Vague Functional Dependencies and Resolution Principle

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Abstract: The goal of this paper is to show that the process of deriving of new vague functional dependencies from given ones may be automated. To achieve this, we join fuzzy formulas to vague functional dependencies. Thus, to prove that a vague functional dependency follows from a set of vague functional dependencies, becomes the same as to prove that the corresponding fuzzy formula is valid whenever the fuzzy formulas from the corresponding set of fuzzy formulas are valid.

Key–Words: Vague functional dependencies, fuzzy formulas, valuations, resolution principle

1 Preliminaries

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$.

Suppose that $V(U_i)$ is the family of all vagues sets in U_i, $i \in I$.

Here, we say that V_i is a vague set in U_i, if

$$V_i = \{\langle u, [t_{V_i}(u), 1 - f_{V_i}(u)]\rangle \ : \ u \in U_i\},$$

where $t_{V_i} : U_i \to [0, 1]$, $f_{V_i} : U_i \to [0, 1]$ are functions such that $t_{V_i}(u) + f_{V_i}(u) \leq 1$ for all $u \in U_i$.

We also say that $[t_{V_i}(u), 1 - f_{V_i}(u)] \subseteq [0, 1]$ is the vague value joined to $u \in U_i$.

A vague relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $V(U_1) \times V(U_2) \times ... \times V(U_n)$.

A tuple t of r is denoted by

$$(t[A_1], t[A_2], ..., t[A_n]).$$

Here, we consider the vague set $t[A_i]$ as the value of the attribute A_i on t.

Let $Vag(U_i)$ be the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping $SE_i : Vag(U_i) \times Vag(U_i) \to [0, 1]$, such that $SE_i(x, x) = 1$, $SE_i(x, y) = SE_i(y, x)$, and $SE_i(x, z) \geq$ $\max_{y \in Vag(U_i)} (\min(SE_i(x, y), SE_i(y, z)))$ for all $x, y, z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let

$$\begin{aligned} A_i &= \{\langle u, [t_{A_i}(u), 1 - f_{A_i}(u)]\rangle \ : \ u \in U_i\} \\ &= \{a_u^i \ : \ u \in U_i\}, \\ B_i &= \{\langle u, [t_{B_i}(u), 1 - f_{B_i}(u)]\rangle \ : \ u \in U_i\} \\ &= \{b_u^i \ : \ u \in U_i\} \end{aligned}$$

be two vague sets in U_i.

The similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$\begin{aligned} &SE(A_i, B_i) \\ &= \min\Big\{ \min_{a_u^i \in A_i} \Big\{ \max_{b_u^i \in B_i} \Big\{ SE_i\Big([t_{A_i}(u), 1 - f_{A_i}(u)], \\ &\quad [t_{B_i}(u), 1 - f_{B_i}(u)] \Big) \Big\} \Big\}, \\ &\quad \min_{b_u^i \in B_i} \Big\{ \max_{a_u^i \in A_i} \Big\{ SE_i\Big([t_{B_i}(u), 1 - f_{B_i}(u)], \\ &\quad [t_{A_i}(u), 1 - f_{A_i}(u)] \Big) \Big\} \Big\} \Big\}. \end{aligned}$$

Now, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between tuples t_1 and t_2 on the attribute set X is defined by

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

For various definitions of similarity measures, see, [13], [5], [4], [11] and [12].

2 Vague functional dependencies

In [10], we introduced a new definition of vague functional dependencies.

Thus, if X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta \in [0, 1]$ is a number, then, the vague relation instance r on $R(A_1, A_2, ..., A_n)$ is said to satisfy the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r,

$$SE_Y(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}.$$

Here, we write $X \to_V Y$ instead of $X \overset{\theta}{\to}_V Y$ if $\theta = 1$.

θ is called the linguistic strength of the vague functional dependency $X \overset{\theta}{\to}_V Y$ (see, [20] in the case of fuzzy functional dependencies).

For various definitions of vague functional dependencies, see, [13], [16] and [22].

3 Inference rules

The following rules are the main inference rules for vague functional dependencies described above (see, [10]).

VF1 Inclusive rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \overset{\theta_2}{\longrightarrow}_V Y$ holds.

VF2 Reflexive rule for VFDs: If $X \supseteq Y$, then $X \to_V Y$ holds.

VF3 Augmentation rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, then $X \cup Z \overset{\theta}{\to}_V Y \cup Z$ holds.

VF4 Transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

The following rules are additional inference rules for vague functional dependencies.

VF5 Union rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $X \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Y \cup Z$ holds also true.

