The Variance Theorem for Finite Boundaries Theory and Application

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Abstract: Diffusion and heat conduction are very important processes. Starting from this, the necessity of formulating the variance theorem for finite boundaries is shown and its proof is presented. After this, the results for the momenta of the binomial, the Lévy and the Cauchy distributions are calculated in order to fulfill the quality gate.

Key–Words: Quality gate, diffusion, finite momenta, variance theorem

1 Introduction

The investigation of the structure of complex systems and their dynamical properties are a part of the main topics in science in our days. Such frameworks / patterns with movements of their basic ingredients are characterised by

- a large density of elementary units,
- strong interactions between these units,
- a non-predictable or anomalous evolution in the course of time.

The study of all these properties play an important role in exact and live sciences where glasses, liquid crystals, polymers, proteins, biopolymers, organisms, ecosystems and economy are included.

In [1] the relationship between propagation processes in such complex systems and the statistical momenta are discussed. The physical examples of such transport are the diffusion and the heat conduction. Motivated by this, momenta in infinite boundaries are introduced and defined. A second not less important result in [1] is the formulation of the quality gate, that in order to guarantee the quality of results, two or three different and independent ways have to be presented. This as background, the momentum generating function was introduced, too. All these features are applied to different forms of the Lévy distribution and the Cauchy distribution, too. Due to the fact of infinite boundaries, these results can lead to misunderstandings in their interpretation, as discussed in [1].

Furthermore the question was answered, what happed if the momenta are applied to a Fourier convolution, which is the general solution of a linear partial differential equation. Starting from

$$\begin{aligned}&\left\langle x^i, f(x) * g(x)\right\rangle = \\ &\int_{-\infty}^{\infty} x^i \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, \mathrm{d}\xi\, \mathrm{d}x = \\ &\int_{-\infty}^{\infty} g(\xi) \int_{-\infty}^{\infty} x^i\, f(x-\xi)\, \mathrm{d}x\, \mathrm{d}\xi,\end{aligned} \tag{1}$$

by substitution $x - \xi = y \quad \Leftrightarrow \quad \mathrm{d}x = \mathrm{d}y$ and the application of the binomial theorem follows:

$$\begin{aligned}&\left\langle x^i, f(x) * g(x)\right\rangle = \\ &\int_{-\infty}^{\infty} g(\xi) \int_{-\infty}^{\infty} \sum_{j=0}^{i} \binom{i}{j} y^j\, \xi^{i-j}\, f(y)\, \mathrm{d}y\, \mathrm{d}\xi =\end{aligned}$$

$$\sum_{j=0}^{i} \binom{i}{j} \int_{-\infty}^{\infty} g(\xi)\, \xi^{i-j} \int_{-\infty}^{\infty} f(y)\, y^j \,\mathrm{d}y \,\mathrm{d}\xi \;= \sum_{j=0}^{i} \binom{i}{j} \left\langle x^{i-j}, g(x) \right\rangle \left\langle x^{j}, f(x) \right\rangle . \tag{2}$$

For the first three characteristic momenta results

$$\mu\left(f(x) * g(x)\right) \;= \mu\left(f(x)\right) + \;\; \mu\left(g(x)\right), \tag{3}$$

$$\sigma^2\left(f(x) * g(x)\right) \;= \sigma^2\left(f(x)\right) \,+\, \sigma^2\left(g(x)\right), \tag{4}$$

$$\sigma^3\left(f(x) * g(x)\right) \;= \sigma^3\left(f(x)\right) \,+\, \sigma^3\left(g(x)\right). \tag{5}$$

For higher momenta also such kind of relations can be derived

$$\sigma^4\left(f(x) * g(x)\right) \;= \sigma^4\left(f(x)\right) \,+\, 6\,\sigma^2\left(f(x)\right)\sigma^2\left(g(x)\right) \,+\, \sigma^4\left(g(x)\right) \tag{6}$$

$$\sigma^5\left(f(x) * g(x)\right) \;= \sigma^5\left(f(x)\right) \,+\, 10\,\sigma^3\left(f(x)\right)\sigma^2\left(g(x)\right) \,+\, 10\,\sigma^2\left(f(x)\right)\sigma^3\left(g(x)\right) \,+\, \sigma^5\left(g(x)\right) \tag{7}$$

Here, the basic definitions of the momentum with infinite boundaries of the forms

$$m_i = \int_{-\infty}^{\infty} x^i\, F(x)\,\mathrm{d}x = \left\langle x^i, F(x) \right\rangle \tag{8}$$

and

$$\mu_i = \int_{-\infty}^{\infty} \left(x - \frac{m_1}{m_0} \right)^i \frac{F(x)}{m_0}\,\mathrm{d}x \tag{9}$$

are used. The second expression indicates the i-th central momentum, $F(x)$ a distribution function, m_1 and m_0 represent the first momentum and the norm respectively.

A second chapter of this publication motivates the introduction of momenta with finite boundaries and the formulation of the variance theorem with finite boundaries, which are defined and discussed in the third part. A fourth chapter investigates the two forms of Lévy's distribution and the Cauchy distribution with respect to the momenta in finite boundaries on two different ways in order to fulfill the quality gate.

At least a conclusion summarises the results here.

2 Motivating Examples for the Variance Theorem for Finite Boundaries

This section is dedicated to show the necessity of an extension of the momentum definition from infinite boundaries to finite ones. A first example comming from heat conduction shows, that the interpretation of the results for infinite boundaries gives missunderstandings. A second sample is dedicated to a continuous binomial distribution and motivates finite boundaries here. A last example discusses the case of anomalous diffusion and shows, that with infinite boundaries momenta do not exist.

These motivations go along with the results from [1] that for two versions of the Lévy distribution and the Cauchy distribution the momenta in infinite boundaries do not exsit.

2.1 A Heat Conduction Problem

Consider the heat conduction equation of the form

$$\frac{\partial T(t,x)}{\partial t} - a\frac{\partial^2 T(t,x)}{\partial x^2} = \frac{s(t,x)}{\varrho_{\text{mass}} c_p}, \tag{10}$$

where $T(t,x)$ describes the temperatur difference in $[K]$, a the heat conductivity. $s(t,x)$ represents a control quantity in order to take outside influences into account, which are for example sources, sinks, initial conditions, boundary conditions etc. Furthermore initial conditions are given by

$$T(0,x) = \kappa\, x \tag{11}$$

with some constant κ.

Applying now the Fourier convolution

$$\int_{-\infty}^{\infty} f\left(x-\xi\right)\delta(\xi) d\xi = \int_{-\infty}^{\infty} f\left(\xi\right)\delta(x-\xi) d\xi = f(x) \tag{12}$$

to the propagator[1] of the heat equation, the result is

$$\int_{-\infty}^{\infty} \frac{\exp\left(-\frac{\xi^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}\,\kappa\left(x-\xi\right)\kappa\,\mathrm{d}\xi \;=\; \kappa\, x.$$

This means, that a linear gradient of temperature in infinite ranges is a stable stationary solution of the heat equation (10) for $s(t,x) = 0$. This solution also is a stationary solution in finite ranges with a constant heating well at one side of the wall and a constant cooling well at the other side of the wall [2]. Without such wells the integration limits must be set to zero at the position of the calorimetric temperature. Then, by

[1] The authors understand by this a fundamental solution, starting by Dirac's delta function.

these results for a homogeneous wall turns out

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{\exp\left(-\frac{\xi^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}\,\kappa\,(x-\xi)\,\mathrm{d}\xi \;= \tag{13}$$
$$\frac{\kappa\,\sqrt{a\,t}}{\sqrt{\pi}}\left(\exp\left(-\frac{(2\,x+d)^2}{16\,a\,t}\right) - \exp\left(-\frac{(2\,x-d)^2}{16\,a\,t}\right)\right) +$$
$$\frac{\kappa\,x}{2}\left(\mathrm{erf}\left(\frac{2\,x+d}{\sqrt{16\,a\,t}}\right) - \mathrm{erf}\left(\frac{2\,x-d}{\sqrt{16\,a\,t}}\right)\right).$$

The so–called error function $\mathrm{erf}(x)$ is related to Euler's incomplete gamma function, which can be calculated by the power series of the confluent hypergeometric function [3]. This example shows, that it is necessary to expand the definition of momenta and the formulation of the variance theorem to the situation of finite boundaries. This becomes more evident if theoretical results are compared to experimental data, which are available in a measurement window. This is discussed in a forthcomming paper in more detail.

