Multivariate Spatial Autoregressive Three-Stage Least Squares Fixed Effect Panel Simultaneous Models and Estimation of their Parameters

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Abstract: Models which describe a two-way flow of influence among dependent variables are called simultaneous equation models. Simultaneous equation models using panel data, especially for fixed effect where there are spatial autoregressive with exact solutions, still few of their development and require to be developed. This paper proposed feasible generalized least squares-three-stage least squares (FGLS-3SLS) to find all the estimators with exact solution. The proposed estimators are proved to be consistent.

Key-Words: simultaneous equation models, panel data, fixed effect, spatial autoregressive, FGLS, 3SLS, consistent

1 Introduction

Single-equation methods and system methods are two methods to find the estimators of parameter in simultaneous equation models. Single-equation methods are applied to one equation of the system at a time meanwhile system methods are applied to all equations of the system simultaneously as revealed by [15]. The latter are the methods which are much more efficient than the former because they use much more informations [15].

Three-stage least squares (3SLS) and full information maximum likelihood (FIML) are solution techniques of system methods. However, the estimators of 3SLS are more robust than of FIML [8]. Consequently, solution technique by means of 3SLS is much more advantageous than the one by FIML because it is both time saving and cost saving.

Unfortunately, the limited observations can be an obstacle to obtain the estimators of parameter of simultaneous equation models. However, we have still a chance to overcome these problems by means of panel data. One of many advantages of panel data is their ability to increase the sampel size [4,10,11].

Model which contains spatial correlation among dependent variables can be evaluated by spatial autoregressive model [1]. In this solution, we use first-order queen contiguity to find row-standardized spatial weight matrix [17] and Moran Index to examine spatial influence [3,23,24]. Some papers about estimation of parameter in simultaneous equation models for fixed effect are revealed in [5], and [16]. But, estimating these parameters has done by simulation.

In this paper, we are motivated to develop simultaneous equation models for fixed effect panel data with one-way error component by means of 3SLS solutions, especially for spatial correlation among dependent variables. The objective of this paper is to obtain the closed-form and numerical approximation estimators of parameter models and to prove their consistency, especially for closed-form estimators.

2 Models Development

We refer to [10] with m simultaneous equations models in m endogenous variables, namely

$$\mathbf{y}_h = \mathbf{1}\mu_h + \mathbf{X}_h\boldsymbol{\alpha}_h + \mathbf{Y}_{-h}\boldsymbol{\beta}_{-h} + \mathbf{u}_h, \quad (1)$$

for $h = 1,2,3,\cdots,m$, where $\mathbf{y}_h$ denotes the hth endogenous vector, $\mathbf{X}_h$ denotes the hth matrix of observations including (for example k_h) exogenous variables, $\mathbf{Y}_{-h}$ denotes the $-h$th matrix of observations including endogenous explanatory variables except the hth endogenous explanatory variables, μ_h denotes the hth mean parameter, $\boldsymbol{\alpha}_h$ denotes the hth parameters vector of exogenous variables, $\boldsymbol{\beta}_{-h}$ denotes the $-h$th parameters vector of endogenous explanatory variables, $\mathbf{u}_h$ denotes the hth random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$ (homoscedasticity) in which σ_h^2 denotes the unknown hth error variance and $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, and $\mathbf{1}$ denotes the unit vector. In this context, we suppose that (1) are over identified.

The next model is fixed effect panel data regression models with one way error component [4,11], namely

$$\mathbf{y}_j = \mathbf{1}\mu + \mathbf{X}_j\boldsymbol{\alpha} + \mathbf{1}\gamma_j + \mathbf{u}_j, \quad (2)$$

for $j = 1,2,3,\cdots,T$, where $\mathbf{y}_j$ denotes the jth time period endogenous vector, $\mathbf{X}_j$ denotes the jth time period matrix of observations including (for example k_h) exogenous variables, μ denotes the mean parameter, $\boldsymbol{\alpha}$ denotes the parameters vector of exogenous variables, γ_j denotes the jth time period time specific effect parameter, $\mathbf{u}_j$ denotes the jth time period random error vector assumming mean vector $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I}_n$, σ^2 denotes the unknown error variance. Equation (2) has one restriction, namely $\sum_{j=1}^{T}\gamma_j = 0.$

If equations (1) and (2) are combined, the following equation is obtained

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}, \quad (3)$$

for $h = 1,2,3,\cdots,m$, $j = 1,2,3,\cdots,T$, where $\mathbf{y}_{hj}$ denotes the jth time period hth endogenous vector, $\mathbf{X}_{hj}$ denotes the jth time period hth matrix including (for example k_h) exogenous variables, $\mathbf{Y}_{-hj}$ denotes the jth time period $-h$th matrix including endogenous explanatory variables except the jth time period hth endogenous explanatory variables, γ_{hj} denotes the jth time period hth time specific effect parameter, $\mathbf{u}_{hj}$ denotes the jth time period hth random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$. There is one restriction, namely $\sum_{j=1}^{T}\gamma_j = 0.$

The furthermore model is spatial autoregressive model which refers to [1], namely:

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{X}\boldsymbol{\alpha} + \rho\mathbf{W}\mathbf{y} + \mathbf{u}, \quad (4)$$

where $\mathbf{y}$ denotes the endogenous vector, $\mathbf{X}$ denotes the matrix of observations including (for example k) exogenous variables, ρ denotes the spatial autoregressive parameter, $\mathbf{W}$ denotes the row-standardized spatial weight matrix, and $\mathbf{u}$ denotes the random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I}_n$.

If (3) contains spatial influence and the spatial influence comes only through the endogenous variables, then we can adopt models in equations (4) and obtain new form equations as follows:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \rho_h\mathbf{W}\mathbf{y}_{hj} + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}. \quad (5)$$

Equation (5) can be simplified as follows:

$$\mathbf{A}_h\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}, \quad (6)$$

for $h = 1,2,3,\cdots,m$, $j = 1,2,3,\cdots,T$, where $\mathbf{A}_h = \mathbf{I}_n - \rho_h\mathbf{W}$, ρ_h denotes the hth spatial autoregressive parameter, and $\mathbf{u}_{hj}$ denotes the jth time period hth random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$. There is one restriction, namely $\sum_{j=1}^{T}\gamma_j = 0.$

We refer to [12] for the properties of kronecker products, [19] for reparameterization, [8,9,15] for 3SLS estimation, [13] for GLS and FGLS, [17] for the use first-order queen contiguity to find the row-standardized spatial weight matrix, [3,23,24] for examining spatial influence by means of Moran Index and [18] for consistency.

For the solution of (6) by 3SLS, we obtain the following equation:

$$\begin{aligned}\mathbf{X}_{*j}^t\mathbf{A}_h\mathbf{y}_{hj} &= \mathbf{X}_{*j}^t\mathbf{1}\mu_h + \mathbf{X}_{*j}^t\mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{X}_{*j}^t\mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} \\ &+ \mathbf{X}_{*j}^t\mathbf{1}\gamma_{hj} + \mathbf{X}_{*j}^t\mathbf{u}_{hj},\end{aligned} \quad (7)$$

but the restriction $\sum_{j=1}^{T}\gamma_{hj} = 0$ will not be achieved. This is due to $\mathbf{X}_{*j}^t$ having in general, different values of the matrix of observations in every jth time period. This paper overcomes the restrictive problem by means of average value approach of the matrix of observations [20-22]. We use this approach because the estimator of the mean is unbiased, consistent, and efficient as revealed by [8-10,15].

