$ so that $l=2$ and $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{0} h_1(B)\right]=\left[mp-2(2)\right]E[h_1(B)]$, i.e.,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{0} tr\left(\Psi B^{-1}\right)\right]=$$

$$\left[mp-2(2)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)\right].$$

Next let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)\right]=E\left[\left(tr\left(\Psi B^{-1}\right)\right)h_2(B)\right]$

where $h_2(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{-2} tr\left(\Psi^{-1}B\right)$. Then $h_2(xB)=x^3h_2(B)$ so that $l=3$ and $E\left[\left(tr\left(\Psi B^{-1}\right)\right)h_2(B)\right]=\left[mp-2(3)\right]E[h_2(B)]$, i.e.,

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)\right]=$$

$$\left[mp-2(3)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-2} tr\left(\Psi^{-1}B\right)\right]$$

Similarly

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-2)} tr\left(\Psi^{-1}B\right)\right] =$$
$$\left[mp-2(k)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)} tr\left(\Psi^{-1}B\right)\right]$$

and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)} tr\left(\Psi^{-1}B\right)\right] =$$
$$\left[mp-2(k+1)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right].$$

That is

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{\left[mp-2(k+1)\right]}E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)} tr\left(\Psi^{-1}B\right)\right],$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{\left[mp-2(k)\right]}E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-2)} tr\left(\Psi^{-1}B\right)\right],$$

.−.−.−.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-2} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{\left[mp-2(3)\right]}E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)\right],$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{\left[mp-2(2)\right]}E\left[tr\left(\Psi B^{-1}\right)\right].$$

Finally we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right]$$
$$=\frac{1}{\left[mp-2(k+1)\right]}\times\frac{1}{\left[mp-2(k)\right]}\times\cdots$$
$$\times\frac{1}{\left[mp-2(3)\right]}\times\frac{1}{\left[mp-2(2)\right]}E\left[tr\left(\Psi^{-1}B\right)\right],$$

which can be written as

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right]$$
$$=\frac{1}{2^k\left[(mp/2)-(k+1)\right]}\times\frac{1}{\left[(mp/2)-k\right]}\times\cdots$$
$$\times\frac{1}{\left[(mp/2)-3\right]}\times\frac{1}{\left[(mp/2)-2\right]}E\left[tr\left(\Psi^{-1}B\right)\right]$$

which can be represented by

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{2^k}\frac{1}{\left((mp/2)-2\right)^{\{k\}}}E\left[tr\left(\Psi^{-1}B\right)\right],$$

Since,

$$E\left[tr\left(\Psi^{-1}B\right)\right] = tr\Psi^{-1}E(B) = tr\Psi^{-1}\left(\frac{1}{m-p-1}\Psi\right)$$
$$=\frac{trI_p}{m-p-1}=\frac{p}{m-p-1},$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{1}{2^k}\frac{1}{\left((mp/2)-2\right)^{\{k\}}}\frac{p}{m-p-1}.$$

Since,

$$x^{\{n\}} = x(x-1)\cdots(x-n+1) =$$
$$(x-n+1)(x-n+2)\cdots(x-1)x = (x-n+1)_{\{n\}},$$

we have $\left((mp/2)-2\right)^{\{k\}} = ((mp/2)-2-k+1)_{\{k\}}$, and the above can be represented as

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} tr\left(\Psi^{-1}B\right)\right] =$$
$$\frac{2^{-k}p}{(m-p-1)\left((mp/2)-k-1\right)_{\{k\}}},$$
$$m > \max\{p+1,\ 4/p\}.$$

9. $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} | B^{-1}|^{h}\right] =$

$$2^{k+hp}\ \left((mp/2)+hp\right)_{\{k\}}\frac{\Gamma_p(m/2+h)}{\Gamma_p(m/2)}|\Psi|^{-h}.$$