2.2 The Contiuous Binomial Distribution

In order to get a continuous function, interpolating the binomial distribution of the formulation

$$T(x,\,t) \;= \tag{14}$$
$$\frac{1}{2\,v\,\Delta t}\left(\frac{p}{p+q}\right)^{\frac{x+v\,t}{2\,v\,\Delta t}}\left(\frac{q}{p+q}\right)^{\frac{v\,t-x}{2\,v\,\Delta t}}\begin{pmatrix}\frac{t}{\Delta t}\\ \frac{x+v\,t}{2\,v\,\Delta t}\end{pmatrix}$$
$$=\;\frac{1}{2\,v\,\Delta t}\,\frac{p^{\frac{x+v\,t}{2\,v\,\Delta t}}\,q^{\frac{v\,t-x}{2\,v\,\Delta t}}}{(p+q)^{\frac{t}{\Delta t}}}\,\frac{\left(\frac{t}{\Delta t}\right)!}{\left(\frac{x+v\,t}{2\,v\,\Delta t}\right)!\,\left(\frac{v\,t-x}{2\,v\,\Delta t}\right)!}.$$

the use of $n! \approx \left(\frac{n}{\mathrm{e}}\right)^n\sqrt{2\,\pi\,n}$, which is Stirling's formula [4], approximates the factorial. This leads to the following term, where $k=(p+u)\,n$ is substituted to get a good approximation of the binomial distribution for $n\,p \gg 1$ and $p \geq q$ and $p+q=1$:

$$\frac{1}{n}\,\log\left(p^k\,q^{n-k}\begin{pmatrix}n\\k\end{pmatrix}\sqrt{2\,\pi\,n\,(p+u)\,(q-u)}\right)$$
$$\approx\;\log\left(\left(\frac{p}{p+u}\right)^{p+u}\left(\frac{q}{q-u}\right)^{q-u}\right). \tag{15}$$

This result causes a definition range $-p \leq u \leq q$ for the approximated continuous binomial distribution

$$f(u) \;=$$
$$p^{(p+u)\,n}\,q^{(q-u)\,n}\begin{pmatrix}n\\(p+u)\,n\end{pmatrix} \approx \tag{16}$$
$$\frac{\left(\left(\frac{p}{p+u}\right)^{p+u}\left(\frac{q}{q-u}\right)^{q-u}\right)^n}{\sqrt{2\,\pi\,n\,(p+u)\,(q-u)}}.$$

It is easy to see, that the definition range corresponds to $0 \leq k \leq n$ for the discrete binomial distribution. This approximation is valid without an additional condition like a constant $a=\frac{v^2\,\Delta t}{2}$ for equation

$$T(x,t) \;= \tag{17}$$
$$\frac{\begin{pmatrix}\frac{t}{\Delta t}\\ \frac{x+v\,t}{2\,v\,\Delta t}\end{pmatrix}}{2\,v\,\Delta t\,2^{\frac{t}{\Delta t}}} \approx \frac{\exp\left(-\frac{x^2}{2\,v^2\,\Delta t\,t}\right)}{2\,v\,\Delta t\sqrt{\frac{\pi}{2}\,\frac{t}{\Delta t}}} = \frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}},$$

solving the difference equation of a random walk with equal propabilities [1]

$$T(t,x) = \tag{18}$$
$$\frac{1}{2}\,T\,(x-v\,\Delta t, t-\,\Delta t)+\frac{1}{2}\,T\,(x+v\,\Delta t, t-\Delta t)\,.$$

2.3 Anomalous Diffusion

In [1] was shown, that for a classical Brownian motion the mean squared displacement

$$\left\langle x^2, f(t,x)\right\rangle = \sigma^2 = 2\,a\,t \tag{19}$$

grows linearly in time in absence of an external bias. In the continuum limit, this kind of motion can be described by the diffusion equation

$$\frac{\partial P(t,x)}{\partial t} \;=\; a\,\frac{\partial^2 P(t,x)}{\partial x^2}, \tag{20}$$

where $P(t,x)$ is the probability density function. Assuming the initial condition $P(0,x)=\delta(x)$ with $\delta(x)$ as Dirac's delta function, the fundamental solution becomes [16]

$$P(t,x) \;=\; \frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}. \tag{21}$$

The diffusion constant fulfills the Einstein–Stokes relation $a=\frac{k_B T}{m\,\eta}$, where $k_B T$ is the Boltzmann energy at temperature T, m the mass of the test particle and η the friction coefficient. Thus, a relation between microscopic and macroscopic quantities was found and used to determine the Avogadro number (see [6]).

In various experiments, deviations from this linear behaviour (19) are observed (see [7, 8, 9, 10, 11, 12, 13]), which can be summarized by the application of a power law form [14, 15, 5]:

$$\left\langle x^2, f(x,t)\right\rangle \;=\; \sigma^2 \;=\; 2\,a_\xi\,t^\xi. \tag{22}$$

The related differential equations, owning such kind of mean squared displacement, are not usual diffusion equations. They can be constructed by replacing the usual partial derivatives by fractional ones like

$$\frac{\partial^\alpha}{\partial t^\alpha} P(t,x) - \frac{\partial^\alpha}{\partial t^\alpha} P(0,x) + a\,\mathcal{R}_x^{-\beta}\left(P(t,x)\right) = 0. \quad (23)$$

Here,

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = {}_h^C\mathcal{D}_t^\alpha\left(f(t)\right) = \quad (24)$$
$$\left(\frac{\mathrm{d}\cdot}{\mathrm{d}t}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_h^t (t-\tau)^{n-\alpha-1}\, f(\tau)\,\mathrm{d}\tau.$$

represents the Caputo derivative for $n-1 \le \alpha \le n$ [17], and

$$\frac{\partial^\beta}{\partial x^\beta} f(x) = -\mathcal{R}_x^{-\beta}\left(f(x)\right) =$$
$$-\frac{2^\beta\,\Gamma\left(\frac{\beta+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(-\frac{\beta}{2}\right)} \int_{-\infty}^{\infty} \frac{f(\xi)}{|x-\xi|^{\beta+1}}\,\mathrm{d}\xi \quad (25)$$

is the Riesz operator with $0 < \beta < 2$ [18].
The solution of this fractional diffusion equation can be found by [18]

$$P(t,x) = \quad (26)$$
$$\frac{\mathcal{H}_{3,2}^{1,2}\left[\frac{2^\beta\,a\,t^\alpha}{|x|^\beta}\middle| \begin{matrix} \{\{0,1\},\{\frac{1}{2},\frac{\beta}{2}\}\} & | & \{\{0,\frac{\beta}{2}\}\} \\ \{\{0,1\}\} & | & \{\{0,\alpha\}\} \end{matrix}\right]}{\sqrt{\pi}\,|x|} =$$
$$\frac{\mathcal{H}_{2,3}^{2,1}\left[\frac{|x|^\beta}{2^\beta\,a\,t^\alpha}\middle| \begin{matrix} \{\{1,1\}\} & | & \{\{1,\alpha\}\} \\ \{\{\frac{1}{2},\frac{\beta}{2}\},\{1,1\}\} & | & \{\{1,\frac{\beta}{2}\}\} \end{matrix}\right]}{\sqrt{\pi}\,|x|}$$

