Vague Multivalued Dependencies and Resolution Principle

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Abstract: In this paper we show that the process of deriving of new vague functional or new vague multivalued dependencies from given ones may be automated. In order to achieve our goal, we associate fuzzy formulas to vague functional and vague multivalued dependencies. In this way, to prove that a vague functional or a vague multivalued dependency follows from a set of vague functional and vague multivalued dependencies becomes the same as to prove that the corresponding fuzzy formula is a logical consequence of the corresponding set of fuzzy formulas.

Key–Words: Vague functional dependencies, vague multivalued dependencies, fuzzy formulas, valuations, resolution principle

1 Introduction

In [12], we proved that the implication:

if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, then r satisfies the dependency $X \overset{\theta}{\rightarrow}_V Y$,

is equivalent to the implication:

if $i_{r',\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$, then $i_{r',\beta}((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)) > \frac{1}{2}$.

Here, $R(A_1, A_2, ..., A_n)$ is a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$, C is a set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$, $X \overset{\theta}{\rightarrow}_V Y$ is a vague functional dependency on $\{A_1, A_2, ..., A_n\}$, C' is the set of fuzzy formulas (with respect to valuation $i_{r',\beta}$) joined to C, and $(\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)$ is the fuzzy formula (with respect to $i_{r',\beta}$) joined to $X \overset{\theta}{\rightarrow}_V Y$.

In the second implication, r' is a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$ is a number.

As it follows from [12], the equivalence given above means that the knowledge that $(\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)$ is valid if $\mathcal{K}$ is valid, $\mathcal{K} \in C'$, is enough to know when $X \overset{\theta}{\rightarrow}_V Y$ follows from C.

In order to prove that $(\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)$ is valid when all $\mathcal{K} \in C'$ are valid, however, one usually uses the resolution principle. There, as it is known, our steps can be fully automated.

The main purpose of this paper is to generalize the result given by the equivalence stated above, in order to include vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$ as well.

In particular, C will denote a set of vague functional and vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$, and $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$ will denote a vague functional resp. a vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$.

2 Preliminaries

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$.

Suppose that $V(U_i)$ is the family of all vagues sets in U_i, $i \in I$.

Here, we say that V_i is a vague set in U_i, if

$$V_i = \{\langle u, [t_{V_i}(u), 1 - f_{V_i}(u)]\rangle \ : \ u \in U_i\},$$

where $t_{V_i} : U_i \to [0, 1]$, $f_{V_i} : U_i \to [0, 1]$ are functions such that $t_{V_i}(u) + f_{V_i}(u) \leq 1$ for all $u \in U_i$.

We also say that $[t_{V_i}(u), 1 - f_{V_i}(u)] \subseteq [0, 1]$ is the vague value joined to $u \in U_i$.

A vague relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $V(U_1) \times V(U_2) \times ... \times V(U_n)$.

A tuple t of r is denoted by

$$(t[A_1], t[A_2], ..., t[A_n]).$$

Here, we consider the vague set $t[A_i]$ as the value of the attribute A_i on t.

Let $Vag(U_i)$ be the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping $SE_i : Vag(U_i) \times Vag(U_i) \to [0,1]$, such that $SE_i(x,x) = 1$, $SE_i(x,y) = SE_i(y,x)$, and $SE_i(x,z) \geq \max_{y \in Vag(U_i)} (\min(SE_i(x,y), SE_i(y,z)))$ for all $x, y, z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let

$$\begin{aligned} A_i &= \{\langle u, [t_{A_i}(u), 1 - f_{A_i}(u)]\rangle \ : \ u \in U_i\} \\ &= \{a_u^i \ : \ u \in U_i\}, \\ B_i &= \{\langle u, [t_{B_i}(u), 1 - f_{B_i}(u)]\rangle \ : \ u \in U_i\} \\ &= \{b_u^i \ : \ u \in U_i\} \end{aligned}$$

be two vague sets in U_i.

The similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$\begin{aligned} &SE(A_i, B_i) \\ &= \min\Big\{ \min_{a_u^i \in A_i} \Big\{ \max_{b_u^i \in B_i} \Big\{ SE_i\Big([t_{A_i}(u), 1 - f_{A_i}(u)], \\ &[t_{B_i}(u), 1 - f_{B_i}(u)] \Big) \Big\} \Big\}, \\ &\min_{b_u^i \in B_i} \Big\{ \max_{a_u^i \in A_i} \Big\{ SE_i\Big([t_{B_i}(u), 1 - f_{B_i}(u)], \\ &[t_{A_i}(u), 1 - f_{A_i}(u)] \Big) \Big\} \Big\} \Big\}. \end{aligned}$$

Now, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between tuples t_1 and t_2 on the attribute set X is defined by

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

For various definitions of similarity measures, see, [16], [5], [4], [14] and [15].

3 Vague functional and vague multivalued dependencies

Recently, in [10] and [11], we introduced new definitions of vague functional and vague multivalued dependencies.

If X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta \in [0,1]$ is a number, then, the vague relation instance r on $R(A_1, A_2, ..., A_n)$ is said to satisfy the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r,

$$SE_Y(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}.$$

Vague relation instance r is said to satisfy the vague multivalued dependency $X \overset{\theta}{\to\to}_V Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min\{\theta, SE_X(t_1, t_2)\}, \\ SE_Y(t_3, t_1) &\geq \min\{\theta, SE_X(t_1, t_2)\}, \\ &SE_{\{A_1, A_2, ..., A_n\} \setminus (X \cup Y)}(t_3, t_2) \\ &\geq \min\{\theta, SE_X(t_1, t_2)\}. \end{aligned}$$

We write $X \to_V Y$ resp. $X \to\to_V Y$ instead of $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$ if $\theta = 1$.

As in [12], θ is called the linguistic strength of the vague functional (vague multivalued) dependency $X \overset{\theta}{\to}_V Y$ ($X \overset{\theta}{\to\to}_V Y$).

Note that the authors in [24] first introduced the formal definitions of fuzzy functional and fuzzy multivalued dependencies which are given on the basis of conformance values.

For various definitions of vague functional and vague multivalued dependencies, see, [16], [19], [26] and [20].

4 Inference rules

The following list contains the inference rules for vague functional and vague multivalued dependencies (see, [10], [11]).

VF1 Inclusive rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \overset{\theta_2}{\longrightarrow}_V Y$ holds.

VF2 Reflexive rule for VFDs: If $X \supseteq Y$, then $X \to_V Y$ holds.

VF3 Augmentation rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, then $X \cup Z \overset{\theta}{\to}_V Y \cup Z$ holds.

VF4 Transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF5 Union rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $X \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Y \cup Z$ holds also true.

VF6 Pseudo-transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $W \cup Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $W \cup X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF7 Decomposition rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, and $Z \subseteq Y$, then $X \overset{\theta}{\to}_V Z$ also holds.

VM1 Inclusive rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \to\overset{\theta_2}{\longrightarrow}_V Y$ holds.

VM2 Complementation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V Q$ holds, where $Q = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

VM3 Augmentation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, and $W \supseteq Z$, then $W \cup X \to\overset{\theta}{\to}_V Y \cup Z$ also holds.

VM4 Transitivity rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $Y \to\overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus Y$ holds true.

VM5 Replication rule: If $X \overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V Y$ holds.

VM6 Coalescence rule for VFDs and VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y, we have that $W \overset{\theta_2}{\longrightarrow}_V Z$ holds true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ also holds true.

VM7 Union rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $X \to\overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \cup Z$ holds true.

VM8 Pseudo-transitivity rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $W \cup Y \to\overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $W \cup X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus (W \cup Y)$ holds also true.

VM9 Decomposition rule for VMVDs: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $X \to\overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \cap Z$, $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \setminus Z$, and $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus Y$ hold also true.

VM10 Mixed pseudo-transitivity rule: If $X \to\overset{\theta_1}{\longrightarrow}_V Y$ and $X \cup Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z \setminus Y$ holds true.

By Theorems 4 and 5 in [10], and Theorems 2 and 3 in [11], the inference rules VF1-VF7 and VM1-VM10 are sound.