As a consequence of this approach, we can write (7) as follows:

$$\begin{aligned}\bar{\mathbf{X}}_{*}^t\mathbf{A}_h\mathbf{y}_{hj} &= \bar{\mathbf{X}}_{*}^t\mathbf{1}\mu_h + \bar{\mathbf{X}}_{*}^t\mathbf{X}_{hj}\boldsymbol{\alpha}_h + \bar{\mathbf{X}}_{*}^t\mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} \\ &+ \bar{\mathbf{X}}_{*}^t\mathbf{1}\gamma_{hj} + \bar{\mathbf{X}}_{*}^t\mathbf{u}_{hj},\end{aligned} \quad (8)$$

which can be rewritten to obtain new forms of vectors and matrices as follows:

$$\bar{\mathbf{X}}_{**}^t\mathbf{A}\mathbf{y}_j = \bar{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\mu} + \bar{\mathbf{X}}_{**}^t\mathbf{Z}_j\boldsymbol{\theta} + \bar{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\gamma}_j + \bar{\mathbf{X}}_{**}^t\mathbf{u}_j, \quad (9)$$

where $\mathbf{Z}_j = \left[\mathbf{X}_j \;\vdots\; \mathbf{Y}_{-j}\right]$ and $\boldsymbol{\theta}^t = \left[\boldsymbol{\alpha}^t \;\vdots\; \boldsymbol{\beta}_-^t\right]$ having dimensions $mn \times \left(\sum_{h=1}^{m} k_h + m(m-1)\right)$ and $\left(\sum_{h=1}^{m} k_h + m(m-1)\right) \times 1$, respectively.

Explanation of the vectors and matrices from equations (7)-(9) are $\bar{\mathbf{X}}_{**}$ denotes the $mn \times m\sum_{h=1}^{m} k_h$ diagonal matrix whose submain diagonal is $\bar{\mathbf{X}}_*$, $\bar{\mathbf{X}}_* = \frac{1}{T}\sum_{j=1}^{T} \mathbf{X}_{*j}$ where $\mathbf{X}_*$ denotes the $n \times \sum_{h=1}^{m} k_h$ matrix including all the exogenous variables in the system, $\mathbf{A}$ denotes the $mn \times mn$ diagonal matrix whose submain diagonal is the $n \times n$ matrix $\mathbf{A}_h$, $\mathbf{y}_j$ denotes the $mn \times 1$ vector including all of the $n \times 1$ vectors $\mathbf{y}_{hj}$, $\mathbf{G}$ denotes the $mn \times m$ diagonal matrix whose submain diagonal is $\mathbf{1}$, $\boldsymbol{\mu}$ denotes the $m \times 1$ vector including all of μ_h, $\mathbf{X}_j$ denotes the $mn \times \sum_{h=1}^{m} k_h$ diagonal matrix whose submain diagonal is the $n \times k_h$ matrix $\mathbf{X}_{hj}$, $\boldsymbol{\alpha}$ denotes the $\sum_{h=1}^{m} k_h \times 1$ vector including all of the $k_h \times 1$ vectors $\boldsymbol{\alpha}_h$, $\mathbf{Y}_{-j}$ denotes the $mn \times m(m-1)$ diagonal matrix whose submain diagonal is the $n \times (m-1)$ matrix $\mathbf{Y}_{-hj}$, $\boldsymbol{\beta}_-$ denotes the $m(m-1) \times 1$ vector including all of the $(m-1) \times 1$ vectors $\boldsymbol{\beta}_{-h}$, $\boldsymbol{\gamma}_j$ denotes the $m \times 1$ vector including all of γ_{hj}, and $\mathbf{u}_j$ denotes the $mn \times 1$ vector including all of the $n \times 1$ vectors $\mathbf{u}_{hj}$, as well as n denotes the sampel size of observations. For $j = 1,2,3,\cdots,T$, the restriction $\sum_{j=1}^{T} \gamma_{hj} = 0$ is changed $\sum_{j=1}^{T} \boldsymbol{\gamma}_j = \mathbf{0}$.

3 Estimating the Parameters

Now, we pay attention to equation (9). Estimation all of the parameter models is done in three stages. At the first-stage, we estimate all the endogenous explanatory variables in the system in every time period as follows:

$$\mathbf{y}_{hj} = \mathbf{X}_j^* \boldsymbol{\alpha}_{hj} + \mathbf{v}_{hj}, \tag{10}$$

where $\mathbf{X}_j^*$ denotes the matrix of observations including intercept and all the exogenous variables in the system in every jth time period, $\boldsymbol{\alpha}_{hj}$ denotes the hth parameter vector of the exogenous variables in the system in every jth time period, and $\mathbf{v}_{hj}$ denotes the hth error random vector in every jth time period assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_{v_h}^2 \mathbf{I}_n$ in which $\sigma_{v_h}^2$ denotes the unknown v_hth error variance.

Estimator for $\boldsymbol{\alpha}_{hj}$ is obtained by minimizing residual sum of squares $\left(\mathbf{v}_{hj}^t \mathbf{v}_{hj}\right)$ in least squares method. To minimize this residual sum of squares, we first differentiate with respect to $\boldsymbol{\alpha}_{hj}$, then by setting this derivative equal to zero, we obtain the estimator of $\boldsymbol{\alpha}_{hj}$ which is given by

$$\hat{\boldsymbol{\alpha}}_{hj} = \left(\mathbf{X}_j^{*t} \mathbf{X}_j^*\right)^{-1} \mathbf{X}_j^{*t} \mathbf{y}_{hj}. \tag{11}$$

Next, we estimate $\mathbf{y}_{hj}$ by

$$\hat{\mathbf{y}}_{hj} = \mathbf{X}_j^* \hat{\boldsymbol{\alpha}}_{hj}, \tag{12}$$

and then we obtain

$$\hat{\mathbf{Y}}_{-1j} = \left[\hat{\mathbf{y}}_{2j} \quad \hat{\mathbf{y}}_{3j} \quad \hat{\mathbf{y}}_{4j} \quad \cdots \quad \hat{\mathbf{y}}_{mj}\right],$$
$$\hat{\mathbf{Y}}_{-2j} = \left[\hat{\mathbf{y}}_{1j} \quad \hat{\mathbf{y}}_{3j} \quad \hat{\mathbf{y}}_{4j} \quad \cdots \quad \hat{\mathbf{y}}_{mj}\right],$$
$$\hat{\mathbf{Y}}_{-3j} = \left[\hat{\mathbf{y}}_{1j} \quad \hat{\mathbf{y}}_{2j} \quad \hat{\mathbf{y}}_{4j} \quad \cdots \quad \hat{\mathbf{y}}_{mj}\right], \cdots,$$
$$\hat{\mathbf{Y}}_{-mj} = \left[\hat{\mathbf{y}}_{1j} \quad \hat{\mathbf{y}}_{2j} \quad \hat{\mathbf{y}}_{3j} \quad \cdots \quad \hat{\mathbf{y}}_{m-1,j}\right].$$

