On dual of the split-off matroids

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Abstract: Azadi et al. [Generalization of Splitting off Operation to Binary Matroids, Electronic Notes in Discrete Math, 15 (2003), 186–188] have generalized the splitting off (or in short split-off) operation on graphs to binary matroids. The dual of a split-off matroid is not always equal to the split-off of dual of the original matroid. In this paper, for a given matroid M and two elements x and y from $E(M)$, we first characterize the cobases of the split-off matroid M_{xy} in terms of the cobases of the matroid M. Then, by using the set of cobases of M_{xy} and the set of bases (Azadi characterized this set) of $(M^*)_{xy}$, we characterize those binary matroids for which $(M_{xy})^* = (M^*)_{xy}$. Indeed, for a binary matroid M on a set E with $x, y \in E$, we prove that $(M_{xy})^* = (M^*)_{xy}$ if and only if $M = N \oplus N'$ where N is an arbitrary binary matroid and N' is $U_{0,2}$ or $U_{2,2}$ such that $x, y \in E(N')$.

Key–Words: Binary matroid, Uniform matroid, Direct sum, Split-off operation

1 Introduction

The matroid notation and terminology used here will follow Oxley [5]. A matroid M is a pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets E having the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are two members of $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The collection $\mathcal{I}$ forms the *independent sets* of M, and the set E is called the *ground set* of M. We shall often write $\mathcal{I}(M)$ for $\mathcal{I}$ and $E(M)$ for E, particularly when several matroids are being considered. A maximal independent set in M is called *a basis* or *a base* of M. If $\mathcal{B}$ (or $\mathcal{B}(M)$) be the collection of all bases of M, the matroid M can be defined in terms of its bases and is denoted by the pair $(E, \mathcal{B})$. The collection $\mathcal{B}$ has the following properties:

(B1) $\mathcal{B}$ is non-empty.

(B2) If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

An alternate version of **(B2)** says that if B_1 and B_2 are bases of a matroid M, then $|B_1| = |B_2|$. For a given matroid M with a basis B, the *rank* of M is the cardinality of B and is denoted by $r(M)$. Let M be a matroid on the ground set E and $\mathcal{B}^*(M)$ be $\{E(M) - B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid on $E(M)$. The matroid, whose ground set is $E(M)$ and whose set of bases is $\mathcal{B}^*(M)$, is called the *dual* of M and is denoted by M^*. The bases of M^* are called *cobases* of M and the rank of M^* is called the *corank* of M and is denoted by $r^*(M)$. Clearly, $r^*(M) = |E(M)| - r(M)$.

Let $\mathbb{F}$ be a field and let $E \subseteq \mathbb{F}^k$ be a finite set of vectors. Then a *linear matroid* is a matroid whose bases are the maximal linearly independent sets of vectors in E over $\mathbb{F}$. A *binary matroid* is a linear matroid over the finite field $GF(2)$. The matroid just obtained from the matrix A is called the *vector matroid* of A.

Two matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ are isomorphic if there exists bijection $\phi : 2^{(E_1)} \longrightarrow 2^{(E_2)}$ such that $X \in \mathcal{B}_1$ if and only if $\phi(X) \in \mathcal{B}_2$. If M and M' are two isomorphic matroids, then $r(M) = r(M')$ and $r^*(M) = r^*(M')$.

The definitions of two important classes of matroids and duals of them, which will be used in this paper are formally described as follows.

Definition 1. *Let m and n be non-negative integers with $m \leq n$. Let E be an n-element set and $\mathcal{B}$ be the collection of m-element subsets of E. Then this matroid called the* uniform matroid *on n-element set and denoted by $U_{m,n}$.*

Clearly, the dual of $U_{m,n}$ is $U_{n-m,n}$.