VF6 Pseudo-transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $W \cup Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $W \cup X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF7 Decomposition rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, and $Z \subseteq Y$, then $X \overset{\theta}{\to}_V Z$ also holds.

The fact that the rules VF5-VF7 are additional inference rules, means that these rules follow from the rules VF1-VF4.

According to Theorems 4 and 5 in [10], the inference rules VF1-VF7 are sound.

This, in the case of the inference rule VF1, for example, means that r satisfies $X \overset{\theta_2}{\longrightarrow}_V Y$, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies the vague functional dependency $X \overset{\theta_1}{\longrightarrow}_V Y$.

4 Completeness

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

By Theorem 7 in [10], the set $\{VF1, VF2, VF3, VF4\}$ is complete set.

This means that there exists a vague relation instance r^* on $R(A_1, A_2, ..., A_n)$ (r^* is denoted by r in [10]), such that r^* satisfies $A \overset{1\theta}{\to}_V B$ if $A \overset{1\theta}{\to}_V B$ belongs to $\mathcal{V}^+$, and violates $X \overset{\theta}{\to}_V Y$, where $X \overset{\theta}{\to}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$ which is not an element of the closure $\mathcal{V}^+$ of $\mathcal{V}$.

The closure $\mathcal{V}^+$ of $\mathcal{V}$ is the set of all vague functional dependencies that can be derived from $\mathcal{V}$ by repeated applications of the inference rules VF1-VF4, where $\mathcal{V}$ is some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$.

In [10], $r^* = \{t_1, t_2\}$ is given by Table I.

Table 1:

	attributes of $X^+(\theta, \mathcal{V})$	other attributes
t_1	$V_1, V_1, ..., V_1$	$V_1, V_1, ..., V_1$
t_2	$V_1, V_1, ..., V_1$	$V_2, V_2, ..., V_2$

$X^+(\theta, \mathcal{V})$ is the closure of X with respect to $\mathcal{V}$, i.e., $X^+(\theta, \mathcal{V})$ is the set of attributes $A \in \{A_1, A_2, ..., A_n\}$, such that $X \overset{\theta}{\to}_V A$ belongs to $\mathcal{V}^+$.

For the sake of simplicity it is assumed that $U_1 = U_2 = ... = U_n = \{u\} = U$.

The vagues sets V_1 and V_2 in U are given by

$$\begin{aligned} V_1 &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle \,:\, u \in U\} \\ &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle\} = \{\langle u, a\rangle\} \end{aligned}$$

and

$$\begin{aligned} V_2 &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle \,:\, u \in U\} \\ &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle\} = \{\langle u, b\rangle\}. \end{aligned}$$

It is assumed that $SE(a, b) = \theta'$.

$\theta' \in \left(\theta'', \theta\right)$ is fixed, where

$$\theta'' = \max_{A \xrightarrow{1\theta}_V B \in \mathcal{V}^+ \,:\, {}_1\theta_l(\mathcal{V}) < \theta} \{{}_1\theta_l(\mathcal{V})\}.$$

Here, ${}_1\theta_l(\mathcal{V})$ denotes the limit strength of the dependency $A \xrightarrow{1\theta}_V B$ with respect to $\mathcal{V}$, i.e., ${}_1\theta_l(\mathcal{V})$ belongs to $[0,1]$, $A \overset{1\theta_l(\mathcal{V})}{\rightarrow}_V B$ belongs to $\mathcal{V}^+$, and $\theta_2 \le {}_1\theta_l(\mathcal{V})$ for each $A \xrightarrow{\theta_2}_V B$ that belongs to $\mathcal{V}^+$.

5 Fuzzy implications

Recall the following definitions (see, e.g., [19]).

A mapping $N : [0,1] \to [0,1]$ is a fuzzy negation if $N(0) = 1$, $N(1) = 0$, and $N(x) \ge N(y)$ for $x \le y$.

A mapping $\mathcal{C} : [0,1]^2 \to [0,1]$ is a conjunction on the unit interval if $\mathcal{C}(0,0) = \mathcal{C}(0,1) = \mathcal{C}(1,0) = 0$, $\mathcal{C}(1,1) = 1$, and $\mathcal{C}(x,z) \le \mathcal{C}(y,z)$, $\mathcal{C}(z,x) \le \mathcal{C}(z,y)$ for $x \le y$.

A mapping $T : [0,1]^2 \to [0,1]$ is a triangular norm (t-norm for short), if $T(x,1) = x$, $T(x,y) = T(y,x)$, $T(x, T(y,z)) = T(T(x,y), z)$, and $T(x,y) \le T(x,z)$ for $y \le z$.