At the second-stage, we estimate parameters of $\mu_h, \boldsymbol{\alpha}_h, \boldsymbol{\beta}_{-h}$, dan γ_{hj} to obtain $\hat{\mathbf{u}}_{hj}^*$ of (6). We first substitute $\mathbf{Y}_{-hj}$ by $\hat{\mathbf{Y}}_{-hj}$ in (6), where $\mathbf{Y}_{-hj} = \hat{\mathbf{Y}}_{-hj} + \hat{\mathbf{V}}_{-hj}$ and obtain new equations as follows:

$$\mathbf{A}_h \mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{Z}_{hj} \boldsymbol{\theta}_h + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}^*, \tag{13}$$

where $\mathbf{Z}_{hj} = \left[\mathbf{X}_{hj} \;\vdots\; \hat{\mathbf{Y}}_{-hj}\right]$ and $\boldsymbol{\theta}_h^t = \left[\boldsymbol{\alpha}_h^t \;\vdots\; \boldsymbol{\beta}_{-h}^t\right]$ having dimensions $n \times (k_h + m - 1)$ and $1 \times (k_h + m - 1)$, respectively, and $\mathbf{u}_{hj}^*$ denotes the composite random error with $\mathbf{u}_{hj}^* = \hat{\mathbf{V}}_{-hj} \boldsymbol{\beta}_{-h} + \mathbf{u}_{hj}$. By using the results of (12), we apply least squares method to find the parameter estimators of $\mu_h, \boldsymbol{\theta}_h$, and γ_{hj}. Because the matrix in the right-hand side is less than full rank, to obtain the estimator of $\boldsymbol{\theta}_h$, we use $n \times n$ dimensional transformation matrix $\mathbf{Q}$ in which $\mathbf{Q1} = \mathbf{0}$. We note in passing that $\mathbf{Q} = \mathbf{I}_n - \frac{1}{n}\mathbf{11}^t$ is symmetrical and idempotent. Premultiplying (13) by $\mathbf{Q}$ we have

$\mathbf{QA}_h\mathbf{y}_{hj} = \mathbf{QZ}_{hj}\boldsymbol{\theta}_h + \mathbf{Qu}^*_{hj}$ and by means of least squares method the estimator of $\boldsymbol{\theta}_h$ is as follows:

$$\hat{\boldsymbol{\theta}}_h = \left[\sum_{j=1}^{T}\mathbf{Z}^t_{hj}\mathbf{QZ}_{hj}\right]^{-1}\left[\sum_{j=1}^{T}\mathbf{Z}^t_{hj}\mathbf{QA}_h\mathbf{y}_{hj}\right]. \quad (14)$$

By (13) the estimators of μ_h and γ_{hj} are

$$\hat{\mu}_h = \frac{\mathbf{1}^t}{nT}\left[\mathbf{A}_h\sum_{j=1}^{T}\mathbf{y}_{hj} - \left(\sum_{j=1}^{T}\mathbf{Z}_{hj}\right)\hat{\boldsymbol{\theta}}_h\right], \quad (15)$$

$$\hat{\gamma}_{hj} = \frac{1}{n}\left(\mathbf{1}^t\mathbf{A}_h\mathbf{y}_{hj} - n\hat{\mu}_h - \mathbf{1}^t\mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h\right), \quad (16)$$

respectively.

By (14) to (16) we can estimate $\mathbf{u}^*_{hj}$ as follows

$$\hat{\mathbf{u}}^*_{hj} = \mathbf{A}_h\mathbf{y}_{hj} - \mathbf{1}\left(\hat{\mu}_h + \hat{\gamma}_{hj}\right) - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h. \quad (17)$$

But, in case ρ_h is not known, we can estimate it by means of concentrated log-likelihood.

The likelihood function of $\mathbf{u}_{hj}$, $i = 1,2,3,\cdots,n$, $j = 1,2,3,\cdots,T$, denoted by L_h is as follows: $L_h = \prod_{j=1}^{T}\left(2\pi\sigma_h^2\right)^{-\frac{n}{2}}\exp\left(-\frac{1}{2\sigma_h^2}\mathbf{u}^t_{hj}\mathbf{u}_{hj}\right)$, and by Jacobian transformation, we obtain the natural logarithm of L_h as

$$\ln L_h = -\frac{nT}{2}\ln(2\pi\sigma_h^2) - \frac{1}{2\sigma_h^2}\sum_{j=1}^{T}\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right)^t \times\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right) + T\ln\left\|\mathbf{A}_h\right\|,$$

where $\left\|\mathbf{A}_h\right\|$ is the absolute of the determinant of $\mathbf{A}_h$.

We take derivative for σ_h^2. Setting this derivative equal to zero, we obtain the estimator of σ_h^2, namely

$$\hat{\sigma}_h^2 = \frac{1}{nT}\sum_{j=1}^{T}\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right)^t\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right). \quad (18)$$

By (18), we obtain concentrated log-likelihood as follows:

$$\ln L_h^{con} = C - \frac{nT}{2}\ln\left(\frac{1}{nT}\sum_{j=1}^{T}\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right)^t \times\left(\mathbf{A}_h\mathbf{y}_{hj} - \mathbf{a}_{hj}\right)\right) + T\ln\left\|\mathbf{A}_h\right\|, \quad (19)$$

where $C = -\frac{nT}{2}\ln(2\pi) - \frac{nT}{2}$.

Let $\mathbf{W}$ have eigenvalues $\omega_1,\omega_2,\cdots,\omega_n$. The acceptable spatial autoregressive parameter is $\frac{1}{\omega_{\text{minimum}}} < \rho_h < 1$ [2]. We use numerical method for $\ln L_h^{con}$ to find estimator of ρ_h, namely method of forming sequence of ρ_h by means of R program [20-22]. Its procedure is as follows.

1. We make sequence values of ρ_h, where ρ_h = seq(start value, end value, increasing).
2. For every $\mathbf{y}_{hj}$ and $\mathbf{a}_{hj}$, $h = 1,2,3,\cdots m$, we insert values of ρ_h in (19). Because the values of $\mathbf{a}_{hj}$ are unknown, we use the estimator, $\hat{\mathbf{a}}_{hj}$, where $\hat{\mathbf{a}}_{hj} = \mathbf{1}\left(\hat{\mu}_h + \hat{\gamma}_{hj}\right) + \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h$, with $\mathbf{Z}_{hj} = \left[\mathbf{X}_{hj} \;\vdots\; \hat{\mathbf{Y}}_{-hj}\right]$.
3. Finding the value of ρ_h that gives the largest $\ln L_h^{con}$.

Based on the estimate $\hat{\rho}_h$, the equations (14) to (16) can be rewritten as follows:

$$\hat{\boldsymbol{\theta}}_h = \left[\sum_{j=1}^{T}\mathbf{Z}^t_{hj}\mathbf{QZ}_{hj}\right]^{-1}\left[\sum_{j=1}^{T}\mathbf{Z}^t_{hj}\mathbf{Q}\hat{\mathbf{A}}_h\mathbf{y}_{hj}\right], \quad (20)$$

$$\hat{\mu}_h = \frac{\mathbf{1}^t}{nT}\left[\hat{\mathbf{A}}_h\sum_{j=1}^{T}\mathbf{y}_{hj} - \left(\sum_{j=1}^{T}\mathbf{Z}_{hj}\right)\hat{\boldsymbol{\theta}}_h\right], \quad (21)$$

$$\hat{\gamma}_{hj} = \frac{1}{n}\left(\mathbf{1}^t\hat{\mathbf{A}}_h\mathbf{y}_{hj} - n\hat{\mu}_h - \mathbf{1}^t\mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h\right), \quad (22)$$

respectively, where $\hat{\mathbf{A}}_h = \mathbf{I}_n - \hat{\rho}_h\mathbf{W}$.