Definition 2. *Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. Let $E = E_1 \cup E_2$ and*

$$\mathcal{B} = \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1) \quad and \quad B_2 \in \mathcal{B}(M_2)\}. \tag{1}$$

Then $(E, \mathcal{B})$ is a matroid and called the direct sum *of M_1 and M_2 and is denoted by $M_1 \oplus M_2$.*

Clearly, if $M = M_1 \oplus M_2$, then $M^* = M_1^* \oplus M_2^*$.

Splitting off operation for graphs was introduced by Lovasz [4] as follows. Let G be a graph and $x = vv_1$, $y = vv_2$ be two adjacent non-loop edges in G. Let G_{xy} be the graph obtained from G by adding the edge v_1v_2 and deleting the edges x and y. The transition from G to G_{xy} is called a *splitting off* (or in short *split-off*) operation. The split-off operation has important applications in graph theory [3], [4]. Applying this operation is a well-known and useful method for solving problems in graph connectivity and it may decrease the edge connectivity of the graph.

Shikare, Azadi and Waphare [6],[7] extended the notion of the split-off operation from graphs to binary matroids as follows.

Definition 3. *Let M be a binary matroid on a set E and let $x, y \in E$. Let A be a matrix that represents M over $GF(2)$ and A_{xy} be the matrix obtained from A by adjoining an extra column, with label α, which is the sum of the columns corresponding to x and y, and then deleting the two columns corresponding to x and y. Let M_{xy} be the vector matroid of the matrix A_{xy}. The transition from M to M_{xy} is called a* split-off operation *and the matroid M_{xy} is referred to the* split-off matroid.

Definition 4. *Two non-loop (a loop is a minimal singleton set which is not independent) elements x and y from binary matroid M are called* equivalent*, if every basis of M contains at least one of x and y. Note that two coloops of M (loops of M^*) are equivalent.*

For a given binary matroid M and two elements x and y from $E(M)$, we denote by $x \sim y$ the two equivalent elements x and y and, otherwise, we denote by $x \nsim y$. The next proposition provides a useful characterization of bases of M_{xy} in terms of the bases of M.

Proposition 5. ([6]) *Let M be a binary matroid on a set E and $x, y \in E$ such that $\alpha \notin E$. Let $\mathcal{B}$ and $\mathcal{B}_{xy}$ be the set of bases of M and M_{xy}, respectively. Then*

(i) *If $x \sim y$, then $\mathcal{B}_{xy} = \mathcal{B}_1 \cup \mathcal{B}_2$ where*

- $\mathcal{B}_1 = \Big\{B - \{x\} : B \in \mathcal{B},\ x \in B \text{ and } y \notin B\Big\} = \Big\{B - \{y\} : B \in \mathcal{B},\ x \notin B \text{ and } y \in B\Big\}$;
- $\mathcal{B}_2 = \Big\{(B - \{x, y\}) \cup \{\alpha\} : B \in \mathcal{B} \text{ and } x, y \in B\Big\}$.

(ii) *If $x \nsim y$, then $\mathcal{B}_{xy} = \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \mathcal{B}'_3$ where*

- $\mathcal{B}'_1 = \Big\{B : B \in \mathcal{B} \text{ and } x, y \notin B\Big\}$;
- $\mathcal{B}'_2 = \Big\{(B - \{x\}) \cup \{\alpha\} : B \in \mathcal{B},\ x \in B,\ y \notin B \text{ and } (B - \{x\}) \cup \{y\} \notin \mathcal{B}\Big\}$;
- $\mathcal{B}'_3 = \Big\{(B - \{y\}) \cup \{\alpha\} : B \in \mathcal{B},\ x \notin B,\ y \in B \text{ and } (B - \{y\}) \cup \{x\} \notin \mathcal{B}\Big\}$.

By the last proposition, we deduce that if $x \sim y$ in M, then $r(M_{xy}) = r(M) - 1$, otherwise $r(M_{xy}) = r(M)$.