Note that t-norm is a conjunction on the unit interval.

A mapping $S : [0,1]^2 \to [0,1]$ is a triangular co-norm (t-co-norm for short), if $S(x,0) = x$, $S(x,y) = S(y,x)$, $S(x, S(y,z)) = S(S(x,y), z)$, and $S(x,y) \le S(x,z)$ for $y \le z$.

As in the case of t-norms, the disjunction in fuzzy logic is often modeled by t-co-norms.

A mapping $I : [0,1]^2 \to [0,1]$ is a fuzzy implication if $I(0,0) = I(0,1) = I(1,1) = 1$, and $I(1,0) = 0$.

S-implications are the short for strong implications.

An S-implication is generated from a fuzzy negation and a t-co-norm. The idea stems from the proposition in classical binary logic:

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q).$$

Thus, an S-implication is defined by

$$I(x,y) = S(N(x), y),$$

$x, y \in [0,1]$, where S is a t-co-norm, and N is a fuzzy negation.

R-implications are short for residual implications.

An R-implication is generated from a conjunction on the unit interval. The idea comes from the equality in classical set theory:

$$(X \setminus A) \cup B = X \setminus (A \setminus B) = \bigcup_{A \cap Z \subseteq B} Z.$$

Hence, an R-implication is defined by

$$I(x,y) = \sup\{t \in [0,1] \;:\; \mathcal{C}(x,t) \le y\},$$

$x, y \in [0,1]$, where $\mathcal{C}$ is a conjunction on the unit interval.

QL-implications are the short for quantum logic implications.

A QL-implication is generated from a strong fuzzy negation, a t-co-norm, and a t-norm.

The idea follows from the equivalency in classical binary logic:

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee (p \wedge q)).$$

Consequently, a QL-implication is defined by

$$I(x,y) = S(N(x), T(x,y)),$$

$x, y \in [0,1]$, where S is a t-co-norm, N is a strong fuzzy negation, and T is a T-norm.

In this paper we shall apply the following operators:

$$\begin{aligned} T_M(x,y) &= \min\{x,y\}, \\ S_M(x,y) &= \max\{x,y\}, \\ I_L(x,y) &= \min\{1 - x + y, 1\}. \end{aligned} \tag{1}$$

T_M is the minimum t-norm, S_M is the maximum t-co-norm, and I_L is the Lukasiewicz fuzzy implication.

Note that the Lukasiewicz fuzzy implication I_L is quite general fuzzy implication. Namely, it is an S-implication since

$$I_L(x,y) = S_L(N_0(x), y),$$

for $N_0(x) = 1 - x$, and $S_L(x,y) = \min\{x+y, 1\}$.

I_L is an R-implication since

$$I_L(x,y) = \sup\{t \in [0,1] \ : \ T_L(x,y) \leq y\}$$

for $T_L(x,y) = \max\{x+y-1, 0\}$.

Finally, I_L is a QL-implication since

$$I_L(x,y) = S_L(N_0(x), T_M(x,y)).$$

For various works on S, R and QL-implications, see, [1], [2], [14], [21], [18], [15], [17].

For detailed study on fuzzy implications, we refer to [3].

6 Valuations

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Let $r = \{t_1, t_2\}$ be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0,1]$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let $A_k \in \{A_1, A_2, ..., A_n\}$.

Since the values $t_1[A_k]$ and $t_2[A_k]$ of the attribute A_k on tuples t_1 and t_2, respectively, are vague sets in U_k, we may write

$$\begin{aligned}
& t_1[A_k] \\
&= \left\{\left\langle u, \left[t_{t_1[A_k]}(u), 1 - f_{t_1[A_k]}(u)\right]\right\rangle \ : \ u \in U_k\right\} \\
&= \{\langle u, u_1\rangle \ : \ u \in U_k\} \\
&= \left\{a_u^1 \ : \ u \in U_k\right\}, \\
& t_2[A_k] \\
&= \left\{\left\langle u, \left[t_{t_2[A_k]}(u), 1 - f_{t_2[A_k]}(u)\right]\right\rangle \ : \ u \in U_k\right\} \\
&= \{\langle u, u_2\rangle \ : \ u \in U_k\} \\
&= \left\{a_u^2 \ : \ u \in U_k\right\}.
\end{aligned}$$

Now, we are free to calculate the similarity measure $SE(t_1[A_k], t_2[A_k])$ between the vague sets $t_1[A_k]$ and $t_2[A_k]$.