with the momenta in infinite boundaries

$$m_i = \quad (27)$$
$$\frac{\left(1+(-1)\right)^i\, 2^i\, t^{\frac{\alpha i}{\beta}}\, a^{\frac{i}{\beta}}\, \Gamma\left(\frac{1+i}{2}\right)\Gamma\left(-\frac{i}{\beta}\right)\Gamma\left(\frac{i+\beta}{\beta}\right)}{\sqrt{\pi}\,\beta\,\Gamma\left(-\frac{i}{2}\right)\Gamma\left(1+\frac{i\alpha}{\beta}\right)} =$$
$$\frac{\left(1+(-1)\right)^i\, 2^{i-1}\, t^{\frac{\alpha i}{\beta}}\, a^{\frac{i}{\beta}}\, \Gamma\left(\frac{1+i}{2}\right)\Gamma\left(1-\frac{i}{\beta}\right)\Gamma\left(\frac{i+\beta}{\beta}\right)}{\sqrt{\pi}\,\Gamma\left(1-\frac{i}{2}\right)\Gamma\left(1+\frac{i\alpha}{\beta}\right)}.$$

For $i \to 2$, the only way to get a sensible momentum is $\beta \to 2$. If $\beta < 2$, the second momentum diverges, for $\beta > 2$ and $0 < i < \beta$ it becomes zero.

All these examples show, that momenta with finite boundaries are needed.

3 The Variance Theorem with Finite Boundaries

In the previous part was shown, that there are at least two reasons for introducing momenta with finite boundaries:

- from the continuous binomial distribution was shown, that the area of definition of this function has to be restricted due to the square root in the denominator.
- from the anomalous diffusion and the distribution function, discussed in [1], finite boundaries for the momenta have become necessary in order to get well defined expressions for them.

But there is also a third reason, restricting the definition of the momenta to finite boundaries. This is comparing momenta of mathematical models with momenta of experimental data. The main reason here is, that the measurements did not run through the whole space $-\infty < x < \infty$. A more detailed description of this situation can be found in a forthcomming paper.

The consideration here is started by

Definition 1 *Let X be a continous random variable with the related distribution function $F(x)$ for $a \le x \le b$. Otherwise is $F(x) = 0$. Then is*

$$m_i = \int_a^b x^i\, F(x)\,\mathrm{d}x = \left\langle x^i, F(x)\right\rangle_{\textit{finite}} \quad (28)$$

for the i-th momentum and

$$\mu_i = \int_a^b \left(x - \frac{m_1}{m_0}\right)^i \frac{F(x)}{m_0}\,\mathrm{d}x \quad (29)$$

for the i-th central momentum.

The characteristic momenta norm $n = m_0$, expectation value $\mu = \frac{m_1}{m_0}$, variance $\sigma^2 = \frac{m_2}{m_0} - \frac{m_1^2}{m_0^2}$, and asymmetry $\sigma^3 = \frac{m_3}{m_0} - 3\,\frac{m_2\,m_1}{m_0^2} + 2\,\frac{m_1^3}{m_0^3}$ can be defined through (28).

With this definition at hand, it is possible to go on by considering the Fourier convolution of functions, where $g(\xi)$ is defined in an interval $a \le \xi \le b$. Outside of it $g(\xi)$ is zero. This leads to

$$\int_a^b f(x-\xi)\,g(\xi)\,\mathrm{d}\xi = \int_{x-b}^{x-a} f(u)\,g(x-u)\,\mathrm{d}u \quad (30)$$

with $u = x - \xi$. Then is $du = -d\xi$. Calculating the momenta for such a Fourier convolution, which is a solution of a linear partial differential equation, results via the Fubini theorem are

$$\begin{aligned} m_i &= \int_{-\infty}^{\infty} x^i \int_a^b f(x-\xi)\,g(\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \int_a^b \int_{-\infty}^{\infty} x^i\, f(x-\xi)\,g(\xi)\,\mathrm{d}x\,\mathrm{d}\xi. \quad (31) \end{aligned}$$

Now, the substitution $x-\xi=u$ generates a term $f(u)$ and leads to finite boundaries of the inner momentum integral, because also $f(u)$ is zero outside the finite boundaries $a \leq u \leq b$:

$$\begin{aligned} m_i &= \int_a^b g(\xi) \int_{-\infty}^{\infty} (\xi+u)^i f(u)\,\mathrm{d}u\,\mathrm{d}\xi \\ &= \int_a^b g(\xi) \int_a^b (u+\xi)^i f(u)\,\mathrm{d}u\,\mathrm{d}\xi. \quad (32) \end{aligned}$$

The *binomial theorem* for non–negative integer i can be applied, which discouples the integrals:

$$\begin{aligned} m_i &= \int_a^b g(\xi) \int_a^b \sum_{j=0}^{i} \binom{i}{j} u^j\,\xi^{i-j} f(u)\,\mathrm{d}u\,\mathrm{d}\xi \\ &= \sum_{j=0}^{i} \binom{i}{j} \left(\int_a^b f(u)\,u^j\,\mathrm{d}u\right) \left(\int_a^b g(\xi)\,\xi^{i-j}\,\mathrm{d}\xi\right) \\ &= \sum_{j=0}^{i} \binom{i}{j} \left\langle x^j, f(x)\right\rangle \left\langle x^{i-j}, g(x)\right\rangle \\ &= \left\langle x^i, f(x) * g(x)\right\rangle. \quad (33) \end{aligned}$$

This result is formally identical with the result for infinite boundaries (see equation (36) in [1]). The consequence is to discuss momenta within the ranges of the measured data only, which will never go to infinity. The three fundamental relations of non–linear characteristic momenta (3) to (7) stay valid also for any finite boundaries of the measured data. Now, both the theoretical functions and the measured data need the same boundaries to be compared correctly.

This leads to using the interpolation of the binomial distribution by Euler's Gamma function and the several continuous approximations like (16) for $n\,p \gg 0$ and $-p-\frac{1}{n} < u < q+\frac{1}{n}$ and $p+q=1$, which is *not* self–similar for all n, in finite boundaries for the discussion of statistical or diffusion models. This is done more precisely, than an investigation via the Gaussian distribution:

$$T(t,x) = \frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}. \quad (34)$$

4 The Application of the Momenta and the Variance Theorem for Finite Boundaries

In order to fulfill the quality gate, introduced in [1], an adaption of the momentum generating function to finite boundaries has to be done, too. In order to do this, the beginning is by

Definition 2 *If X is a random variable, then a function $g(x)$, definied as:*

$$\begin{aligned} g(t) = E\left(e^{tX}\right) = \sum_{j=0}^{\infty} \frac{m_j t^j}{j!} = \\ E\left(\sum_{j=0}^{\infty} \frac{X^j t^j}{j!}\right) = \sum_{j=1}^{\infty} e^{t x_j} p(x_j) \quad (35) \end{aligned}$$

is called the momentum generating function in the discrete case.
In the continuum is:

$$g(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx.$$

Taking finite boundaries into account the momentum generating function can be regarded as

$$g(t) = E\left(e^{tX}\right) = \int_{\alpha}^{\beta} e^{tX} f_X(x) dx$$

In order to show the application of this definition, the case of the normal distribution of heat conduction is considered, which is of the form

$$T(t,x) = \frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}.$$

The calculation of the momenta in infinite boundaries can be regarded as

$$M(m,-\infty,\infty) = \frac{\left((-1)^m + 1^m\right) a^{\frac{m}{2}} t^{\frac{m}{2}} \Gamma\left(\frac{1+m}{2}\right)}{\sqrt{\pi}}, \quad (36)$$

from which follows

$$\begin{aligned} m=0: &\quad M(0,-\infty,\infty) = 1, \\ m=1: &\quad M(1,-\infty,\infty) = 0, \\ m=2: &\quad M(2,-\infty,\infty) = 2\,a\,t, \\ m=3: &\quad M(3,-\infty,\infty) = 0. \quad (37) \end{aligned}$$

Discussing this situation with the momentum generating function, results

$$\begin{aligned} &M(m,-\infty,\infty) = \\ &\frac{\partial^m}{\partial\tau^m}\left(\int_{-\infty}^{\infty} \mathrm{e}^{\tau\,x} \frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}} dx\right) = \quad (38) \\ &\frac{\partial^m}{\partial\tau^m}\left(\exp\left(a\,t\,\tau^2\right)\right). \end{aligned}$$

Here, (37) follows by taking the limit $\tau \to 0$.