The furthermore, the equation (17) can be rewritten as follows:

$$\hat{\mathbf{u}}^*_{hj} = \hat{\mathbf{A}}_h\mathbf{y}_{hj} - \mathbf{1}\left(\hat{\mu}_h + \hat{\gamma}_{hj}\right) - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h. \quad (23)$$

We then use (23) and (18) to find the estimated covariance matrix of the estimator $\hat{\mathbf{u}}^*_{hj}$, namely

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \hat{\sigma}_{13} & \cdots & \hat{\sigma}_{1m} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 & \hat{\sigma}_{23} & \cdots & \hat{\sigma}_{2m} \\ \hat{\sigma}_{31} & \hat{\sigma}_{32} & \hat{\sigma}_3^2 & \cdots & \hat{\sigma}_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m1} & \hat{\sigma}_{m2} & \hat{\sigma}_{m3} & \cdots & \hat{\sigma}_m^2 \end{bmatrix}, \; \hat{\sigma}_h^2 = \hat{\sigma}_{hh^*} \text{ if } h = h^*$$

with $\hat{\sigma}_{hh^*} = \frac{1}{nT}\sum_{j=1}^{T}\hat{\mathbf{u}}^{*t}_{hj}\hat{\mathbf{u}}^*_{h^*j}$,

where $\hat{\sigma}_h^2$ denotes the hth estimated error variance, $\hat{\sigma}_{hh^*}$ denotes the h^*th and the hth estimated error covariance, and $\hat{\boldsymbol{\Sigma}}$ denotes $m \times m$ estimated covariance matrix.

From (9), we have error covariance matrix $\text{var}\left(\bar{\mathbf{X}}^t_{**}\mathbf{u}_j\right) = \bar{\mathbf{X}}^t_{**}\,\text{var}\left(\mathbf{u}_j\right)\bar{\mathbf{X}}_{**}$. This covariance shows that random errors are heteroscedastic, where $\text{var}\left(\mathbf{u}_j\right) = E\left(\mathbf{u}_j\mathbf{u}^t_j\right)$ for $h = h^* = 1,2,3,\cdots,m$,

$\mathbf{u}_j^t = \left[\mathbf{u}_{1j}^t \quad \mathbf{u}_{2j}^t \quad \mathbf{u}_{3j}^t \quad \cdots \quad \mathbf{u}_{mj}^t\right]$,

$\mathbf{u}_{hj}^t = \left[u_{h1j} \quad u_{h2j} \quad u_{h3j} \quad \cdots \quad u_{hnj}\right]$,

in which we assumed that

$$E\left(u_{hij}u_{h^*i^*j}\right) = \begin{cases}\sigma_{hh^*} & \text{if } i = i^* \\ 0 & \text{if } i \neq i^*\end{cases}$$

so that $E\left(\mathbf{u}_{hj}\mathbf{u}_{h^*j}^t\right) = \sigma_{hh^*}\mathbf{I}_n$. We obtain $\mathrm{var}\left(\mathbf{u}_j\right) = \mathbf{\Sigma} \otimes \mathbf{I}_n$ with $mn \times mn$ as its dimension. Consequently, $\mathrm{var}\left(\bar{\mathbf{X}}_{**}^t\mathbf{u}_j\right) = \mathbf{\Sigma} \otimes \bar{\mathbf{X}}_*^t\bar{\mathbf{X}}_* = \mathbf{\Sigma}_\#$ which is $m\sum_{h=1}^m k_h \times m\sum_{h=1}^m k_h$ symmetrical matrix.

Because $\mathbf{\Sigma}$ is unknown, we use the estimator of $\mathbf{\Sigma}_\#$. The estimator of $\mathbf{\Sigma}_\#$ is as follows:

$\hat{\mathbf{\Sigma}}_\# = \hat{\mathbf{\Sigma}} \otimes \bar{\mathbf{X}}_*^t\bar{\mathbf{X}}_*$.

In the above results, we see that the error variance in equation (9) is not constant and the matrix in the right-hand side is less than full rank. For the last-stage, we overcome those problems again by means of reparameterization and GLS. The estimators are as follows:

$$\hat{\boldsymbol{\theta}} = \left[\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\hat{\mathbf{A}}\mathbf{y}_j, \quad (24)$$

$$\hat{\boldsymbol{\mu}} = \left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}}\sum_{j=1}^T\left(\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{Z}_j\hat{\boldsymbol{\theta}}\right), \quad (25)$$

$$\hat{\boldsymbol{\gamma}}_j = \left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}}\left(\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j\hat{\boldsymbol{\theta}}\right), \quad (26)$$

where $\hat{\mathbf{H}} = \bar{\mathbf{X}}_{**}\hat{\mathbf{\Sigma}}_\#^{-1}\bar{\mathbf{X}}_{**}^t$ and $\hat{\mathbf{M}} = \mathbf{G}\left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}} - \mathbf{I}_{mn}$. They have dimensions $mn \times mn$, respectively.

In this paper, the estimators of $\boldsymbol{\theta}, \boldsymbol{\alpha}$, and $\boldsymbol{\gamma}_j$ are called the *estimators* of feasible generalized least squares-multivariate spatial autoregressive three-stage least squares fixed effect panel simultaneous models (FGLS-MSAR3SLSFEPSM).

4 Properties of Estimators

Theorem (Consistency). *If* $\bar{\mathbf{X}}_{**}^t\mathbf{A}\mathbf{y}_j = \bar{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\mu} + \bar{\mathbf{X}}_{**}^t\mathbf{Z}_j\boldsymbol{\theta} + \bar{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\gamma}_j + \bar{\mathbf{X}}_{**}^t\mathbf{u}_j$ *as defined in* (9), then $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\mu}}$, *and* $\hat{\boldsymbol{\gamma}}_j$ *are consistent estimators.*

Proof. Recall (9). This can be rewritten as $\mathbf{A}\mathbf{y}_j = \mathbf{G}\boldsymbol{\mu} + \mathbf{Z}_j\boldsymbol{\theta} + \mathbf{G}\boldsymbol{\gamma}_j + \mathbf{u}_j$. However, we use the estimate $\hat{\rho}_h$. The equation (9) can be rewritten as $\hat{\mathbf{A}}\mathbf{y}_j = \mathbf{G}\boldsymbol{\mu} + \mathbf{Z}_j\boldsymbol{\theta} + \mathbf{G}\boldsymbol{\gamma}_j + \mathbf{u}_j$. Estimators of equation (9) are as follows:

$$\hat{\boldsymbol{\theta}} = \left[\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\hat{\mathbf{A}}\mathbf{y}_j$$

$$= \boldsymbol{\theta} + \left[\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}\left[\sum_{j=1}^T \mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{u}_j\right], \hat{\mathbf{M}}\mathbf{G} = \mathbf{0},$$

$$\hat{\boldsymbol{\mu}} = \left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^T\mathbf{G}^t\hat{\mathbf{H}}\left(\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{Z}_j\hat{\boldsymbol{\theta}}\right)$$

$$= \boldsymbol{\mu} + \left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\left(\sum_{j=1}^T\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) + \sum_{j=1}^T\mathbf{G}^t\hat{\mathbf{H}}\mathbf{u}_j\right),$$

where $\sum_{j=1}^T\boldsymbol{\gamma}_j = \mathbf{0}$,

$$\hat{\boldsymbol{\gamma}}_j = \left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}}\left(\hat{\mathbf{A}}\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j\hat{\boldsymbol{\theta}}\right)$$

$$= \left(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\right) + \left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) + \boldsymbol{\gamma}_j$$

$$+ \left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{u}_j.$$

We refer to [6-9,14,15,18]. Asymptotic expectation and variance of $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\mu}}$, and $\hat{\boldsymbol{\gamma}}_j$ are as follows:

$$\bar{E}\left\{\hat{\boldsymbol{\theta}}\right\} = \lim_{\substack{n\to\infty \\ T\to\infty}} E\left\{\hat{\boldsymbol{\theta}}\right\} = \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{nT}\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}$$

$$\times\left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{nT}\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}E\left\{\mathbf{u}_j\right\}\right]$$

$$= \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{nT}\mathbf{S}\right]^{-1}\left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{nT}\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\times\mathbf{0}\right]$$

$$= \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}}\bar{\mathbf{S}}\right]^{-1}\times\mathbf{0} = \boldsymbol{\theta} + \left[\bar{\mathbf{S}}\right]^{-1}\times\mathbf{0} = \boldsymbol{\theta},$$

where $\mathbf{S}$ and $\bar{\mathbf{S}}$ are constant nonsingular matrices.