We have,

$$\begin{aligned}
& SE(t_1[A_k], t_2[A_k]) \\
&= \min\left\{\min_{a_u^1 \in t_1[A_k]}\left\{\max_{a_u^2 \in t_2[A_k]}\left\{SE_k\Big(u_1, u_2\Big)\right\}\right\},\right. \\
&\quad \left.\min_{a_u^2 \in t_2[A_k]}\left\{\max_{a_u^1 \in t_1[A_k]}\left\{SE_k\Big(u_2, u_1\Big)\right\}\right\}\right\}.
\end{aligned}$$

It is now straight forward to check if $SE(t_1[A_k], t_2[A_k]) \geq \beta$ or $SE(t_1[A_k], t_2[A_k]) < \beta$.

In the first resp. the second case, we may put $i_{r,\beta}(A_k)$ to be some value in the interval $\left(\frac{1}{2}, 1\right]$ resp. $\left[0, \frac{1}{2}\right]$.

Obviously, the value $i_{r,\beta}(A_k) \in [0,1]$ depends on r and β.

Actually, each time we have a two-element vague relation instance r on $R(A_1, A_2, ..., A_n)$, and a number $\beta \in [0,1]$, we are able to define the values $i_{r,\beta}(A_k) \in [0,1]$, $k \in \{1, 2, ..., n\}$.

Thus, we introduce a valuation joined to r and β, as a mapping $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0,1]$, such that

$$\begin{aligned}
i_{r,\beta}(A_k) &> \frac{1}{2} \ \text{ if } \ SE(t_1[A_k], t_2[A_k]) \geq \beta, \\
i_{r,\beta}(A_k) &\leq \frac{1}{2} \ \text{ if } \ SE(t_1[A_k], t_2[A_k]) < \beta,
\end{aligned}$$

$k \in \{1, 2, ..., n\}$.

Since $i_{r,\beta}(A_k) \in [0,1]$ for $k \in \{1, 2, ..., n\}$, it follows that the attributes A_k, $k \in \{1, 2, ..., n\}$ become fuzzy formulas with respect to $i_{r,\beta}$.

Let A_i, $A_j \in \{A_1, A_2, ..., A_n\}$.

We define the fuzzy formulas: $A_i \wedge A_j$, $A_i \vee A_j$, $A_i \Rightarrow A_j$ with respect to $i_{r,\beta}$ by putting

$$\begin{aligned}
i_{r,\beta}(A_i \wedge A_j) &= \min\{i_{r,\beta}(A_i), i_{r,\beta}(A_j)\}, \\
i_{r,\beta}(A_i \vee A_j) &= \max\{i_{r,\beta}(A_i), i_{r,\beta}(A_j)\}, \\
& i_{r,\beta}(A_i \Rightarrow A_j) \\
&= \min\{1 - i_{r,\beta}(A_i) + i_{r,\beta}(A_j), 1\}.
\end{aligned}$$

According to (1), these definitions make sense.

Thus, $A_i \wedge A_j$, $A_i \vee A_j$, and $A_i \Rightarrow A_j$ become fuzzy formulas with respect to $i_{r,\beta}$.

Consequently, if A_i, A_j, $A_k \in \{A_1, A_2, ..., A_n\}$, then $A_i \Rightarrow (A_j \wedge A_k)$, for example, becomes a fuzzy formula with respect to $i_{r,\beta}$, since

$$\begin{aligned} &i_{r,\beta}(A_i \Rightarrow (A_j \wedge A_k)) \\ &= \min\{1 - i_{r,\beta}(A_i) + i_{r,\beta}(A_j \wedge A_k), 1\}, \end{aligned}$$

and A_i, $A_j \wedge A_k$ are already fuzzy formulas with respect to $i_{r,\beta}$.

In particular, $\wedge_{A\in X} A$ and $(\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)$ become fuzzy formulas with respect to $i_{r,\beta}$, where X and Y are some subsets of $\{A_1, A_2, ..., A_n\}$.

Through the rest of the paper, we shall assume that each time some $r = \{t_1, t_2\}$ and some $\beta \in [0, 1]$ are given, the fuzzy formula

$$(\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)$$

with respect to $i_{r,\beta}$, is joined to $X \xrightarrow{\theta}_V Y$, where $X \xrightarrow{\theta}_V Y$ is a vague functional dependency on $\{A_1, A_2, ..., A_n\}$.