Now, the situation is considered for finite boundaries. In this case, the momentum integral is of the form

$$M\left(m,\alpha,\beta\right)=\int_{\alpha}^{\beta}x^m\frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}\mathrm{d}x=\frac{2^{-1+m}a^{\frac{m}{2}}t^{\frac{m}{2}}\left(\Gamma\left(\frac{1+m}{2},\frac{\alpha^2}{4\,a\,t}\right)-\Gamma\left(\frac{1+m}{2},\frac{\beta^2}{4\,a\,t}\right)\right)}{\sqrt{\pi}}.$$

From this is:

$$m=0: \quad M(0,\alpha,\beta)=\frac{\Gamma\left(\frac{1}{2},\frac{\alpha^2}{4\,a\,t}\right)-\Gamma\left(\frac{1}{2},\frac{\beta^2}{4\,a\,t}\right)}{2\sqrt{\pi}}, \tag{39}$$

$$m=1: \quad M(1,\alpha,\beta)=\frac{\sqrt{a\,t}\left(\exp\left(-\frac{\alpha^2}{4\,a\,t}\right)-\exp\left(-\frac{\beta^2}{4\,a\,t}\right)\right)}{\sqrt{\pi}}, \tag{40}$$

$$m=2: \quad M(2,\alpha,\beta)=\frac{2\,a\,t\left(\Gamma\left(\frac{3}{2},\frac{\alpha^2}{4\,a\,t}\right)-\Gamma\left(\frac{3}{2},\frac{\beta^2}{4\,a\,t}\right)\right)}{\sqrt{\pi}}, \tag{41}$$

$$m=3: \quad M(3,\alpha,\beta)=\frac{4\,a^{\frac{3}{2}}\,t^{\frac{3}{2}}\left(\Gamma\left(2,\frac{\alpha^2}{4\,a\,t}\right)-\Gamma\left(2,\frac{\beta^2}{4\,a\,t}\right)\right)}{\sqrt{\pi}}. \tag{42}$$

By taking the double of the limits $\alpha\to-\infty,\beta\to\infty$ for even m of these expressions leads to the results (37).

In order to fulfill the quality gate, the momentum generating function with finite boundaries is used. This leads to

$$M(m,\alpha,\beta)=\frac{\partial^m}{\partial\tau^m}\left(\int_{\alpha}^{\beta}\exp\tau\,x\frac{\exp\left(-\frac{x^2}{4\,a\,t}\right)}{\sqrt{4\,\pi\,a\,t}}dx\right)=\frac{\partial^m}{\partial\tau^m}\left(\frac{1}{2}\exp\left(a\,t\,\tau^2\right)\left(-\frac{\Gamma\left(\frac{1}{2},\frac{(-\alpha+2\,a\,t\,\tau)^2}{4\,a\,t}\right)}{\sqrt{\pi}}+\frac{\Gamma\left(\frac{1}{2},\frac{(-\beta+2\,a\,t\,\tau)^2}{4\,a\,t}\right)}{\sqrt{\pi}}\right)\right).$$

By taking into account $m=0,1,2,3$, the results from (39) can be reproduced, and the quality gate is fulfilled. Determining the double of the limits $\alpha\to-\infty$ and $\beta\to\infty$ for even m after the differentiation the same results like those in (37) can be found, and the quality gate is fulfilled.

4.1 The Lévy-Distribution

For the Lévy distribution [20]

$$f(x)=\sqrt{\frac{\sigma}{2\,\pi}}\,\frac{\exp\left(-\frac{\sigma}{2\,(x-\mu)}\right)}{(x-\mu)^{\frac{3}{2}}}, \tag{43}$$

now the momenta can be calculated by the momentum integral, resulting

$$\int_{\mu}^{\infty}x^m\sqrt{\frac{\sigma}{2\,\pi}}\,\frac{\exp\left(-\frac{\sigma}{2\,(x-\mu)}\right)}{(x-\mu)^{\frac{3}{2}}}\mathrm{d}x=\frac{\mu^m\sqrt{\sigma}}{2}\left(-\frac{2\,\Gamma(\frac{1}{2}-m)\,{}_1F_1\left(\frac{1}{2}-m,\frac{3}{2},\frac{\sigma}{2\mu}\right)}{\sqrt{\mu}\,\Gamma(-m)}+\frac{\sqrt{2}\,{}_1F_1\left(-m,\frac{1}{2},\frac{\sigma}{2\mu}\right)}{\sqrt{\sigma}}\right).$$

This leads to the expressions

$$\begin{aligned} m_0 &= n=1,\\ m_1 &= \mu-\sigma,\\ m_2 &= \mu^2-2\,\mu\,\sigma+\frac{\sigma^2}{3},\\ m_3 &= \mu^3-3\,\mu^2\,\sigma+\mu\,\sigma^2-\frac{\sigma^3}{15}. \end{aligned}$$

Here can be seen, that the first momentum lies out of the definition area, which is $D=\{x|x>\mu\}$.

The second way to get results for the momenta, is using the momentum generating function. Here, the integral

$$\sqrt{\sigma}\int_{\mu}^{\infty}\frac{\exp\left(\mathrm{t}\,\mathrm{x}-\frac{\sigma}{2(\mathrm{x}-\mu)}\right)}{(x-\mu)^{\frac{3}{2}}}\mathrm{d}x$$

has to be solved. For the momenta follows:

$$M(m)=\lim_{t\to0}\left(\frac{\partial^m}{\partial t^m}e^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\right)$$

and

$$\begin{aligned} M(0) &= \lim_{t\to0}\left(\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\right)=1,\\ M(1) &= \lim_{t\to0}\left(\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\,\mu-\frac{\mathrm{i}\,\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\sqrt{\sigma}}{\sqrt{2\,t}}\right),\\ M(2) &= \lim_{t\to0}\left(\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\,\mu^2-\frac{\mathrm{i}\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\sqrt{\sigma}}{2\sqrt{2}\,t^{\frac{3}{2}}}-\frac{\mathrm{i}\sqrt{2}\,\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\,\mu\sqrt{\sigma}}{\sqrt{t}}-\frac{\mathrm{e}^{t\,\mu-\mathrm{i}\sqrt{2\,t\,\sigma}}\,\sigma}{2\,t}\right). \end{aligned}$$

From this it can be seen, that only the norm exsists. The higher order momenta possess a singularity at $t \to 0$ and have no finite values. Here, from the quality gate's point of view is following, that these results are of formal nature and have no meaning. This example shows, that the application of the quality gate is necessary to avoid a misinterpreation of results.