Proof. Let

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k} | B^{-1}|^{h}\right] = E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1} h_1(B)\right]$$

where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}|B^{-1}|^h$. Then $h_1(xB)=x^{-k-hp+1}\ h_1(B)$ so that by Corollary 2.1, we have $l=-(k+hp-1)$ and $E\left[tr\left(\Psi B^{-1}\right)h_1(B)\right]=$ $[mp-2\{-(k+hp-1)\}]E\left[h_1(B)\right],$ i.e.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^k|B^{-1}|^h\right]=$$
$$[mp+2(k-1+hp)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}|B^{-1}|^h\right].$$

Similarly we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-1}|B^{-1}|^h\right]=$$
$$[mp+2(k-2+hp)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-2}|B^{-1}|^h\right],$$

.−.−.−.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-2)}|B^{-1}|^h\right]=$$
$$[mp+2\{k-(k-1)+hp\}]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)}|B^{-1}|^h\right]$$

and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-(k-1)}|B^{-1}|^h\right]=$$
$$[mp+2(k-k+hp)]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k}|B^{-1}|^h\right]$$

Proceeding thus, we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^k|B^{-1}|^h\right]$$
$$=[mp+2(k-1+hp)][mp+2(k-2+hp)]\cdots$$
$$[mp+2\{k-(k-1)+hp\}][mp+2(k-k+hp)].$$
$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k}|B^{-1}|^h\right]$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^k|B^{-1}|^h\right]$$
$$=[(mp/2)+(k-1+hp)][(mp/2)+(k-2+hp)]\cdots$$
$$[(mp/2)+\{k-(k-1)+hp\}][(mp/2)+(k-k+hp)]$$
$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{k-k}|B^{-1}|^h\right]$$

which can be represented by

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^k|B^{-1}|^h\right]=$$
$$2^k\ \left((mp/2)+hp\right)_{\{k\}}E\left[|B|^{-h}\right]$$

Now, from ( ) we have

$$E\left(|B|^{-h}\right)=\frac{\Gamma_p(m/2+h)}{2^{-hp}\ \Gamma_p(m/2)}|\Psi|^{-h}.$$

Since $B^{-1}=A$, we also have

$$E\left(|B|^{-h}\right)=E\left(|A|^h\right)=2^{hp}\frac{\Gamma_p((m/2)+h)}{\Gamma_p(m/2)}|\Sigma|^h,$$

which is equivalent to what we have before.

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^k|B^{-1}|^h\right]=$$
$$2^k\ \left((mp/2)+hp\right)_{\{k\}}\frac{\Gamma_p(m/2+h)}{2^{-hp}\ \Gamma_p(m/2)}|\Psi|^{-h}$$

10. $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k}|B^{-1}|^h\right]=$

$$\frac{\Gamma_p(m/2+h)}{2^{hp+k}\left(mp/2-k+ph\right)_k\Gamma_p(m/2)}\ |\Psi|^{-h}$$

Proof. Let $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^0|B^{-1}|^h\right]=E\left[\left(tr\left(\Psi B^{-1}\right)\right)h_1(B)\right]$ where $h_1(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{-1}|B^{-1}|^h$. Then $h_1(xB)=\left(tr\left(\Psi\left(xB\right)^{-1}\right)\right)^{-1}|\left(xB\right)^{-1}|^h=x^{1-ph}h_1(B)$ so that $l=1-ph$ and $E\left[\left(tr\left(\Psi B^{-1}\right)\right)^0 h_1(B)\right]=\left[mp-2(1-ph)\right]E[h_1(B)]$, i.e.,

$$E\left[tr\left(\Psi B^{-1}\right)^0|B^{-1}|^h\right]=$$
$$\left[mp-2(1-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1}|B^{-1}|^h\right].$$

Let

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1}|B^{-1}|^{h}\right]=E\left[\left(tr\left(\Psi B^{-1}\right)\right)h_2(B)\right]$$

where $h_2(B)=\left(tr\left(\Psi B^{-1}\right)\right)^{-2}|B^{-1}|^{h}$. Then

$$h_2(xB)=\left(tr\left(\Psi\left(xB\right)^{-1}\right)\right)^{-2}|\left(xB\right)^{-1}|^{h}=x^{2-ph}\ h_2(B)$$

so that $l=2-ph$ and

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1}h_2(B)\right]=\left[mp-2(2-ph)\right]E[h_2(B)]$$

