Self-Organization of Two-Contours Dynamical System with Common Node and Cross Movement

ALEXANDER TATASHEV[a], MARINA YASHINA[b]
Moscow Automobile and Road Construction, State Technical University (MADI),
Department of Higher Mathematics,
Leningradsky Prospect 64, 125319 Moscow, RUSSIA
a-tatashev@yandex.ru[a], yash-marina@yandex.ru[b]

Abstract: This paper considers a dynamical system of Buslaev contour network type, containing two contours. There are N_i cells in the contour i, $i = 1, 2$. There is a common point of all contours. This point is called a node. There are M particles in the system. At any time $t = 0, 1, 2, \ldots$, each particle occupies a cell. No cell can be occupied by more than one particle simultaneously. The particles move in a given direction. At any step, each particle moves onto one cell forward if the cell ahead is vacant. If two particles come to the node simultaneously, then a competition of these particles occurs, and only one particle moves. This particle is chosen in accordance with a deterministic or stochastic competition resolution rule. After completing the movement in the contour i, the particle moves in the contour j with probability α_{ij} , $i, j = 1, 2$. We say that the system is in the state of free movement if all particles move without delays at the present moment and in the future. We have obtained the conditions for the system to result in a state of free movement over a time interval with a finite expectation.

Key–Words: Dynamical systems, cellular automata, traffic models, self-organization, contour networks

1 Introduction

In [1]–[3], movement of particles on a one-dimensional lattice was studied. It was assumed that, at any discrete moment, each particle moves onto a cell forward if the cell ahead is vacant. The main studied characteristic is the average velocity of particles. We say that the system is in a state of free movement if all particles move at any step in the present moment and in the future. In [1]–[3], it was proved that a necessary and sufficient condition for the system to result in a state of free movement is that $\rho \le 1/2$, where ρ is the density of particles, i.e., the number of particles related to the number of cells. It is noted in [3] that the movement of particles in this model corresponds to the rule of elementary cellular automaton 184 (CA 184 or ECA 184) in Wolfram classification, [4]. Some generalizations of this model were studied in [5], [6]. In [6], this system was interpreted in terms of exclusion processes.

In [7], two-dimensional traffic model (BML model) was introduced. In this model, particles move on a toroidal lattice in accordance with a rule similar to the rule 184. In [8]–[11], conditions for self-organization (the system results in a state of free movement from any initial state) and collapse (no particle moves after a moment) have been obtained. In [12], the concept of cluster movement in traffic model was introduced. In this model, the neighboring particles form a cluster, and these particles move simultaneously. It was noted in [13] that, in the cluster movement model, the particles move in accordance with the rule ECA 240. There is a version of the cluster movement model with continuous state space and time scale. In the continuous version, the clusters are moving segments. In [14], problems are introduced related to mixed traffic models combining deterministic and stochastic approaches, and an approach to traffic flow modeling on networks is also discussed.

In [15], the concept of contour network (Buslaev contour network) was introduced. The supporter of the contour network consists of contours. There are common points of neighboring contours. These points are called nodes. There are particles in the contours. The particles move on the contours in accordance with the rule ECA 184 or ECA 240. Delays occur at the nodes. These delays are due to that no particles (clusters) can move through the node simultaneously. If particles (clusters) come to the node simultaneously, then a competition occurs, and only one of the competing particles move. This particle is chosen in accordance with a given deterministic or stochastic rule. Contour networks with continuous state space and discrete time are also studied. In [16], the concept of the spectrum of a deterministic contour network. The spectrum is a set of cyclic trajectories in the state space and related average velocities of particles (clusters) for different initial states and fixed parameters. In contrast to the systems considered in [1]–[3], [5], the average velocities of particles in contour systems depend on the initial state. If the competition resolution rule is stochastic, then the average velocity can depend on the realization of the stochastic process for a fixed initial state. The main problem is to study the spectrum and, in particular, find the values of average velocities. Conditions of system resulting in a state of free movement or collapse are only studied.

Analytical results were obtained for regular periodic structures (for example, in [15]–[22]), for two-contours systems [23]–[25]), for systems with aunique node, [26], [27]. In these works, mainly, systems are studied such that, at any contour, there is aunique particle or a unique cluster. For a system such that there is a unique node (this node is a common cell), in [13], the following was proved. If the particles move in accordance with the rule ECA 184 (individual movement), then the system results in a state of free movement over a finite time (over a time with a finite expectation in the case of a stochastic competition resolution rule) if and only if the total number of particles on the contours is not more than half number of cells in a contour (it is assumed that the number of cells in any contour is the same). This is compatible with results proved in [2], [3] for an isolated contour. In [13], [15]–[27], contour networks were studied such that, in this networks, particles (clusters) move on their own contours, i.e., the particles cannot pass from a contour to the other contour. The paper [28] considers a contour network with two-dimensional periodical structure. In this network particles can pass from a contour to another contour. This system was studied in [28] by simulation.