Hence, it follows that r satisfies $X \to\overset{\theta}{\to}_V Y$ if r satisfies $X \overset{\theta}{\to}_V Y$ (see, VM5), where r is a vague relation instance on $R(A_1, A_2, ..., A_n)$.

5 Fuzzy implications

A mapping $I : [0,1]^2 \to [0,1]$ is a fuzzy implication if $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.

The most important classes of fuzzy implications are: S-implications, R-implications and QL-implications (strong, residual, quantum logic implications, respectively).

For precise definitions and description of S- , R- , QL-implications, as well as for the definitions of various additional fuzzy implications, see, [23] and [3].

In this paper (as in [12]), we use the following operators:

$$\begin{aligned} T_M(x,y) &= \min\{x,y\}, \\ S_M(x,y) &= \max\{x,y\}, \\ I_L(x,y) &= \min\{1-x+y,1\}, \end{aligned} \tag{1}$$

where T_M is the minimum t-norm (t-norms are usually applied to model fuzzy conjunctions), S_M is the maximum t-co-norm (fuzzy disjunctions are often modeled by t-co-norms), and I_L is the Lukasiewicz fuzzy implication.

The Lukasiewics fuzzy implication is an S- , an R- and a QL-fuzzy implication at the same time (see, [23], [3]).

Some of the works that deal with S- ,R- and QL-implications are the following: [1], [2], [17], [25], [22], [18], [21].

6 Valuations

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Let $r = \{t_1, t_2\}$ be a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$ be a number.

Suppose that the similarity measures SE_i, SE and SE_X are given as above.

Let $A_k \in \{A_1, A_2, ..., A_n\}$.

We calculate the similarity measure $SE(t_1[A_k], t_2[A_k])$ between the vague sets $t_1[A_k]$ and $t_2[A_k]$.

We check whether or not $SE(t_1[A_k], t_2[A_k]) \geq \beta$.

If $SE(t_1[A_k], t_2[A_k]) \geq \beta$, we put $i_{r,\beta}(A_k)$ to be some value in the interval $\left(\frac{1}{2}, 1\right]$.

Otherwise, if $SE(t_1[A_k], t_2[A_k]) < \beta$, we put $i_{r,\beta}(A_k)$ to be some value in the interval $\left[0, \frac{1}{2}\right]$.

We say that $i_{r,\beta}$ is a valuation joined to r and β.

Thus, $i_{r,\beta}$ is a function defined on $\{A_1, A_2, ..., A_n\}$ with values in $[0, 1]$.

More precisely, $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0, 1]$,

$$i_{r,\beta}(A_k) > \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) \geq \beta,$$
$$i_{r,\beta}(A_k) \leq \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) < \beta,$$

$k \in \{1, 2, ..., n\}$.

Note that the fact that $i_{r,\beta}(A_k) \in [0, 1]$ for $k \in \{1, 2, ..., n\}$ yields that the attributes A_k, $k \in \{1, 2, ..., n\}$ are actually fuzzy formulas now (with respect to $i_{r,\beta}$).

Having in mind (1), we define

$$i_{r,\beta}(A \wedge B) = \min\{i_{r,\beta}(A), i_{r,\beta}(B)\},$$
$$i_{r,\beta}(A \vee B) = \max\{i_{r,\beta}(A), i_{r,\beta}(B)\},$$
$$i_{r,\beta}(A \Rightarrow B) = \min\{1 - i_{r,\beta}(A) + i_{r,\beta}(B), 1\}$$

for A, $B \in \{A_1, A_2, ..., A_n\}$.

Since T_M, S_M and I_L are functions defined on $[0, 1]^2$ with values in $[0, 1]$, it follows that $A \wedge B$, $A \vee B$ and $A \Rightarrow B$, A, $B \in \{A_1, A_2, ..., A_n\}$, are also fuzzy formulas with respect to $i_{r,\beta}$.

Consequently, $((A \wedge B) \Rightarrow C) \vee D$, where A, B, C, $D \in \{A_1, A_2, ..., A_n\}$, for example, is a fuzzy formula with respect to $i_{r,\beta}$.