$$\text{asy.var}\left\{\hat{\boldsymbol{\theta}}\right\} = \text{asy.var}\left\{\left[\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}\left[\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{u}_j\right]\right\}$$

$$= \left[\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}\left[\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\left(\mathbf{\Sigma}\otimes\mathbf{I}_n\right)\right.$$

$$\left.\times\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]\left[\sum_{j=1}^T\mathbf{Z}_j^t\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1},$$

where $\hat{\mathbf{H}}$ and $\hat{\mathbf{H}}\hat{\mathbf{M}}$ are symmetrical. Now,

$$\lim_{\substack{n\to\infty\\T\to\infty}} \text{asy.var}\left\{\hat{\boldsymbol{\theta}}\right\} = \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1} \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}\hat{\mathbf{M}}\right]$$

$$\times\left\{\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left(\boldsymbol{\Sigma}\otimes\mathbf{I}_n\right)\right\}\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j$$

$$\times\left[\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]^{-1}$$

$$=\left[\mathbf{S}\right]^{-1}\left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}\hat{\mathbf{M}}\times\mathbf{0}\times\hat{\mathbf{H}}\hat{\mathbf{M}}\mathbf{Z}_j\right]$$

$$\times\left[\lim_{\substack{n\to\infty\\T\to\infty}} \overline{\mathbf{S}}\right]^{-1}$$

$$=\mathbf{S}^{-1}\times\mathbf{0}\times\left[\overline{\mathbf{S}}\right]^{-1}$$

$$=\mathbf{0}.$$

This shows that $\hat{\boldsymbol{\theta}}$ is asymptotically unbiased estimator. If $n\to\infty$ or $T\to\infty$ or both of $n\to\infty$ and $T\to\infty$, then $\text{asy.var}\left\{\hat{\boldsymbol{\theta}}\right\}\to\mathbf{0}$. Therefore, $\hat{\boldsymbol{\theta}}$ is a consistent estimator. Next,

$$\overline{E}\left\{\hat{\boldsymbol{\mu}}\right\} = \lim_{\substack{n\to\infty\\T\to\infty}} E\left\{\hat{\boldsymbol{\mu}}\right\}$$

$$=\boldsymbol{\mu}+\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\right)\left(\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\right.$$

$$\left.\times\left(\boldsymbol{\theta}-\lim_{\substack{n\to\infty\\T\to\infty}} E\left\{\hat{\boldsymbol{\theta}}\right\}\right)+\lim_{\substack{n\to\infty\\T\to\infty}}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}E\left\{\mathbf{u}_j\right\}\right)$$

$$=\boldsymbol{\mu}+\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\frac{1}{n}\mathbf{S}_1\right]^{-1}\right)\left(\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\left(\boldsymbol{\theta}-\boldsymbol{\theta}\right)\right.$$

$$\left.+\lim_{\substack{n\to\infty\\T\to\infty}}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\times\mathbf{0}\right)$$

$$=\boldsymbol{\mu}+\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\overline{\mathbf{S}}_1\right]^{-1}\right)\times\mathbf{0}$$

$$=\boldsymbol{\mu}+\mathbf{0}\times\mathbf{0}$$

$$=\boldsymbol{\mu},$$

where $\mathbf{S}_1$ and $\overline{\mathbf{S}}_1$ are constant nonsingular matrices. We have

$$\text{asy.var}\left\{\hat{\boldsymbol{\mu}}\right\} = \text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}$$

$$+\text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{u}_j\right\},$$

$$\text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}=\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}$$

$$\times\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\,\text{asy.var}\left\{\hat{\boldsymbol{\theta}}\right\}\mathbf{Z}_j^t\hat{\mathbf{H}}\mathbf{G}\right]\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1},$$

$$\text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{u}_j\right\}=\left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}$$

$$\times\left[\mathbf{G}^t\hat{\mathbf{H}}\left(\boldsymbol{\Sigma}\otimes\mathbf{I}_n\right)\hat{\mathbf{H}}\mathbf{G}\right]\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1},$$

$$\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}$$

$$=\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\right)\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\right.$$

$$\left.\times\left(\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\left\{\hat{\boldsymbol{\theta}}\right\}\right)\mathbf{Z}_j^t\hat{\mathbf{H}}\mathbf{G}\right]$$

$$\times\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\right)$$

$$=\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\overline{\mathbf{S}}_1\right]^{-1}\right)\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{Z}_j\times\mathbf{0}\times\mathbf{Z}_j^t\hat{\mathbf{H}}\mathbf{G}\right]$$

$$\times\left(\lim_{\substack{n\to\infty\\T\to\infty}} \frac{1}{nT}\left[\overline{\mathbf{S}}_1\right]^{-1}\right)$$

$$=\mathbf{0},$$

$$\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\left\{\left[T\,\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{u}_j\right\}$$

$$=\left[\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\left[\mathbf{G}^t\hat{\mathbf{H}}\left\{\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}\left(\boldsymbol{\Sigma}\otimes\mathbf{I}_n\right)\right\}\hat{\mathbf{H}}\mathbf{G}\right]$$

$$\times\left[\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}\mathbf{G}\right]^{-1}$$

$$=\left[\mathbf{S}_1\right]^{-1}\left[\mathbf{G}^t\hat{\mathbf{H}}\times\mathbf{0}\times\hat{\mathbf{H}}\mathbf{G}\right]\left[\lim_{\substack{n\to\infty\\T\to\infty}}\overline{\mathbf{S}}_1\right]^{-1}$$

$$=\mathbf{S}_1^{-1}\times\mathbf{0}\times\left[\overline{\mathbf{S}}_1\right]^{-1}$$

$$=\mathbf{0}.$$

Consequently,

$$\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\{\hat{\boldsymbol{\mu}}\} = \lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\Bigg\{\left[T\,\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\sum_{j=1}^{T}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\Bigg\} + \lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\Bigg\{\left[T\,\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\sum_{j=1}^{T}\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Bigg\} = \mathbf{0}.$$