This paper considers a two-contours system such that there is a unique common cell of the contours. The lengths of the contours are different in the general case. The particles move on a contour in accordance with the rule ECA 184. Transitions of particles from one of contours to the other contours may occur. The probabilities α_{ij} are given, where α_{ij} is the probability that, after completing the movement in the ith contour, a particle moves in the jth contour, $i, j = 1, 2$. The competition of two particles, located at the node, is resolved in accordance a given deterministic or stochastic rule. We study conditions for the system to result in a state of free movement over a finite time. Section 2 describes the considered dynamical system. Section 3 defines the concepts of free movement and self-organization. Section 4 introduces functions of system state and concepts which are used in proofs. Section 5 gives some information regarding the theory of linear equations in integer numbers. We use these facts in proofs. In Section 6, it is proved that a necessary and sufficient condition for the system to result in a state of free movement from any initial state is that the total number of particles in the system be not greater than half greatest common divisor of N_1, N_2, where N_1, N_2 are numbers of cells on the contours. In Section 7, it is proved the following. Suppose that, after completing movement in any contour, each particle chooses any contour with positive probabilities. Then, if sufficient condition for self-organization, proved in Section 6, does not hold, then the system cannot result in a state of free movement from any initial state.

2 Description of System

We consider a system containing two *contours*. There are N_i cells in the ith contour, $i = 1, 2$. The indexes of these cells are 0, 1, 2,..., N_i - 1, $i = 1, 2$. There is a unique common cell of the contours. This cell is called the *node*. There are M particles in the system. At any time $t = 0, 1, 2,...$, any particle occupies a cell in one of the contours. More than one particle cannot be in the same cell simultaneously. The cell j of the contour i is called the cell (i, j), $j = 1,..., N_i$, $i = 1, 2$. Suppose, at time t, a particle is in the cell (i,j), $0 \leq j \leq N_i - 2$, and the cell $(i,j+1)$ is vacant; then, at time $t + 1$, this particle will be in the cell $j(i, j+1)$, $i = 1, 2$. Let the cell behead a particle be occupied at time t; then the particle is in the same cell at time $t + 1$. If, at time t, a particle is in the cell $(i, N_i - 1)$ (i.e. the particle is before node), $i = 1, 2$, the node is vacant, and the other particles are not at the node, then the particle will be at the node at time $t + 1$. If two particles are at the node (i.e. the cells $(1, N_1-1)$, $(2, N_2 - 1)$ are occupied) simultaneously, then a *competition* occurs. In this case, only one particle moves in accordance with a deterministic or statistic *competition rule*. We assume that, if a competition occurs at time t, then, with probability $0 \leq q_1 \leq 1$, at $t + 1$, the node is occupied by a particle such that this particle was located in the cell $(1, N_1 - 1)$ at time t, and, with probability $q_2 = 1 - q_1$, $t + 1$, the node is occupied by a particle such that this particle was located in the cell $(2, N_2 - 1)$ at time t. After completing the movement in the ith contour and passing the node, a particle begins to move in the jth contour with probability α_{ij}, $\alpha_{i1} + \alpha_{i2} = 1$, $i, j = 1, 2$. The state of system at time t, $t = 0, 1, 2,...$, is the vector

$$x(t) = (x_0; x_{11}, \ldots, x_{1,N_1-1}; x_{21}, \ldots, x_{2,N_2-1}),$$

where $x_0 = 0$ if the node is vacant at time t, and $x_0 =$ 1if the node is occupied; the value x_{ij} equals 0 or 1 depending on that if the cell (i,j), $j = 1,..., N_i - 1$, $i = 1, 2$, is vacant or occupied

$$x_0 + \sum_{j=1}^{N_j}(x_{1j} + x_{2j}) = M.$$

The initial state $x(0)$ is given.

3 State of Free Movement. Self-organization

We say that the system is in a *state of free movement* at time t if, with probability 1, all particles move without delays at present time and in the future. In accordance with this definition, if the system is in a state of free movement, then the system will be in states of free movement at any moment in the future. The property of the system to result in a state of free movement from any initial state is called the *self-organization*.

4 Optional Concepts and Functions of System States

We shall define some functions of the system states and concepts.

Denote by d the greatest common divisor of the numbers N_1 and N_2.

Suppose

$$a(x) = (a_1(x), \ldots, a_{d-1}(x)),$$

where $a_r(x) = 0$ if cells (1, j), (2, j) are occupied for all j such that the remainder of dividing j by d is equal to r, and, otherwise, $a_r(x) = 1$.