, i.e.,

$$E\left[tr\left(\Psi B^{-1}\right)^{-1}|B^{-1}|^{h}\right]=$$

$$\left[mp-2(2-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-2}|B^{-1}|^{h}\right].$$

Similarly, we have

$$E\left[tr\left(\Psi B^{-1}\right)^{0}|B^{-1}|^{h}\right]=$$

$$\left[mp-2(1-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1}|B^{-1}|^{h}\right],$$

$$E\left[tr\left(\Psi B^{-1}\right)^{-1}|B^{-1}|^{h}\right]=$$

$$\left[mp-2(2-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-2}|B^{-1}|^{h}\right],$$

.−.−.−.

$$E\left[tr\left(\Psi B^{-1}\right)^{-(k-2)}|B^{-1}|^{h}\right]=$$

$$\left[mp-2(k-1-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)}|B^{-1}|^{h}\right]$$

and

$$E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}|B^{-1}|^{h}\right]=$$

$$\left[mp-2(k-ph)\right]E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k}|B^{-1}|^{h}\right]$$

That is

$$\frac{1}{\left[mp-2(1-ph)\right]}E\left[tr\left(\Psi B^{-1}\right)^{0}|B^{-1}|^{h}\right]=$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-1}|B^{-1}|^{h}\right],$$

$$\frac{1}{\left[mp-2(2-ph)\right]}E\left[tr\left(\Psi B^{-1}\right)^{-1}|B^{-1}|^{h}\right]=$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-2}|B^{-1}|^{h}\right],$$

.−.−.−.

$$\frac{1}{\left[mp-2(k-1-ph)\right]}E\left[tr\left(\Psi B^{-1}\right)^{-(k-2)}|B^{-1}|^{h}\right]$$

$$=E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-(k-1)}|B^{-1}|^{h}\right]$$

and

$$\frac{1}{\left[mp-2(k-ph)\right]}E\left[tr\left(\Psi B^{-1}\right)^{-(k-1)}|B^{-1}|^{h}\right]=$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k}|B^{-1}|^{h}\right]$$

Finally, we have

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k}|B^{-1}|^{h}\right]$$

$$=\frac{1}{\left[mp-2(k-ph)\right]}\times\frac{1}{\left[mp-2(k-1-ph)\right]}\times\cdots$$

$$\times\frac{1}{\left[mp-2(2-ph)\right]}\times\frac{1}{\left[mp-2(1-ph)\right]}$$

$$E\left[tr\left(\Psi B^{-1}\right)^{0}|B^{-1}|^{h}\right]$$

$$=2^{-k}\frac{1}{\frac{mp}{2}+(ph-k)}\frac{1}{\frac{mp}{2}+(ph-k+1)}\cdots$$

$$\cdots\frac{1}{\frac{mp}{2}+(ph-2)}\frac{1}{\frac{mp}{2}+(ph-1)}E\left[|B^{-1}|^{h}\right]$$

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k}|B^{-1}|^{h}\right]$$

$$=2^{-k}\times\frac{1}{\frac{mp}{2}+(ph-k)}\times\frac{1}{\frac{mp}{2}+(ph-k+1)}\times\cdots$$

$$\times\frac{1}{\frac{mp}{2}+(ph-2)}\times\frac{1}{\frac{mp}{2}+(ph-1)}E(|B|^{-h})$$

which can be evaluated as

$$E\left[\left(tr\left(\Psi B^{-1}\right)\right)^{-k} |B^{-1}|^{h}\right]=$$
$$\frac{\Gamma_p(m/2+h)}{2^{hp+k}\left(mp/2-k+ph\right)_{\{k\}}\Gamma_p(m/2)}\;|\Psi|^{-h}.$$

Acknowledgement

The first author is highly and gratefully indebted to the Prince Mohammad Bin Fahd University, Al Khobar, Saudi Arabia, for providing all necessary research facilities during the preparation of this research paper. The authors acknowledge the excellent research support provided by Kind Fahd University of Petroleum and Minerals, Saudi Arabia especially through a project # FT 2004(23) at the university.

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