Now, for this kind of Lévy distribution (43), the momenta with finite boundaries are calculated. To do this, the expression

$$M(m,\alpha,\beta) = \int_\alpha^\beta x^m \sqrt{\frac{\sigma}{2\pi}}\, \frac{\exp\left(-\frac{\sigma}{2\,(x-\mu)}\right)}{(x-\mu)^{\frac{3}{2}}} \mathrm{d}x \quad (44)$$

has to be solved. Using the substitution $x - \mu = \zeta, d\zeta = dx$ and the binomial theorem [4], the result is:

$$M(m,\alpha,\beta) = \sum_{i=0}^{m} \left(-\frac{2^{-i}\,\mu^{-i+m}\,\sigma^i\,\Gamma(1+m)\,\Gamma(\frac{1}{2}-i, \frac{\sigma}{2\alpha-2\mu})}{\sqrt{\pi}\,\Gamma(1+i)\,\Gamma(1-i+m)} + \frac{2^{-i}\,\mu^{-i+m}\,\sigma^i\,\Gamma(1+m)\,\Gamma(\frac{1}{2}-i, \frac{\sigma}{2\beta-2\mu})}{\sqrt{\pi}\,\Gamma(1+i)\,\Gamma(1-i+m)} \right).$$

For the first four momenta follows

$$m = 0: \quad (45)$$
$$M(0,\alpha,\beta) = \frac{\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right)}{\sqrt{\pi}} - \frac{\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right)}{\sqrt{\pi}},$$

$$m = 1:$$
$$M(1,\alpha,\beta) = \quad (46)$$
$$\frac{1}{2\sqrt{\pi}} \left(-\sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + \sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) + 2\,\mu \left(\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) - \Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) \right) \right),$$

$$m = 2:$$
$$M(2,\alpha,\beta) = \quad (47)$$
$$\frac{1}{4\sqrt{\pi}} \left(-\sigma^2\,\Gamma\left(-\frac{3}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + \sigma^2\,\Gamma\left(-\frac{3}{2}, \frac{\sigma}{2(\beta-\mu)}\right) + 4\,\mu \left(-\sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + \sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) \right) - 4\,\mu^2\,\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + 4\,\mu^2\,\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) \right),$$

$$m = 3:$$
$$M(3,\alpha,\beta) = \quad (48)$$
$$\frac{1}{8\sqrt{\pi}} \left(-\sigma^3\,\Gamma\left(-\frac{5}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + \sigma^3\,\Gamma\left(-\frac{5}{2}, \frac{\sigma}{2(\beta-\mu)}\right) + 2\,\mu \left(-3\,\sigma^2\,\Gamma\left(-\frac{3}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + 3\,\sigma^2\,\Gamma\left(-\frac{3}{2}, \frac{\sigma}{2(\beta-\mu)}\right) + 2\,\mu \left(-3\,\sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + 3\,\sigma\,\Gamma\left(-\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) - 2\,\mu\,\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\alpha-\mu)}\right) + 2\,\mu\,\Gamma\left(\frac{1}{2}, \frac{\sigma}{2(\beta-\mu)}\right) \right) \right) \right)$$

The second way to calculate the momenta, is to use the momentum generating function. Therefore, the integral

$$M(m,\alpha,\beta) = \sqrt{\frac{\sigma}{2\pi}} \int_\alpha^\beta \frac{\mathrm{e}^{t\,x - \frac{\sigma}{2(x-\mu)}}}{(x-\mu)^{\frac{3}{2}}} \mathrm{d}x,$$

has to be solved, which in this form becomes quite difficult. Alternatively, the series representation of the momentum generating function can be used. Due to the fact, that the first four momenta are important in this consideration,

$$\mathrm{e}^{t\,x} = 1 + t\,x + \frac{t^2\,x^2}{2} + \frac{t^3\,x^3}{6} + \frac{t^4\,x^4}{24} + O(t^5\,x^5)$$

is used. With this at hand, the momenta result:

$$M(m,\alpha,\beta) = \frac{\exp\left(-\frac{\sigma}{2\alpha-2\mu}\right)}{2520\sqrt{2\,\pi}} \cdots \quad (49)$$
$$\Big[-2\,t\,\sqrt{\alpha-\mu}\,\sqrt{\sigma}\,(2520 + 420\,t\,(\alpha + 5\,\mu - \sigma) + 28\,t^2\,\left(3\,\alpha^2 + 9\,\alpha\mu + 33\,\mu^2 - \alpha\,\sigma - 14\,\mu\,\sigma\right) + t^3\,\left(15\,\alpha^3 + 279\,\mu^3 + \alpha^2\,(39\,\mu - 3\,\sigma) - 185\,\mu^2\,\sigma - 27\,\mu\,\sigma^2 - \sigma^3 + \alpha\,\left(87\,\mu^2 - 22\,\mu\,\sigma + \sigma^2\right)\right)) +$$
$$\exp\left(-\frac{\sigma}{2\alpha-2\mu}\right)\sqrt{2\,\pi}\,\left(2520 + 2520\,t\,(\mu-\sigma) + 420\,t^2\left(3\mu^2 - 6\mu\sigma + \sigma^2\right) + 28t^3\left(15\mu^3 - 45\mu^2\sigma + 15\mu\sigma^2 - \sigma^3\right) + t^4\left(105\mu^4 - 420\mu^3\sigma + 210\mu^2\sigma^2 - 28\mu\sigma^3 + \sigma^4\right)\right) \left(1 - \frac{\Gamma\left(\frac{1}{2}, \frac{\sigma}{2\alpha-2\mu}\right)}{\sqrt{\pi}}\right) \Big] +$$

$$\frac{\exp\left(-\frac{\sigma}{2\beta-2\mu}\right)}{2520\sqrt{2\pi}}\cdots$$
$$\Big[-2\,t\sqrt{\beta-\mu}\sqrt{\sigma}\,(2520+420\,t\,(\beta+5\mu-\sigma)+$$
$$28\,t^2\left(3\beta^2+9\beta\,\mu+33\mu^2-\beta\,\sigma-14\,\mu\,\sigma\right)+$$
$$t^3\left(15\,\beta^3+279\,\mu^3+\beta^2\,(39\,\mu-3\,\sigma)-185\mu^2\sigma-\right.$$
$$\left.27\,\mu\,\sigma^2-\sigma^3+\beta\left(87\mu^2-22\,\mu\,\sigma+\sigma^2\right)\right))+$$
$$\exp\left(-\frac{\sigma}{2\beta-2\mu}\right)$$
$$\sqrt{2\pi}\,(2520+2520\,t\,(\mu-\sigma)+$$
$$420\,t^2\left(3\mu^2-6\mu\sigma+\sigma^2\right)+$$
$$28t^3\left(15\mu^3-45\mu^2\,\sigma+15\mu\,\sigma^2-\sigma^3\right)+$$
$$t^4\left(105\mu^4-420\mu^3\,\sigma+210\,\mu^2\sigma^2-28\,\mu\,\sigma^3+\sigma^4\right))$$
$$\left(1-\frac{\Gamma\left(\frac{1}{2},\frac{\sigma}{2\beta-2\mu}\right)}{\sqrt{\pi}}\right)\Big]$$

For the norm, equation (45) follows from (49) after $t\to 0$. The first momentum can be found by differentiating (49) once with repect to t and then putting $t\to 0$:

$$M(1,\alpha,\beta)=$$
$$\sqrt{\frac{2}{\pi}}\left(-\exp\left(-\frac{\sigma}{2\alpha-2\mu}\right)\sqrt{\alpha-\mu}+\right.$$
$$\left.\exp\left(-\frac{\sigma}{2\beta-2\mu}\right)\sqrt{\beta-\mu}\right)\sigma-$$
$$\frac{(\mu-\sigma)\,\Gamma\left(\frac{1}{2},\frac{\sigma}{2\alpha-2\mu}\right)}{\sqrt{\pi}}+\frac{(\mu-\sigma)\,\Gamma\left(\frac{1}{2},\frac{\sigma}{2\beta-2\mu}\right)}{\sqrt{\pi}},$$

which leads to the same expression as (46).