2 Cobases of the split-off matroid

In this section, we characterize the cobases of the split-off matroid M_{xy} in terms of the cobases of M. We denote by $(\mathcal{B}_{xy})^*$ the set of cobases of the split-off matroid M_{xy}.

Theorem 6. *Let $M = (E, \mathcal{B})$ be a binary matroid with collection of bases $\mathcal{B}$ and $M^* = (E, \mathcal{B}^*)$ be the dual of M with collection of bases $\mathcal{B}^*$. Let $x, y \in E$ such that $\alpha \notin E$, and let $\mathcal{B}_{xy}$ and $(\mathcal{B}_{xy})^*$ be the collections of bases of M_{xy} and $(M_{xy})^*$, respectively. Then*

(i) *If $x \sim y$ in M, then $(\mathcal{B}_{xy})^* = (\mathcal{B}_1)^* \cup (\mathcal{B}_2)^*$ where*

- $(\mathcal{B}_1)^* = \Big\{(B^* - \{x\}) \cup \{\alpha\} : B^* \in \mathcal{B}^*,\ x \in B^* \text{ and } y \notin B^*\Big\} = \Big\{(B^* - \{y\}) \cup \{\alpha\} : B^* \in \mathcal{B}^*,\ x \notin B^* \text{ and } y \in B^*\Big\}$;
- $(\mathcal{B}_2)^* = \Big\{B^* : B^* \in \mathcal{B}^* \text{ and } x, y \notin B^*\Big\}$.

(ii) *If $x \nsim y$ in M, then $(\mathcal{B}_{xy})^* = (\mathcal{B}'_1)^* \cup (\mathcal{B}'_2)^* \cup (\mathcal{B}'_3)^*$ where*

- $(\mathcal{B}'_1)^* = \Big\{(B^* - \{x, y\}) \cup \{\alpha\} : B^* \in \mathcal{B}^* \text{ and } x, y \in B^*\Big\}$;

- $(\mathcal{B}'_2)^* = \Big\{(B^* - \{x\}) : B^* \in \mathcal{B}^*,\ x \in B^*,\ y \notin B^*$ *and* $(B^* - \{x\}) \cup \{y\} \notin \mathcal{B}^*\Big\}$;

- $(\mathcal{B}'_3)^* = \Big\{(B^* - \{y\}) : B^* \in \mathcal{B}^*,\ x \notin B^*,\ y \in B^*$ *and* $(B^* - \{y\}) \cup \{x\} \notin \mathcal{B}^*\Big\}$.

Proof. Suppose that E' be the ground set of M_{xy}. Clearly, $E - E' = \{x, y\}$ and $E' - E = \{\alpha\}$. To prove (i) and (ii), we shall show that every member of $(\mathcal{B}_{xy})^*$ is a basis of $(M_{xy})^*$ and every basis of $(M_{xy})^*$ is a member of $(\mathcal{B}_{xy})^*$.

(i) Suppose that $x \sim y$ in M, and $B_1^* \in (\mathcal{B}_1)^*$. Then $B_1^* = (B^* - \{x\}) \cup \{\alpha\}$ where $B^* \in \mathcal{B}^*$, $x \in B^*$ and $y \notin B^*$, or $B_1^* = (B^* - \{y\}) \cup \{\alpha\}$ where $B^* \in \mathcal{B}^*$, $x \notin B^*$ and $y \in B^*$. In the first case, $E' - B_1^* = E' - [(B^* - \{x\}) \cup \{\alpha\}] = (B - \{y\})$ where $B = E - B^*$ and so $x \notin B, y \in B$. By Proposition 5(i), $(B - \{y\})$ is a basis of M_{xy}. We conclude that B_1^* is a basis of $(M_{xy})^*$. Similarly, in the second case, B_1^* is a basis of $(M_{xy})^*$. Now suppose $B_2^* \in (\mathcal{B}_2)^*$. Then $B_2^* = B^*$ and $x, y \notin B^*$. Hence $E' - B_2^* = E' - B^* = E' - (E - B^*) = (B - \{x, y\}) \cup \{\alpha\}$ where $B = E - B^*$ and so $x, y \in B$. By Proposition 5(i) again, $(B - \{x, y\}) \cup \{\alpha\}$ is a basis of M_{xy}. We conclude that B_2^* is a basis of $(M_{xy})^*$.