This shows that $\hat{\boldsymbol{\mu}}$ is asymptotically unbiased estimator. If $n\to\infty$ or $T\to\infty$ or both of $n\to\infty$ and $T\to\infty$, then $\text{asy.var}\{\hat{\boldsymbol{\mu}}\}\to\mathbf{0}$. Therefore, $\hat{\boldsymbol{\mu}}$ is a consistent estimator. Now,

$$\begin{aligned}\bar{E}\{\hat{\boldsymbol{\gamma}}_j\} &= \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\gamma}}_j\}\\
&= \Big(\boldsymbol{\mu}-\lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\mu}}\}\Big) + \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j \times\Big(\boldsymbol{\theta}-\lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\theta}}\}\Big) + \boldsymbol{\gamma}_j + \lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{n}\left[\frac{1}{n}\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\mathbf{G}'\hat{\mathbf{H}}E\{\mathbf{u}_j\}\\
&= (\boldsymbol{\mu}-\boldsymbol{\mu}) + [\mathbf{S}_1]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j(\boldsymbol{\theta}-\boldsymbol{\theta}) + \boldsymbol{\gamma}_j + \lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{n}\left[\bar{\mathbf{S}}_1\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\times\mathbf{0}\\
&= \boldsymbol{\gamma}_j.\end{aligned}$$

$$\text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\} = \text{asy.var}\{\hat{\boldsymbol{\mu}}\} + \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\Big\} + \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Big\},$$

$$\text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\Big\} = \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j \times\text{asy.var}\{\hat{\boldsymbol{\theta}}\}\mathbf{Z}_j'\hat{\mathbf{H}}\mathbf{G}\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1},$$

$$\text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Big\} = \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}} \times(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\hat{\mathbf{H}}\mathbf{G}\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1},$$

$$\begin{aligned}\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\Big\} &= \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1} \times\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\Big(\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\{\hat{\boldsymbol{\theta}}\}\Big) \times\mathbf{Z}_j'\hat{\mathbf{H}}\mathbf{G}\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\\
&= [\mathbf{S}_1]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\times\mathbf{0}\times\mathbf{Z}_j'\hat{\mathbf{H}}\mathbf{G}[\mathbf{S}_1]^{-1}\\
&= \mathbf{0},\end{aligned}$$

$$\begin{aligned}&\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Big\}\\
&= \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\Big\{\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{nT}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\Big\} \times\hat{\mathbf{H}}\mathbf{G}\left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{T}\frac{1}{n}\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\\
&= [\mathbf{S}_1]^{-1}\mathbf{G}'\hat{\mathbf{H}}\times\mathbf{0}\times\hat{\mathbf{H}}\mathbf{G}\left[\lim_{\substack{n\to\infty \\ T\to\infty}}\frac{1}{T}\bar{\mathbf{S}}_1\right]^{-1}\\
&= \mathbf{0}\times[\mathbf{0}]^{-1}\\
&= \infty \text{ (infinit)},\end{aligned}$$

therefore, convergenity be satisfied only if $n\to\infty$, namely

$$\begin{aligned}&\lim_{n\to\infty} \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Big\}\\
&= \left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\Big\{\lim_{n\to\infty}\frac{1}{n}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\Big\}\hat{\mathbf{H}}\mathbf{G} \times\left[\lim_{n\to\infty}\frac{1}{n}\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\\
&= [\mathbf{S}_1]^{-1}\mathbf{G}'\hat{\mathbf{H}}\times\mathbf{0}\times\hat{\mathbf{H}}\mathbf{G}\left[\lim_{n\to\infty}\bar{\mathbf{S}}_1\right]^{-1}\\
&= \mathbf{S}_1^{-1}\times\mathbf{0}\times\left[\bar{\mathbf{S}}_1\right]^{-1}\\
&= \mathbf{0}.\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\} = \lim_{n\to\infty} \text{asy.var}\{\hat{\boldsymbol{\mu}}\} + \lim_{n\to\infty} \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{Z}_j\hat{\boldsymbol{\theta}}\Big\} + \lim_{n\to\infty} \text{asy.var}\Big\{\left[\mathbf{G}'\hat{\mathbf{H}}\mathbf{G}\right]^{-1}\mathbf{G}'\hat{\mathbf{H}}\mathbf{u}_j\Big\} = \mathbf{0}.$$

This shows that $\hat{\boldsymbol{\gamma}}_j$ is asymptotically unbiased estimator. If $n\to\infty$, then $\text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\}\to\mathbf{0}$. Therefore, $\hat{\boldsymbol{\gamma}}_j$ is a consistent estimator.

5 Illustration

Suppose there are three endogenous variables y_1, y_2, y_3 and six exogenous variables $x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}$ observed for two time periods and the number of observation being 10 locations (this illustration uses fictitious data and there is no conflict of interest regarding the publication of this paper). Data are presented in Table 1 and Table 2. The equation models are as follows:

$$\begin{aligned} y_{1ij} &= \mu_1 + \alpha_{11}x_{11ij} + \alpha_{12}x_{12ij} + \rho_1 \mathbf{w}_i^t \mathbf{y}_{1j} + \beta_{12}y_{2ij} \\ &\quad + \beta_{13}y_{3ij} + \gamma_{1j} + u_{1ij}, \quad u_{1ij} \sim N(0,\sigma_1^2), \\ y_{2ij} &= \mu_2 + \alpha_{21}x_{21ij} + \alpha_{22}x_{22ij} + \rho_2 \mathbf{w}_i^t \mathbf{y}_{2j} + \beta_{21}y_{1ij} \\ &\quad + \beta_{23}y_{3ij} + \gamma_{2j} + u_{2ij}, \quad u_{2ij} \sim N(0,\sigma_2^2), \\ y_{3ij} &= \mu_3 + \alpha_{31}x_{31ij} + \alpha_{32}x_{32ij} + \rho_3 \mathbf{w}_i^t \mathbf{y}_{3j} + \beta_{31}y_{1ij} \\ &\quad + \beta_{32}y_{2ij} + \gamma_{3j} + u_{3ij}, \quad u_{3ij} \sim N(0,\sigma_3^2), \end{aligned} \tag{27}$$

where

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1,10} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2,10} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3,10} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{10,1} & w_{10,2} & w_{10,3} & \cdots & w_{10,10} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^t \\ \mathbf{w}_2^t \\ \mathbf{w}_3^t \\ \vdots \\ \mathbf{w}_{10}^t \end{bmatrix} = \left[\mathbf{w}_i^t \right], \quad i = 1,2,3,\cdots,10,$$

and

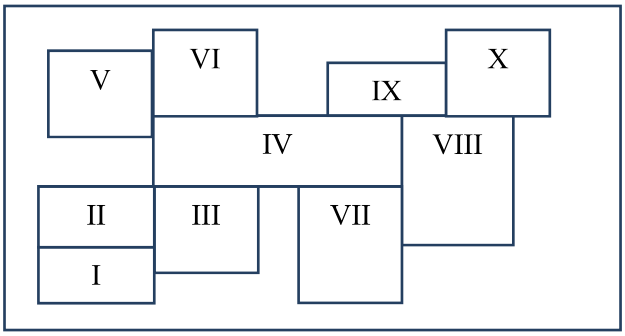

Fig.1: Illustration of the 10 neighboring locations.