The second momentum follows from (49) after a double differentiation with respect to t and after setting $t\to 0$:

$$M(2,\alpha,\beta)=$$
$$\frac{1}{3\sqrt{\pi}}\exp\left(-\frac{\sigma}{2\alpha-2\mu}-\frac{\sigma}{2\beta-2\mu}\right)$$
$$\left(\sqrt{2\sigma}\exp\left(\frac{\sigma}{2\alpha-2\mu}\right)\sqrt{\beta-\mu}\,(\beta+5\mu-\sigma)+\right.$$
$$\left.\sqrt{2\sigma}\exp\left(\frac{\sigma}{2\beta-2\mu}\right)\sqrt{\alpha-\mu}\,(-\alpha-5\mu+\sigma)\right)-$$
$$\exp\left(\frac{\sigma}{2\alpha-2\mu}+\frac{\sigma}{2\beta-2\mu}\right)\left(3\mu^2+6\mu\sigma+\sigma^2\right)$$
$$\left(\Gamma\left(\frac{1}{2},\frac{\sigma}{2\alpha-2\mu}\right)-\Gamma\left(\frac{1}{2},\frac{\sigma}{2\beta-2\mu}\right)\right).$$

Comparing this result with (47) shows the accordance of both expressions.

At last, the third momentum can be determined by differentiation of (49) three times with respect to t and afterwards $t\to 0$. The result is

$$M(3,\alpha,\beta)=$$
$$\frac{1}{15\sqrt{\pi}}\exp\left(-\frac{\sigma}{2\alpha-2\mu}-\frac{\sigma}{2\beta-2\mu}\right)\cdots$$
$$\left(-\sqrt{2\sigma}\exp\left(\frac{\sigma}{2\beta-2\mu}\right)\sqrt{\alpha-\mu}\right.$$
$$\left(3\alpha^2+9\alpha\mu+33\mu^2-\alpha\sigma-14\sigma\mu+\sigma^2\right)+$$
$$\sqrt{2\sigma}\exp\left(\frac{\sigma}{2\alpha-2\mu}\right)\sqrt{\beta-\mu}$$
$$\left.\left(3\beta^2+9\beta\mu+33\mu^2-\beta\sigma-14\sigma\mu+\sigma^2\right)\right)$$
$$-\exp\left(\frac{\sigma}{2\alpha-2\mu}+\frac{\sigma}{2\beta-2\mu}\right)$$
$$\left(15\mu^3-45\mu^2\sigma+15\mu\sigma^2-\sigma^3\right)$$
$$\left(\Gamma\left(\frac{1}{2},\frac{\sigma}{2\alpha-2\mu}\right)-\Gamma\left(\frac{1}{2},\frac{\sigma}{2\beta-2\mu}\right)\right).$$

Comparing this result with (48) shows the congruence of both expressions. It should be mentioned here, that:

$$\begin{aligned}
M(m,\alpha,\beta) &= \lim_{t\to 0}\left(\frac{\partial^m}{\partial\,t^m}\left[\int_\alpha^\beta e^{tx}f(x)dx\right]\right)\\
&= \lim_{t\to 0}\left(\frac{\partial^m}{\partial\,t^m}\left[\int_\alpha^\beta\sum_{i=0}^\infty\frac{(tx)^i}{i!}f(x)dx\right]\right)\\
&= \lim_{t\to 0}\left(\int_\alpha^\beta\frac{\partial^m}{\partial\,t^m}\left[\sum_{i=0}^\infty\frac{(tx)^i}{i!}f(x)\right]dx\right).
\end{aligned}$$

From this follows:

$$\begin{aligned}
M(0,\alpha,\beta) &= \int_\alpha^\beta f(x)dx,\\
M(1,\alpha,\beta) &= \int_\alpha^\beta xf(x)dx,\\
M(2,\alpha,\beta) &= \int_\alpha^\beta x^2f(x)dx,\\
M(3,\alpha,\beta) &= \int_\alpha^\beta x^3f(x)dx.
\end{aligned}$$

The second Lévy distribution to be considered, is of the form [21]

$$L(x)=\frac{\sigma^k}{\Gamma(k)}\frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}}. \tag{50}$$

In order to determine the momenta, the integral of the form

$$M(m,\mu,\infty)=\int_\mu^\infty x^m\frac{\sigma^k}{\Gamma(k)}\frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}}dx$$

is to be considered. After the substitution $(x-\mu)=\zeta, dx=d\zeta$ and the application of the binomial theorem results with $k>m, \sigma>0$ and $k>0$

$$M(m,\mu,\infty)= \frac{\mu^{-k+m}\sigma^k\Gamma(-k)\Gamma(k-m)\,{}_1F_1(k-m,1+k,\frac{\sigma}{\mu})}{\Gamma(k)\Gamma(-m)}+ \mu^m\,{}_1F_1(-m,1-k,\frac{\sigma}{\mu}),$$

from which follows:

$$\begin{aligned} M(0,\mu,\infty) &= 1,\\ M(1,\mu,\infty) &= \mu+\frac{\sigma}{-1+k},\\ M(2,\mu,\infty) &= \mu^2+\frac{2\mu\sigma}{-1+k}+\frac{\sigma^2}{2-3k+k^2},\\ M(3,\mu,\infty) &= \mu^3+\frac{3\mu^2\sigma}{-1+k}+\frac{3\mu\sigma^2}{2-3k+k^2}+\\ &\quad \frac{\sigma^3}{-6+11k-6k^2+k^3}. \end{aligned}$$

Here, the results for $m\geq 1$ own only formal character due to the fact, that the condition $k>m, k>0$ and $m>0$ does not coincide.

The second way to investigate the Lévy distribution, is to use the momentum generating function. This leads to

$$\begin{aligned} M(m,\mu,\infty) &= \int_\mu^\infty e^{tx}\frac{\sigma^k}{\Gamma(k)}\frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}}\mathrm{d}x\\ &= \frac{2\,i^k\,e^{t\,\mu}\,t^{\frac{k}{2}}\,\sigma^{\frac{k}{2}}\,K_\nu[k,2\,i\,\sqrt{t\,\sigma}]}{\Gamma[k]} \end{aligned}$$

where K_ν is the modified Bessel function of second kind, defined, by the differential equation $z^2y''+zy'-(z^2+\nu^2)y=0$ [22].

For this expression, the non–existence of the momenta can be seen for $m\geq 1$.

In contrast to the calculations above, now the momenta for (50) are determined for finite boundaries. In order to do this, the integral:

$$M(m,\alpha,\beta)=\frac{\sigma^k}{\Gamma(k)}\int_\alpha^\beta x^m\,\frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x\,-\,\mu)^{1+k}}\mathrm{d}x$$

has to be solved. After a substitution $x-\mu=\zeta, \mathrm{d}x=\mathrm{d}\zeta$ and the application of the binomial theorem the result is:

$$M(m,\alpha,\beta)= \sum_{i=0}^m\frac{-\mu^i\,\sigma^{-i+m}\,\Gamma(1+m)\,\Gamma(i+k-m,\frac{\sigma}{\alpha-\mu})}{\Gamma(1+i)\,\Gamma(k)\,\Gamma(1-i+m)}+ \sum_{i=0}^m\frac{\mu^i\,\sigma^{-i+m}\,\Gamma(1+m)\,\Gamma(i+k-m,\frac{\sigma}{\beta-\mu})}{\Gamma(1+i)\,\Gamma(k)\,\Gamma(1-i+m)}$$