Conversely, let $(B_{xy})^*$ be a basis of $(M_{xy})^*$. Then $E' - (B_{xy})^* = B_{xy}$ is a basis of M_{xy}. By using Proposition 5(i), one of the following two cases occurs.

(1) $B_{xy} = (B - \{x\})$ where B is a basis of M, and $x \in B$ and $y \notin B$, or $B_{xy} = (B - \{y\})$ where B is a basis of M, and $x \notin B$ and $y \in B$. Therefore $(B_{xy})^* = E' - B_{xy} = E' - (B - \{x\}) = (B^* - \{y\}) \cup \{\alpha\}$ or $(B_{xy})^* = E' - B_{xy} = E' - (B - \{y\}) = (B^* - \{x\}) \cup \{\alpha\}$ where $B^* = E - B$. In the first case, $x \notin B^*$ and $y \in B^*$, and in the second case, $x \in B^*$ and $y \notin B^*$.

(2) $B_{xy} = (B - \{x, y\}) \cup \{\alpha\}$ where B is a basis of M, and $x, y \in B$. Therefore $(B_{xy})^* = E' - B_{xy} = E' - [(B - \{x, y\}) \cup \{\alpha\}] = B^*$ where $B^* = E - B$ and $x, y \notin B^*$.

By (1) and (2), we conclude that every basis of $(M_{xy})^*$ satisfies (i).

(ii) Suppose that $x \nsim y$ in M, and $B_1^* \in (\mathcal{B}'_1)^*$. Then $B_1^* = (B^* - \{x, y\}) \cup \{\alpha\}$ where $B^* \in \mathcal{B}^*$ and $x, y \in B^*$. Hence $E' - B_1^* = E' - [(B^* - \{x, y\}) \cup \{\alpha\}] = E - B^* = B$ where B is a basis of M and so $x, y \notin B$. By Proposition 5(ii), B is a basis of M_{xy}. Therefore B_1^* is a basis of $(M_{xy})^*$. Now suppose $B_2^* \in (\mathcal{B}'_2)^*$. Then $B_2^* = (B^* - \{x\})$ where $B^* \in \mathcal{B}^*$ and $x \in B^*$, $y \notin B^*$ and $(B^* - \{x\}) \cup \{y\} \notin \mathcal{B}^*$. Clearly, if $(B^* - \{x\}) \cup \{y\} \notin \mathcal{B}^*$, then $(B - \{y\}) \cup \{x\} \notin \mathcal{B}$. Hence $E' - B_2^* = E' - (B^* - \{x\}) = E' - [(E - B) - \{x\}] = (B - \{y\}) \cup \{\alpha\}$ where B is a basis of M and so $x \notin B, y \in B$. By Proposition 5(ii) again, $(B - \{y\}) \cup \{\alpha\}$ is a basis of M_{xy}. Thus B_2^* is a basis of $(M_{xy})^*$. Similarly, one can show that when $B_3^* \in (\mathcal{B}'_3)^*$, B_3^* is a basis of $(M_{xy})^*$.

Conversely, let $(B_{xy})^*$ be a basis of $(M_{xy})^*$. Then $E' - (B_{xy})^* = B_{xy}$ is a basis of M_{xy}. By using Proposition 5(ii), one of the following three cases occurs.

(a) $B_{xy} = B$ where $B \in \mathcal{B}$ and $x, y \notin B$. Therefore $(B_{xy})^* = E' - B = E' - (E - B^*) = (B^* - \{x, y\}) \cup \{\alpha\}$ where $B^* \in \mathcal{B}^*$ and so $x, y \in B^*$.