Then from Fig.1, we obtain row-standardized spatial weight matrix as follows:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{1}{7} & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

The formulation of Moran Index is as follows:

$$I_{hj} = \frac{\sum_{i=1}^{10}\sum_{i^*=1}^{10} w_{ii^*}\left(y_{hij} - \bar{y}_{hj}\right)\left(y_{hi^*j} - \bar{y}_{hj}\right)}{\sum_{i=1}^{10}\left(y_{hij} - \bar{y}_{hj}\right)^2} = \frac{\mathbf{y}_{hj}^{*t}\mathbf{W}\mathbf{y}_{hj}^{*}}{\mathbf{y}_{hj}^{*t}\mathbf{y}_{hj}^{*}},$$

for $h = 1,2,3, \; j = 1,2,$

where $\bar{y}_{hj} = \frac{1}{10}\sum_{i=1}^{10} y_{hij}$ and $\mathbf{y}_{hj}^* = \mathbf{y}_{hj} - \bar{y}_{hj}\mathbf{1}.$

Table 1: Data for endogenous variables

Time	Location	Endogenous Variables		
		y_1	y_2	y_3
1	1	15	25	20
	2	17	28	23
	3	14	27	21
	4	12	26	24
	5	18	29	22
	6	19	28	26
	7	20	31	29
	8	13	33	31
	9	14	29	28
	10	16	27	25
2	1	16	24	21
	2	17	29	27
	3	15	27	23
	4	14	26	22
	5	17	30	28
	6	20	29	27
	7	18	32	31
	8	14	32	30
	9	15	29	27
	10	18	26	23

Note: data illustration (fictitious data)

Table 2: Data for exogenous variables

Time	Loca-tion	Exogenous Variables x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}
1	1	45	51	46	49	47	48
	2	40	55	42	56	45	53
	3	41	56	40	58	42	51
	4	42	58	43	57	40	55
	5	47	57	45	58	42	51
	6	46	54	44	55	43	54
	7	45	56	47	54	49	57
	8	43	57	46	59	45	52
	9	47	59	48	60	46	59
	10	44	52	43	53	44	52
2	1	50	65	51	64	53	65
	2	51	66	52	67	54	63
	3	59	66	58	68	57	71
	4	58	64	59	66	54	73
	5	57	63	60	62	56	61
	6	61	67	61	68	60	67
	7	63	68	62	65	61	64
	8	62	68	64	66	59	71
	9	64	69	65	68	53	59
	10	58	65	57	69	58	67

Note: data illustration (fictitious data)

If there is at least one $I_{hj} > E(I)$, then we conclude that there is a spatial influence for the equation models.

$\bar{y}_{11} = 15.80;\ \bar{y}_{21} = 28.30;\ \bar{y}_{31} = 24.90;$
$\bar{y}_{22} = 28.40;\ \bar{y}_{32} = 25.90;$
$I_{11} = \text{-}0.2442;\ I_{21} = 0.0539;\ I_{31} = 0.4586;$
$I_{12} = \text{-}0.2317;\ I_{22} = \text{-}0.0878;\ I_{32} = \text{-}0.1078;$
and $E(I_{hj}) = E(I) = \frac{-1}{n-1} = \frac{-1}{10-1} = \text{-}0.1111.$

Based on the above result, by means of R Program version 3.0.3, we obtain that there is a spatial influence for the equation models.

We then continue to estimate parameters by means of FGLS-3SLS. For the first-stage, we estimate all the endogenous expalanatory variables in the system in every time period and the results are as in Table 3.

Table 3: Estimated values for endogenous explanatory variables

Time	Loca-tion	Endogenous explanatory variables y_1-estimate	y_2-estimate	y_3-estimate
1	1	16.5625	26.5828	21.7290
	2	15.0373	28.5890	25.1588
	3	16.1904	27.6672	20.7955
	4	12.3775	26.0621	23.9145
	5	16.1804	28.3403	22.0819
	6	17.2918	26.9959	24.8593
	7	18.7007	29.1060	26.5129
	8	12.3543	31.5231	28.7592
	9	16.4805	31.4345	30.8012
	10	16.8246	26.6991	24.3877
2	1	15.5100	25.9073	23.1069
	2	17.3247	27.0314	24.7638
	3	15.8433	26.3597	22.8089
	4	13.3019	25.4562	21.0773
	5	17.3259	29.7492	27.8872
	6	18.1930	30.2621	28.3106
	7	18.0785	31.1379	29.8797
	8	14.7653	32.0924	30.1569
	9	14.9671	29.3371	27.3870
	10	18.6902	26.6667	23.6217

For the second-stage we estimate $\mathbf{\Sigma}_{\#}$. But, we first estimate spatial autoregressive by means of equation (19). By $\mathbf{W}$, we have the acceptable spatial autoregressive parameter to be $\text{-}1.6242 < \rho_h < 1$. By the method of forming sequence of ρ_h with increasing 0.01, we obtain

1. $\rho_h = \text{seq(-1.6142, 0.99, 0.01)}$.
2. For every $\mathbf{y}_{hj}$ and $\mathbf{a}_{hj}$, $h = 1,2,3$, we insert values of ρ_h to (19). Because $\mathbf{a}_{hj}$ unknown, we use the estimate, $\hat{\mathbf{a}}_{hj}$, where $\hat{\mathbf{a}}_{hj} = \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) + \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h$, with $\mathbf{Z}_{hj} = [\mathbf{X}_{hj} \ \vdots \ \hat{\mathbf{Y}}_{-hj}]$.
3. We obtain $\hat{\rho}_1 = 0.2658$, $\hat{\rho}_2 = \text{-}1.6042$ and $\hat{\rho}_3 = \text{-}1.5842$ that gives the largest $\ln L_1^{con}$, $\ln L_2^{con}$ and $\ln L_3^{con}$, respectively.

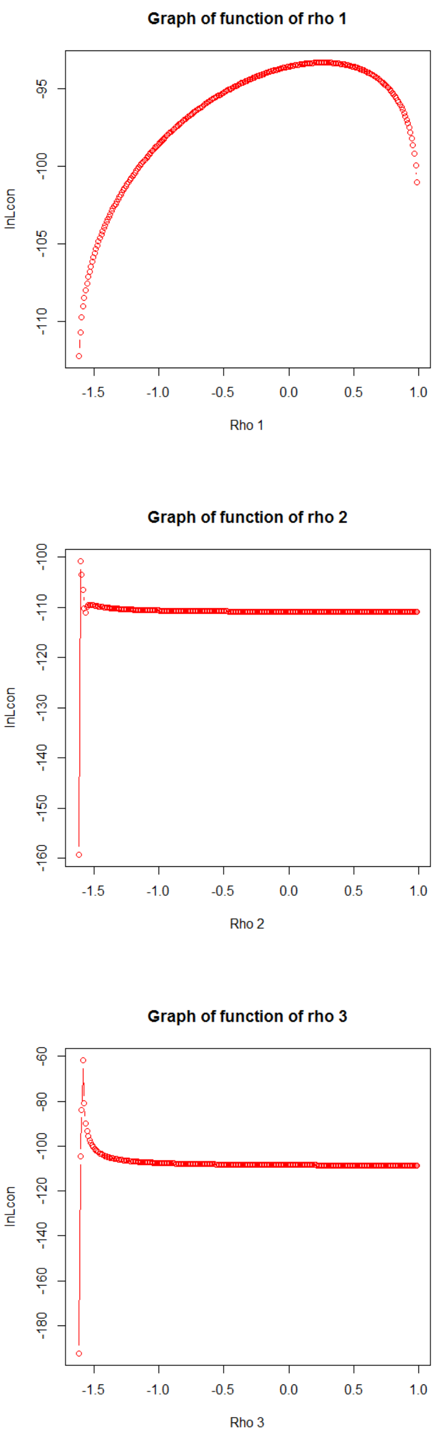

Fig.2: Graphs of function of rho

And by the method of forming sequence of ρ_h with increasing 0.01, we can also make graphs between the values of rho and the values of concentrated log-likelihood as presented in Fig.2.