Denote by $G(x)$ the set of numbers $r_1, \ldots, r_k$ such that $a_{r_s}(x) = 1$, $a_{r_s+1}(x) = 0$ (addition by modulo r). Suppose r_s belongs to the set $G(x)$, and the set $C_s(x)$ corresponds to the value r_s such that the set $C_s(x)$ contains the values $r_s, r_{s-1}, \ldots, r_{s-l+1}$ (addition and substraction by modulo d) satisfying the following condition. The equalities $a_{r_s}(x) = a_{r_s-1}(x) = \ldots = a_{r_s-l+1}(x) = 0$, $a_{r_s-l_s} = 1$, s = 1, 2,..., k, hold for these values. The set $C_s(x)$ is called the supporter of the ith *0-cluster*, and the value l is called the *length* of this 0-cluster; *1-clusters* and the *length* of a *1-cluster* are defined analogously. Denote by $l_{\max}(x)$ the maximum length of 0-cluster under the assumption that the system is in the state x. Denote by $r_0(x)$ the number r such that $a_r(x) = 1$ and the number r + 1 belongs to the 0-cluster supporter of the length $l_{\max}(x)$. If more than one value of r satisfies this condition, then we assume that $r_0(x)$ is the minimum of these values.

Let us consider the set of particles such that, at time t, these particles are located in cells satisfying the following condition. The remainder of dividing the particle index by d is equal to $r_0(x(t))$. These particles are called the leading particles.

Denote by $m_r(x)$ the number of particles satisfying the following condition. The remainder of dividing these particles indexes by d is equal to r.

The value

$$S(x) = \sum_{r=0}^{d-1} m_r(x)(r_0(x) - r) \qquad (1)$$

(subtraction by modulo d) is called *the total distance between all the particles and the leading particles for the system state x.*

5 Optional Concepts and Functions of System States

Suppose a and b are integer positive numbers, c is an integer number. and d is the greatest common divisor of numbers a and b. A necessary and sufficient condition for integer non-negative numbers z_1 and z_2, satisfying the equation

$$az_1 - bz_2 + c = 0 \qquad (2)$$

to exist is that the greatest common divisor of numbers a and b to be a divisor of the number c, [29].

6 Sufficient Condition for Self-Organization

In this section, we formulate and prove a theorem regarding a sufficient condition for self-organization of the system.

Lemma 1 *The value of* $l_{\max}(x(t+1))$ *is not greater than the value of* $l_{\max}(x(t))$.

Proof: Suppose the set of values $r_{i_0}(x(t)) + 1$, $r_{i_0}(x(t)) + 2, \ldots, r_{i_0}(x(t)) + l_0$ is the supporter of a 0-cluster. The values $r_{i_0}(x(t))$ and $r_{i_0}(x(t)) + l_0 + 1$ belong to the supporters of 1-clusters. Let us consider the set of particles such that, at time t, these particles are in the cells with indexes satisfying the following condition. The remainder of dividing cell index by d is equal to $r_{i_0}(x(t))$. At least one of these particles moves at time t. Let us consider the set of particles such that these particles are in the cells with indexes satisfying the following condition. The remainder of dividing particle index by d is equal to $r_{i_0}(x(t)) + l_0 + 1$. If, at time t, all particles of this set move, then, at time t + 1, there is the 0-cluster $r_{i_0}(x(t)) + 2, r_{i_0}(x(t)) + 3, \ldots, r_{i_0}(x(t)) + l_0 + 1$ of length l_0. If at least one particle of this set does not move, then, at time t + 1, there is the cluster $r_{i_0}(x(t))+2, r_{i_0}(x(t))+3, \ldots, r_{i_0}(x(t))+l_0$ of length $l_0 - 1$. Thus, at time t + 1, the length of any 0-cluster is less than the length of the 0-cluster at time t, or the length does not change. From this, Lemma 1 follows.
□

Lemma 2 *The length of any emerging cluster is equal to 1.*

Proof: A new 0-cluster may appear only if r exists such that $a_r(x(t)) = 1$, $a_{r+1}(x(t)) = 1$, (addition by modulo d) $a_r(x(t+1)) = 1$, $a_{r+1}(x(t+1)) = 0$. We also have $a_{r+2}(x(t)) = 1$, and therefore the length of the 0-cluster is equal to 1. Lemma 2 has been proved.
□

Lemma 3 *The function* $l_{\max}(x)$ *is a non-increasing function of time.*

Lemma 3 follows from Lemmas 1 and 2.

Lemma 4 *If*

$$r(x(t_0 + 1)) = r(x(t_0)) + 1, \qquad (3)$$

then $S(x(t_0 + 1)) \geq S(x(t_0))$, *and* $S(t + 1) > S(t)$ *if, at time* t, *at least one particle moves.*

Proof: If (3) holds, then, at time t, at least one leading particle does not move, and any other particle either move at one cell in the direction of movement or does not move. From this and (1), we get Lemma 4. □

Denote by d the greatest common divisor of the numbers N_1 and N_2.