From this follows:

$$m=0: \quad M(0,\alpha,\beta)=\frac{\Gamma(k,\frac{\sigma}{\beta-\mu})-\Gamma(k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}, \tag{51}$$

$$m=1: \quad M(1,\alpha,\beta)= \sigma\,\frac{\Gamma(-1+k,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(-1+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}+ \mu\,\frac{\Gamma(k,\,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(k,\,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)},$$

$$\begin{aligned} m=2:&\\ M(2,\alpha,\beta) =& \;\sigma^2\,\frac{\Gamma(-2+k,\,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(-2+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}+\\ &2\,\mu\,\sigma\,\frac{\Gamma(-1+k,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(-1+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}+\\ &\mu^2\,\frac{\Gamma(k,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)} \end{aligned} \tag{52}$$

$$\begin{aligned} m=3:&\\ M(3,\alpha,\beta) =& \;\sigma^3\,\frac{\Gamma(-3+k,\frac{\sigma}{\beta-\mu})-\Gamma(-3+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}+\\ &3\,\mu\,\sigma^2\,\frac{\Gamma(-2+k,\frac{\sigma}{\beta-\mu})\,-\,\Gamma(-2+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}\\ &3\,\mu^2\,\sigma\,\frac{\Gamma(-1+k,\frac{\sigma}{\beta-\mu})-\Gamma(-1+k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)}+\\ &\mu^3\,\frac{\Gamma(k,\frac{\sigma}{\beta-\mu})-\Gamma(k,\frac{\sigma}{\alpha-\mu})}{\Gamma(k)} \end{aligned} \tag{53}$$

Again, the second way to derive the momenta for (50), is to use the momentum generating function, which gives the expression:

$$M(m,\alpha,\beta)= \lim_{t\to 0}\frac{\partial^m}{\partial\,t^m}\int_\alpha^\beta e^{tx}\frac{\sigma^k}{\Gamma(k)}\frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}}\mathrm{d}x.$$

Here, the series expression for the momentum generating function again is used:

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2} + \frac{t^3x^3}{6} + \frac{t^4x^4}{24} + O(t^tx^5).$$

This leads to:

$$\begin{aligned}
&M(m,\alpha,\beta) = \\
&\lim_{t\to 0} \frac{\partial^m}{\partial t^m} (\cdots \\
&\frac{\sigma^k}{24\Gamma[k]} \left(-\sigma^{-k} \left(t^4\,\sigma^4\,\Gamma\left(-4+k,\frac{\sigma}{\alpha-\mu}\right) + \right.\right. \\
&4\,t^3\,(1+t\,\mu)\,\sigma^3\,\Gamma\left(-3+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&12\,t^2\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&12\,t^3\,\mu\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&6\,t^4\,\mu^2\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&24\,t\,\sigma\,\Gamma\left(-1+k;\frac{\sigma}{\alpha-\mu}\right) + \\
&24\,t^2\,\mu\,\sigma\Gamma\left(-1+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&12\,t^3\,\mu^2\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&4\,t^4\,\mu^3\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\alpha-\mu}\right) + \\
&24\,\Gamma\left(k,\frac{\sigma}{\alpha-\mu}\right) + 24\,t\,\mu\,\Gamma\left(k,\frac{\sigma}{\alpha-\mu}\right) + \\
&12\,t^2\,\mu^2\,\Gamma\left(k,\frac{\sigma}{\alpha-\mu}\right) + 4\,t^3\,\mu^3\,\Gamma\left(k,\frac{\sigma}{\alpha-\mu}\right) + \\
&\left.t^4\,\mu^4\,\Gamma\left(k,\frac{\sigma}{\alpha-\mu}\right)\right) + \\
&\sigma^{-k}\left(t^4\,\sigma^4\,\Gamma\left(-4+k,\frac{\sigma}{\beta-\mu}\right) + \right. \\
&4\,t^3\,(1+t\,\mu)\,\sigma^3\,\Gamma\left(-3+k,\frac{\sigma}{\beta-\mu}\right) + \\
&12\,t^2\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\beta-\mu}\right) + \\
&12\,t^3\,\mu\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\beta-\mu}\right) + \\
&6\,t^4\,\mu^2\,\sigma^2\,\Gamma\left(-2+k,\frac{\sigma}{\beta-\mu}\right) + \\
&24\,t\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\beta-\mu}\right) + \\
&24\,t^2\,\mu\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\beta-\mu}\right) + \\
&12\,t^3\,\mu^2\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\beta-\mu}\right) + \\
&4\,t^4\,\mu^3\,\sigma\,\Gamma\left(-1+k,\frac{\sigma}{\beta-\mu}\right) + \\
&24\,\Gamma\left(k,\frac{\sigma}{\beta-\mu}\right) + 24\,t\,\mu\,\Gamma\left(k,\frac{\sigma}{\beta-\mu}\right) + \\
&12\,t^2\,\mu^2\,\Gamma\left(k,\frac{\sigma}{\beta-\mu}\right) + 4\,t^3\,\mu^3\,\Gamma\left(k,\frac{\sigma}{\beta-\mu}\right) + \\
&\left.\left.\left. t^4\,\mu^4\,\Gamma\left(k,\frac{\sigma}{\beta-\mu}\right)\right)\right)\right)
\end{aligned}$$

Differentiating this expression quite often with resprect to t and taking the limit $t \to 0$ of the results, the momenta (51) to (54) can be found.

Alternatively, the integrals for the momenta can be taken of the form:

$$M(0,\alpha,\beta) \quad = \quad \frac{\sigma^k}{\Gamma(k)} \int_\alpha^\beta \frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}} \mathrm{d}x,$$

$$\begin{aligned}
M(1,\alpha,\beta) &= \frac{\sigma^k}{\Gamma(k)} \int_\alpha^\beta x \frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}} \mathrm{d}x, \\
M(2,\alpha,\beta) &= \frac{\sigma^k}{\Gamma(k)} \int_\alpha^\beta x^2 \frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}} \mathrm{d}x, \\
M(3,\alpha,\beta) &= \frac{\sigma^k}{\Gamma(k)} \int_\alpha^\beta x^3 \frac{\exp\left(-\frac{\sigma}{(x-\mu)}\right)}{(x-\mu)^{1+k}} \mathrm{d}x.
\end{aligned}$$

In all cases, the substituion $x-\mu = \zeta, \mathrm{d}x = \mathrm{d}\zeta$ leads to the results (51) to (54).

In these calculations, again the qualitiy gate from [1] is fulfilled in this case by three independent ways.

4.2 The Cauchy Distribution

In [1] was shown, that:

$$\begin{aligned}
&M(m,-\infty,\infty) = \\
&\int_{-\infty}^{\infty} x^m \frac{1}{(x-\mu)^2+\sigma^2} dx = \\
&(1+(-1)^m)\cdots \\
&\left(\frac{\left(1+\frac{1}{\mu}\right)^m \mu^m\,\sigma^m\,\Gamma\left(\frac{1}{2}-\frac{m}{2}\right)\,\Gamma\left(\frac{1}{2}+\frac{m}{2}\right)}{2\pi}\right).
\end{aligned}$$

This means

$$\begin{aligned}
&M(0,-\infty,\infty) = 1, \\
&M(1,-\infty,\infty) \text{ divergent}, \\
&M(2,-\infty,\infty) = -(1+\mu)^2\,\sigma^2, \\
&M(3,-\infty,\infty) \text{ divergent},
\end{aligned}$$

thus only the norm can be used. The second momentum gives a negative value, which is only of formal interest.