(b) $B_{xy} = (B - \{x\}) \cup \{\alpha\}$ where B is a basis of M, and $x \in B, y \notin B$ and $(B - \{x\}) \cup \{y\} \notin \mathcal{B}$. Therefore $(B_{xy})^* = E' - [(B - \{x\}) \cup \{\alpha\}] = (B^* - \{y\})$ where $B^* = E - B$ and $x \notin B^*$, $y \in B^*$, and $(B^* - \{y\}) \cup \{x\} \notin \mathcal{B}^*$.

(c) $B_{xy} = (B - \{y\}) \cup \{\alpha\}$ where B is a basis of M, and $x \notin B, y \in B$ and $(B - \{y\}) \cup \{x\} \notin \mathcal{B}$. Therefore $(B_{xy})^* = E' - [(B - \{y\}) \cup \{\alpha\}] = (B^* - \{x\})$ where $B^* = E - B$ and $x \in B^*$, $y \notin B^*$, and $(B^* - \{x\}) \cup \{y\} \notin \mathcal{B}^*$.

By (a), (b) and (c), we conclude that every basis of $(M_{xy})^*$ satisfies (ii) and this completes the proof of the theorem. □

As an immediate consequence of Theorem 6, we have the following result.

Corollary 7. *Let M be a binary matroid and $x, y \in E(M)$.Then*

(i) if $x \sim y$ in M, then $r^*(M_{xy}) = r^*(M) = |E(M)| - r(M)$.

(ii) if $x \nsim y$ in M, then $r^*(M_{xy}) = r^*(M) - 1 = |E(M)| - r(M) - 1$.

3 When the dual of split-off matroid is equal with the split-off of dual of original matroid?

Let M be a binary matroid, by Theorem 6, we can determine all basis of $(M_{xy})^*$ in terms of the cobases of M and, by Proposition 5, we can determine all bases of the M^*_{xy} in terms of the cobases of M. The next theorem is the main result of this paper.

Theorem 8. *Let M be a binary matroid on a set E. Let $x, y \in E$ such that $\alpha \notin E$. Then $(M_{xy})^* = (M^*)_{xy}$ if and only if $M = N \oplus N'$ where N is an arbitrarily binary matroid and N' is $U_{0,2}$ or $U_{2,2}$*

Proof. Suppose that M be a rank-k binary matroid on a set E with $|E(M)| = n$. Then $r(M^*) = n - k$ and $|E(M_{xy})| = n - 1$. Let $x, y \in E$ such that $\alpha \notin E$ and let $\mathcal{B}^*$ be the collection of cobases of M. Then by Proposition 5, the collection of bases of $(M^*)_{xy}$ is one of the following cases:

(a) If $x \sim y$ in M^*, then $\mathcal{B}^*_{xy} = \mathcal{B}_1 \cup \mathcal{B}_2$ where

-$\mathcal{B}_1 = \Big\{B^* - \{x\} : B^* \in \mathcal{B}^*,\ x \in B^*$ and $y \notin B^*\Big\} = \Big\{B^* - \{y\} : B^* \in \mathcal{B}^*,\ x \notin B^*$ and $y \in B^*\Big\}$;

-$\mathcal{B}_2 = \Big\{(B^* - \{x, y\}) \cup \{\alpha\} : B^* \in \mathcal{B}^*$ and $x, y \in B^*\Big\}$.