From (20) to (22), we obtain

$$\hat{\boldsymbol{\theta}}_1 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_1 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{11} \\ \hat{\alpha}_{12} \\ \hat{\beta}_{12} \\ \hat{\beta}_{13} \end{bmatrix} = \begin{bmatrix} 0.1036 \\ -0.3523 \\ -0.0245 \\ 0.1671 \end{bmatrix};$$

$$\hat{\boldsymbol{\theta}}_2 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_2 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-2} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{21} \\ \hat{\alpha}_{22} \\ \hat{\beta}_{21} \\ \hat{\beta}_{23} \end{bmatrix} = \begin{bmatrix} 0.3064 \\ 0.0138 \\ 0.0601 \\ 0.5034 \end{bmatrix};$$

$$\hat{\boldsymbol{\theta}}_3 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_3 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-3} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{31} \\ \hat{\alpha}_{32} \\ \hat{\beta}_{31} \\ \hat{\beta}_{32} \end{bmatrix} = \begin{bmatrix} -0.0035 \\ 0.4096 \\ 0.1662 \\ 1.5033 \end{bmatrix}$$

$\hat{\mu}_1 = 24.6620;$ $\hat{\mu}_2 = 43.0006;$ $\hat{\mu}_3 = -4.3468$

$\hat{\gamma}_{11} = -1.2130;$ $\hat{\gamma}_{12} = 1.2130;$

$\hat{\gamma}_{21} = 2.5188;$ $\hat{\gamma}_{22} = -2.5188;$

$\hat{\gamma}_{31} = 2.0259;$ $\hat{\gamma}_{32} = -2.0259.$

Table 4: Estimate values for residual errors

Time	Loca-tion	Residual errors		
		u_1-estimate	u_2-estimate	u_3-estimate
1	1	-2.2422	-3.1123	-5.0391
	2	1.6481	-3.0250	-7.3850
	3	-0.6623	-0.7777	-6.3261
	4	-3.0902	0.7119	3.4309
	5	2.6486	0.1127	-2.1118
	6	2.3310	-1.2024	-0.6691
	7	4.5791	3.1835	4.8456
	8	-2.9752	2.6659	4.7162
	9	-1.5409	-2.5683	-1.4361
	10	-0.6962	4.0118	9.9744
2	1	0.3667	-0.6427	-0.9871
	2	1.6319	-0.6768	-0.9082
	3	-1.0635	-2.3864	-4.8065
	4	-2.6379	2.7829	1.8581
	5	-1.0336	-0.6903	0.7926
	6	3.3020	-1.5430	-2.7742
	7	1.6052	2.0132	2.7448
	8	-2.9123	0.2430	-2.4101
	9	-1.1280	-2.1098	2.4048
	10	1.8695	3.0098	4.0858

Next, from (23), we obtain the estimate values for residual errors being presented in Tabel 5.3. We then use the estimate values for residual errors (in Table 4) to find $\hat{\boldsymbol{\Sigma}}$ as follow:

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} 6.7696 & 0.3747 & -0.4183 \\ 0.3747 & 6.4086 & 10.0961 \\ -0.4183 & 10.0961 & 23.7600 \end{bmatrix},$$

and we obtain

$$\hat{\boldsymbol{\Sigma}}_{\#} = \begin{bmatrix} 177{,}677.50 & 210{,}733.62 & 179{,}376.68 & \cdots & -12{,}769.99 \\ 210{,}733.62 & 250{,}442.60 & 212{,}838.98 & \cdots & -15{,}178.10 \\ 179{,}376.68 & 212{,}838.98 & 181{,}177.40 & \cdots & -12{,}894.44 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -12{,}769.99 & -15{,}178.10 & -12{,}894.44 & \cdots & 846{,}645.42 \end{bmatrix},$$

For the last-stage, we estimate the parameters of equation models (27). By (24) to (26), we obtain

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{31} \\ \alpha_{32} \\ \hline \beta_{12} \\ \beta_{13} \\ \beta_{21} \\ \beta_{23} \\ \beta_{31} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} 0.2237 \\ -0.7310 \\ 0.2296 \\ -0.1024 \\ -0.6446 \\ 0.2366 \\ \hline 0.1374 \\ -0.0086 \\ -0.0032 \\ 0.5628 \\ 0.5789 \\ 2.2122 \end{bmatrix};$$

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 41.4296 \\ 53.5816 \\ 11.5225 \end{bmatrix};$$

$$\hat{\boldsymbol{\gamma}}_1 = \begin{bmatrix} \gamma_{11} \\ \gamma_{21} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} -2.4323 \\ 1.3741 \\ -2.8309 \end{bmatrix};$$

$$\hat{\boldsymbol{\gamma}}_2 = \begin{bmatrix} \gamma_{12} \\ \gamma_{22} \\ \gamma_{32} \end{bmatrix} = \begin{bmatrix} 2.4323 \\ -1.3741 \\ 2.8309 \end{bmatrix};$$

and the estimated equation models (27) are

$$\begin{aligned} \hat{y}_{1i1} &= 41.4296 + 0.2237x_{11i1} - 0.7310x_{12i1} \\ &\quad + 0.2658\mathbf{w}_i^t\mathbf{y}_{11} + 0.1374y_{2i1} - 0.0086y_{3i1} \\ &\quad - 2.4323 \\ \hat{y}_{2i1} &= 53.5816 + 0.2296x_{21i1} - 0.1024x_{22i1} \\ &\quad - 1.6042\mathbf{w}_i^t\mathbf{y}_{21} - 0.0032y_{1i1} + 0.5628y_{3i1} \\ &\quad + 1.3741 \\ \hat{y}_{3i1} &= 11.5225 - 0.6446x_{31i1} + 0.2366x_{32i1} \\ &\quad - 1.5842\mathbf{w}_i^t\mathbf{y}_{31} + 0.5789y_{1i1} + 2.2122y_{2i1} \\ &\quad - 2.8309, \\ \hat{y}_{1i2} &= 41.4296 + 0.2237x_{11i2} - 0.7310x_{12i2} \\ &\quad + 0.2658\mathbf{w}_i^t\mathbf{y}_{12} + 0.1374y_{2i2} - 0.0086y_{3i2} \\ &\quad + 2.4323 \\ \hat{y}_{2i2} &= 53.5816 + 0.2296x_{21i2} - 0.1024x_{22i2} \\ &\quad - 1.6042\mathbf{w}_i^t\mathbf{y}_{22} - 0.0032y_{1i2} + 0.5628y_{3i2} \\ &\quad - 1.3741 \\ \hat{y}_{3i2} &= 11.5225 - 0.6446x_{31i2} + 0.2366x_{32i2} \\ &\quad - 1.5842\mathbf{w}_i^t\mathbf{y}_{32} + 0.5789y_{1i2} + 2.2122y_{2i2} \\ &\quad + 2.8309. \end{aligned}$$

6 Conclusion

In this paper, we are motivated to develop simultaneous equation models for fixed effect panel data with one-way error component by means of 3SLS solutions, especially for spatial correlation among dependent variables.

The estimators are called the *estimators* of feasible generalized least squares-multivariate spatial autoregressive three-stage least squares fixed effect panel simultaneous models (FGLS-MSAR3SLSFEPSM). All estimators are consistent estimators.

In future research, we encourage to develop models for both spatial correlation among dependent variables and spatial correlation among errors (general spatial).

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