The second way, using the momentum generating function yields the result:

$$
\begin{aligned}
&M(m,-\infty,\infty) = \\
&\lim_{t\to 0} \frac{\partial^m}{\partial t^m}\left[\int_{-\infty}^{\infty} e^{t\,x}\,\frac{1}{(x+\mu)^2+\sigma^2}\,\mathrm{d}x\right] \\
=\ &\lim_{t\to 0} \frac{\partial^m}{\partial t^m}\cdot \\
&\frac{1}{\pi\,\sigma}\,e^{t\,\mu}\left(-\frac{2\,\pi^{\frac{3}{2}}\,\mathcal{G}_{3,5}^{3,1}\left[\frac{t^2\,\sigma^2}{4},\ \begin{matrix}1,\frac{3}{4},\frac{5}{4}\\ \frac{1}{2},1,1,\frac{3}{4},\frac{5}{4}\end{matrix}\right]}{t}\right)+ \\
&\frac{1}{2}\,\sigma\,\left(-2\,(i\,\pi+\mathrm{CosIntegal}[t,\sigma])\sin(t\,\sigma)\right. \\
&\left.-2\,\cos(t\,\sigma)\,\left(\pi\,+\,2\mathrm{SinIntegral}[t\,\sigma]\right)\right)]
\end{aligned}
$$

with $t<0$. Here G is the Meijer's G function defined by ([23])

$$
\mathcal{G}_{p,q}^{m,n}\left(x\ \middle|\ \begin{matrix}a_1,\cdots a_p\\ b_1,\cdots b_q\end{matrix}\right) = \frac{1}{2\,\pi\,\mathrm{i}}\int \frac{\prod\limits_{j=1}^{m}\Gamma\left(b_j-s\right)\prod\limits_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod\limits_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-s\right)}x^s\mathrm{d}s
$$

with $0\le m\le q$, $0\le n\le p$. By further calculations can be shown, that most expressions diverge by taking the limit $t\to 0$ due to the additional condition $t<0$.

Now, finite boundaries for the momenta are considered. It should be mentioned here, that the way using the momentum integral

$$
M(m,\alpha,\beta) = \frac{\sigma}{\pi}\int_{\alpha}^{\beta}\frac{x^m}{\left(x-\mu\right)^2+\sigma^2}\mathrm{d}x, \tag{54}
$$

became difficult due to the absence of theorems concerning regularized hypergeometric functions. So the way, using the binomial theorem has been done. Starting from (54) after the substitution $x-\mu=z$, $\mathrm{d}x=\mathrm{d}z$ yields:

$$
M(m,\alpha,\beta) = \frac{\sigma}{\pi}\int_{\alpha-\mu}^{\beta-\mu}\frac{(z+\mu)^m}{z^2+\sigma^2}\mathrm{d}z.
$$

The the application of the binomial theorem leads to

$$
M(m,\alpha,\beta) = \frac{\sigma}{\pi}\sum_{j=0}^{m}\mu^{m-j}\binom{m}{j}\int_{\alpha-\mu}^{\beta-\mu}\frac{z^j}{z^2+\sigma^2}\mathrm{d}z.
$$

Solving the integral leads to

$$
\begin{aligned}
&M(m,\alpha,\beta)\ = \\
&\sum_{j=0}^{m}\frac{1}{\pi}\,\mu^{m-j}\,\sigma\binom{m}{j}\Bigg[\cdots \\
&-\frac{(\alpha-\mu)^{1+j}\,{}_2F_1(1,\frac{1+j}{2},\frac{3+j}{2},-\frac{(\alpha-\mu)^2}{\sigma^2})}{(1+j)\,\sigma^2}+ \\
&\frac{(\beta-\mu)^{1+j}\,{}_2F_1(1,\frac{1+j}{2},\frac{3+j}{2},-\frac{(\beta-\mu)^2}{\sigma^2})}{(1+j)\,\sigma^2}\Bigg]
\end{aligned}
$$

From this follows

$$
\begin{aligned}
&M(0,\alpha,\beta) = \\
&\frac{-\arctan\left(\frac{\alpha-\mu}{\sigma}\right)\,+\,\arctan\left(\frac{\beta-\mu}{\sigma}\right)}{\pi},
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&M(1,\alpha,\beta) = \\
&\frac{2\,\mu\left(\arctan\left(\frac{\beta-\mu}{\sigma}\right)\,-\,\arctan\left(\frac{\alpha-\mu}{\sigma}\right)\right)}{2\,\pi}\,+ \\
&\frac{\sigma\left(\ln\left(1\,+\,\left(\frac{\beta-\mu}{\sigma}\right)^2\right)\,-\,\ln\left(1\,+\,\left(\frac{\alpha-\mu}{\sigma}\right)^2\right)\right)}{2\,\pi},
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
&M(2,\alpha,\beta) = \frac{\mu^2-\sigma^2}{\pi}\cdots \\
&\left(\arctan\left(\frac{\beta-\mu}{\sigma}\right)\,-\,\arctan\left(\frac{\alpha-\mu}{\sigma}\right)\right)\,+ \\
&\frac{\sigma}{\pi}\left(\beta-\alpha+\mu\ln\left(1+\left(\frac{\beta-\mu}{\sigma}\right)^2\right)\,-\right. \\
&\left.\mu\ln\left(1+\left(\frac{\alpha-\mu}{\sigma}\right)^2\right)\right).
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
&M(3,\alpha,\beta) = \frac{\mu^3-3\,\mu\,\sigma^2}{\pi}\cdots \\
&\left(\arctan\left(\frac{\beta-\mu}{\sigma}\right)\,-\,\arctan\left(\frac{\alpha-\mu}{\sigma}\right)\right)\,+ \\
&\frac{\sigma}{2\,\pi}\,((\beta-\alpha)(\alpha+\beta+4\,\mu)+ \\
&(3\mu^2-\sigma^2)\ln\left(1+\left(\frac{\beta-\mu}{\sigma}\right)^2\right)- \\
&\left.(3\,\mu^2-\sigma^2)\ln\left(1+\left(\frac{\alpha-\mu}{\sigma}\right)^2\right)\right).
\end{aligned} \tag{58}
$$

In the following, the calculation of the momenta is discussed, using the momentum generating function.

Here the direct way of integrating the product of generating function and Cauchy distribution and afterwards differentiating with respect to t, is not used. Due to the fact, that the integration is with respect to the variable x, integration and differentiation can be swapped. So the first step is to calculate the first three derivatives with respect to t:

$$\begin{aligned} M(m,\alpha,\beta) &= \int_\alpha^\beta \frac{\sigma}{\pi} e^{tx} \frac{1}{(x-\mu)^2+\sigma^2} \mathrm{d}x, \\ \frac{\partial}{\partial t} M(m,\alpha,\beta) &= \int_\alpha^\beta \frac{\sigma}{\pi} \frac{x e^{tx}}{(x-\mu)^2+\sigma^2} \mathrm{d}x, \\ \frac{\partial^2}{\partial t^2} M(m,\alpha,\beta) &= \int_\alpha^\beta \frac{\sigma}{\pi} \frac{x^2 e^{tx}}{(x-\mu)^2+\sigma^2} \mathrm{d}x, \\ \frac{\partial^3}{\partial t^3} M(m,\alpha,\beta) &= \int_\alpha^\beta \frac{\sigma}{\pi} \frac{x^3 e^{tx}}{(x-\mu)^2+\sigma^2} \mathrm{d}x. \end{aligned}$$

Taking the limit of these expressions and evaluating the integrals, the momenta (55) to (58) can be found.

In this section, two ways have been presented in order to calculate the momenta for finite boundaries for the Cauchy distribution and to fulfill the quality gate.

5 Conclusion

In this publication, the introduction of momenta in finite boundaries is motivated by using examples from heat conduction, from the approximation of the binomial distribution, by comparing theoretical investigations to measured data and from anomalous diffusion. The last two features are discussed in a forthcomming paper in more detail.

The definition of momenta is given and the variance theorem for finite boundaries is formulated. These results then are applied to the Gaussian distribution for heat conducting processes, for the binomial distribution of diffusion, for two variants of the Lévy distribution and the Cauchy one. In order to fulfill the quality gate, introduced in [1], the concept of momentum generating functions is extended to finite boundaries, too.

It can be shown at two different ways, that in the case of infinite boundaries the momenta for the considered distribution functions do not exist. In contrast to this, the momenta can be calculated in finite boundaries on two different ways, too. This shows, that the momenta exsist, and the extended formulation of momenta and the variance theorem are useful.

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