7 Main result

Theorem 1. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $\mathcal{V}^+$ be the closure of $\mathcal{V}$, where $\mathcal{V}$ is some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \xrightarrow{\theta}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$ which is not an element of $\mathcal{V}^+$. Let r^* be a vague relation instance on $R(A_1, A_2, ..., A_n)$ joined to $\mathcal{V}^+$ and $X \xrightarrow{\theta}_V Y$ (in the way described above). Then, there exists a two-element vague relation instance $s \subseteq r^*$ on $R(A_1, A_2, ..., A_n)$, such that s satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to $\mathcal{V}^+$, and violates $X \xrightarrow{\theta}_V Y$.*

Proof. Since r^* is a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ such that r^* satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to $\mathcal{V}^+$, and violates $X \xrightarrow{\theta}_V Y$, it is enough to take $s = r^*$.

This completes the proof. □

Theorem 2. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let C be some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that c is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. The following two conditions are equivalent:*

(a) Any vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency c.

(b) Any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency c.

Proof. $(a) \Rightarrow (b)$ Suppose that the condition (a) is satisfied.

Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, such that r satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C.

Since (a) holds true for any vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, it follows that (a) particularly holds true for the vague relation instance r.

Therefore, r satisfies c, i.e., the condition (b) is satisfied.

$(b) \Rightarrow (a)$ Suppose that the condition (b) is satisfied.

Moreover, suppose that the condition (a) is not satisfied.

It follows that there is a vague relation instance r on $R(A_1, A_2, ..., A_n)$, such that r satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C, and violates c.

Suppose that $c \in C^+$, where C^+ is the closure of C.

Since C^+ is the set of all vague functional dependencies on $\{A_1, A_2, ..., A_n\}$ that can be derived from C by repeated applications of the inference rules VF1-VF4, and the inference rules VF1-VF4 are sound by [10, Th. 4], the fact that r satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C yields that r satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C^+.

Consequently, r satisfies c.

This is a contradiction.

We conclude, $c \notin C^+$.

Since $c \notin C^+$, we may, reasoning as earlier, join some vague relation instance r^* on $R(A_1, A_2, ..., A_n)$ to C^+ and c.

By Theorem 1, there exists a two-element vague relation instance $s \subseteq r^*$ on $R(A_1, A_2, ..., A_n)$, such that s satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C^+, and violates c.

Since $C \subseteq C^+$, it follows that s satisfies $A \xrightarrow{1\theta}_V B$ if $A \xrightarrow{1\theta}_V B$ belongs to C, and violates c.

This contradicts the fact that the condition (b) is satisfied.

Hence, the condition (a) is satisfied.

This completes the proof. $\square$

The following theorem holds true.

Theorem 3. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let C be some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\rightarrow}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. The following two conditions are equivalent:*

(a) Any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency $X \overset{\theta}{\rightarrow}_V Y$.

(b) Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$. Suppose that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$, where C' is the set of fuzzy formulas with respect to $i_{r,\beta}$, joined to the elements of C. Then,

$$i_{r,\beta}((\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)) > \frac{1}{2}.$$

Theorem 4. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let C be some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\rightarrow}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. The following two conditions are equivalent:*

(a) Any vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency $X \overset{\theta}{\rightarrow}_V Y$.

(b) Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$. Suppose that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$, where C' is the set of fuzzy formulas with respect to $i_{r,\beta}$, joined to the elements of C. Then,

$$i_{r,\beta}((\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)) > \frac{1}{2}.$$

Proof. Suppose that c that appears in Theorem 2 is given by $X \overset{\theta}{\rightarrow}_V Y$.

Now, the assertion is an immediate consequence of Theorem 2 and Theorem 3.

This completes the proof. $\square$

8 Applications

Example 1. Let $R(A, B, C, D, E)$ be a relation scheme on domains U_1, U_2, U_3, U_4, U_5, where A is an attribute on the universe of discourse U_1,..., E is an attribute on the universe of discourse U_5. Suppose that the following vague functional dependencies on $\{A, B, C, D, E\}$ hold true.

$$\{A, B\} \overset{\theta_1}{\rightarrow}_V C,$$
$$B \overset{\theta_2}{\rightarrow}_V D,$$
$$\{C, D\} \overset{\theta_3}{\rightarrow}_V E.$$

Then, the vague functional dependency $\{A, B\} \overset{\theta}{\rightarrow}_V E$ on $\{A, B, C, D, E\}$ holds also true. Here, $\theta = \min\{\theta_1, \theta_2, \theta_3\}$.

Proof. I One applies the inference rules VF1-VF7. $\square$

Proof. II Follows immediately from Theorem 4. $\square$

9 Remarks

For analogous results in the case of fuzzy functional (and fuzzy multivalued) dependencies, see, [6], [7], [8], [9].

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