(b) If $x \nsim y$ in M^*, then $\mathcal{B}^*_{xy} = \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \mathcal{B}'_3$ where

-$\mathcal{B}'_1 = \Big\{B^* : B \in \mathcal{B}^*$ and $x, y \notin B^*\Big\}$;

-$\mathcal{B}'_2 = \Big\{(B^* - \{x\}) \cup \{\alpha\} : B^* \in \mathcal{B}^*,\ x \in B^*\ y \notin B^*$ and $(B^* - \{x\}) \cup \{y\} \notin \mathcal{B}^*\Big\}$;

-$\mathcal{B}'_3 = \Big\{(B^* - \{y\}) \cup \{\alpha\} : B^* \in \mathcal{B}^*,\ x \notin B^*,\ y \in B^*$ and $(B^* - \{y\}) \cup \{x\} \notin \mathcal{B}^*\Big\}$.

Moreover, By Theorem 6, the collection of bases of $(M_{xy})^*$ is $(B_1)^* \cup (B_2)^*$ when $x \sim y$ in M or it is $(B'_1)^* \cup (B'_2)^* \cup (B'_3)^*$ when $x \nsim y$ in M.

Now suppose that $x \sim y$ in M and M^*. Then by Proposition 5, $r(M_{xy}) = k - 1$ and $r(M^*_{xy}) = n - k - 1$. But, by Corollary 7, $r((M_{xy})^*) = n - k$. We conclude that in this case $(M^*)_{xy}$ is not equal or isomorphic to $(M_{xy})^*$. Similarly, if $x \nsim y$ in M and M^*. Then by Proposition 5, $r(M_{xy}) = k$ and $r((M^*)_{xy}) = n-k$. But by Corollary 7, $r((M_{xy})^*) = n - k - 1$ and so $(M^*)_{xy}$ cannot equal or isomorphic to $(M_{xy})^*$. Suppose that $x \sim y$ in one of M and M^*. Then, By **(a)**, **(b)** and Theorem 6, there are two following cases to have a same collection of bases for $(M^*)_{xy}$ and $(M_{xy})^*$.

(i) $\mathcal{B}_1 = (\mathcal{B}'_2)^* = (\mathcal{B}'_3)^* = \emptyset$.
This means $x \sim y$ in M^* and $x \nsim y$ in M. Therefore, the collections of bases of two matroids $(M^*)_{xy}$ and $(M_{xy})^*$ is $\big\{(B^* - \{x, y\} \cup \{\alpha\} : B^* \in \mathcal{B}^*, x, y \in B^*\big\}$. We conclude that every basis of M^* contains both x and y and so x and y are loops of M and coloops of M^*. Hence, $M = N \oplus U_{0,2}$ and $M^* = N^* \oplus U_{2,2}$ where N is an arbitrary binary matroid.

(ii) $(\mathcal{B}'_1)^* = \mathcal{B}'_2 = \mathcal{B}'_3 = \emptyset$.
This means $x \sim y$ in M and $x \nsim y$ in M^*. Therefore, the collection of bases of two matroids $(M^*)_{xy}$ and $(M_{xy})^*$ is $\big\{(B^* : B^* \in \mathcal{B}^*, x, y \notin B^*\big\}$. We conclude that every basis of M^* does not contain both x and y and so x and y are loops of M^* and coloops of M. Hence, $M = N \oplus U_{2,2}$ and $M^* = N^* \oplus U_{0,2}$ where N is an arbitrary binary matroid.

Conversely, suppose that x and y are loops of M. Then $M = N \oplus U_{0,2}$ where N is a binary matroid. Thus

- First by applying the split-off operation on two elements of $U_{0,2}$, we have $M_{xy} = N \oplus U_{0,1}$ and then by duality, $(M_{xy})^* = N^* \oplus U_{1,1}$.
- First by duality, $M^* = N^* \oplus U_{2,2}$ and then by applying the split-off operation on two elements of $U_{2,2}$, we have $(M^*)_{xy} = N^* \oplus U_{1,1}$.

Similarly, if x and y are coloops of M. Then $M = N \oplus U_{2,2}$ where N is a binary matroid and $x, y \notin E(N)$. Therefore $(M^*)_{xy} = (M_{xy})^* = N^* \oplus U_{0,1}$ and this completes the proof of the theorem. □

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