Theorem 5 *If the condition*

$$2M \leq d, \tag{4}$$

holds, then the system results in a state of free movement over a time interval with a finite expectation.

Proof: If the system is not in a state of free movement at time t_0, then, over a time with a finite expectation, a delay of at least one particle occurs. Suppose that the delay occurs at time $t_1 \geq t_0$. Therefore, in accordance with Lemma 4 either $S(x(t+1)) > S(x(t))$ or the set of particles, leading at time $t+1$, consists of particles such that these particles do not belong to the set of particles, leading at time t. A particle may become leading again only if the value $l_{\max}$ decreases. Therefore, if the system does not result in a state of free movement, the value of $l_{\max}$ will be smaller. Hence, either the system results in a state of free movement or the value $l_{\max}$ becomes equal to 1. However, if (4) holds, the value $l_{\max}$ may be equal to 1 only in the case of $2M = d$. However, if $l_{\max}(x) = 1$, $2M = d$, then x is a state of free movement. Thus the system results in a state of free movement from any initial state. Theorem 1 has been proved. □

7 Necessary Condition for Self-Organization

In Section 7, it is proved that the sufficient condition for self-organisation (Section 6) is also a necessary condition, under some additional restrictions.

Lemma 6 *Suppose the condition*

$$\alpha_{ij} > 0, \ i, j = 1, 2. \tag{5}$$

holds, and, for the state x, there is a vacant cell between any two particles of each particles of each con-tour. Then a necessary and sufficient condition for the state x to be not a state of free movement is the following. There exist the numbers j_1, j_2 such that the difference of the remainders of dividing these numbers by modulo d is equal to −1, 0, *or* 1, *and, under the assumption that the system is in the state x, the cells* $(1, j_1)$ *and* $(2, j_1)$ *are occupied.*

Proof: Suppose, at time t_0 there are particles in the cells $(1, j_1)$, $(2, j_2)$ with indexes j_1, j_2 satisfying the conditions of the lemma. If no particle passes to the other contour, then one of the particles delays (if no delays occurred earlier) at time $t_0 + z_{1,0} N_1$, where $z_{1,0}$ is minimum integer non-negative valuez1such that there exists an integer non-negative number $z_2 = z_{2,0}$ such that one of the equalities

$$N_1 z_{1,0} - N_2 z_{2,0} = j_2 - j_1, \tag{6}$$

$$N_1 z_{1,0} - N_2 z_{2,0} = j_2 - j_1 + 1, \tag{7}$$

$$N_1 z_{1,0} - N_2 z_{2,0} = j_2 - j_1 - 1 \tag{8}$$

hold (addition and subtraction by modulo d on the right-hand side of the equations (6)–(8). Under the conditions of the theorem, the right-hand side of the equation (6), (7), or (8) is divided by d, and, in accordance with the condition for existence of the equation (2) solution (Section 2), the system cannot be in a state of free movement at time t_0. Assume that, at time t_0, two particles are in the same contour, and, after passing the node, one of these particles will be in the same contour, and the other particle transits to the other contour. Let us consider the difference (by modulo d) of the cells indexes such that these cells are occupied by two the particles. If this difference is equal −1, 0, or 1, then, after particles passing the node, the difference will be the same. From this, Lemma 5 follows.

Theorem 7 *Suppose (5) holds and (4) does not hold; then there is no state of free movement.*

Proof: Over a time with finite expectation, the system results in a state, satisfying the condition of Lemma 5. Theorem 2 follows from Lemma 5.

8 Conclusion

We consider a dynamical system. This system contains two contours. There are N_i cells in the ith contour, $i = 1, 2$. There is a common cell of the contours. There are M particles in the system. At any moment $t = 0, 1, 2, \ldots$, each particles occupies a cell. If, at time t, the cell ahead the particle is vacant, then the particle will be in this cell at time $t+1$. If two particle are at the node simultaneously, then only one particle moves. This particle is chosen in accordance with a given deterministic or stochastic competition resolution rule. With probability α_{ij}, after the movement in the ith contour, a particle moves in the jth contour, $i, j = 1, 2$. We study conditions for the system to result in a state of free movement, i.e., a state such that all particles move at current moment and in the future. We have proved that, for any competition resolution rule, the sufficient condition for the system to result in a state of free movement over a time with a finite expectation from any initial state is that the condition $2M \leq d$ to hold, where d is the greatest common divisor of the number N_1, N_2, and, under the assumption $\alpha_{ij} > 0$, i, j= 1, 2, this sufficient condition is a condition for a state of free movement to exist.

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