Namely, this follows from now from the fact that

$$\begin{aligned}
& i_{r,\beta}(((A \wedge B) \Rightarrow C) \vee D) \\
&= \max\{i_{r,\beta}((A \wedge B) \Rightarrow C), i_{r,\beta}(D)\} \\
&= \max\Big\{\min\{1 - i_{r,\beta}(A \wedge B) + i_{r,\beta}(C), 1\}, \\
&\quad i_{r,\beta}(D)\Big\} \\
&= \max\Big\{\min\Big\{1 - \min\{i_{r,\beta}(A), i_{r,\beta}(B)\} + \\
&\quad i_{r,\beta}(C), 1\Big\}, i_{r,\beta}(D)\Big\}.
\end{aligned}$$

In this paper we are interested in the following fuzzy formulas with respect to $i_{r,\beta}$:

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B),$$
$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C)),$$

where X and Y are subsets of $\{A_1, A_2, ..., A_n\}$, and $Z \subseteq \{A_1, A_2, ..., A_n\}$ is given by $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$, where X and Y are given.

Through the rest of the paper we shall assume that each time some $r = \{t_1, t_2\}$ and some $\beta \in [0, 1]$ are given, the fuzzy formula

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$$

resp.

$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$$

with respect to $i_{r,\beta}$ is joined to $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$, where $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$ is a vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$, and $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

7 Preliminary results

The following Theorem is derived in [13]. It will be used in the sequel.

Theorem 1. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $(\mathcal{V}, \mathcal{M})^+$ be the closure of $\mathcal{V} \cup \mathcal{M}$, where $\mathcal{V}$ resp. $\mathcal{M}$ is some set of vague functional resp. vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$ is some vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$ which is not and element of $(\mathcal{V}, \mathcal{M})^+$. Let r^* be a vague relation instance on $R(A_1, A_2, ..., A_n)$ joined to $(\mathcal{V}, \mathcal{M})^+$ and $X \overset{\theta}{\to}_V Y$*

resp. $X \overset{\theta}{\to\to}_V Y$ *(in the way described in [13]). Then, there exists a two-element vague relation instance* $s \subseteq r^*$ *on* $R(A_1, A_2, ..., A_n)$*, such that* s *satisfies* $A \overset{1\theta}{\to}_V B$ *resp.* $A \overset{1\theta}{\to\to}_V B$ *if* $A \overset{1\theta}{\to}_V B$ *resp.* $A \overset{1\theta}{\to\to}_V B$ *belongs to* $(\mathcal{V}, \mathcal{M})^+$*, and violates* $X \overset{\theta}{\to}_V Y$ *resp.* $X \overset{\theta}{\to\to}_V Y$.

8 Auxiliary results

Theorem 2. *Let* $R(A_1, A_2, ..., A_n)$ *be a relation scheme on domains* U_1, U_2,..., U_n, *where* A_i *is an attribute on the universe of discourse* U_i, $i \in I$. *Let* C *be some set of vague functional and vague multivalued dependencies on* $\{A_1, A_2, ..., A_n\}$. *Suppose that* c *is some vague functional or vague multivalued dependency on* $\{A_1, A_2, ..., A_n\}$. *The following two conditions are equivalent:*

(a) *Any vague relation instance on* $R(A_1, A_2, ..., A_n)$ *which satisfies all dependencies in* C*, satisfies the dependency* c.

(b) *Any two-element vague relation instance on* $R(A_1, A_2, ..., A_n)$ *which satisfies all dependencies in* C*, satisfies the dependency* c.

Proof. $(a) \Rightarrow (b)$ Suppose that (a) holds true.

Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C.

Since (a) is valid for any vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, it follows that it is also valid for the instance r.

Hence, r satisfies c, i.e., (b) holds true.

$(b) \Rightarrow (a)$ Suppose that (b) holds true.

Moreover, suppose that (a) does not hold true.

It follows that there is some vague relation instance r on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, and violates c.

Suppose that $c \in C^+$, where C^+ is the closure of C.

Since C^+ is the set of all vague functional and vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$ that can be derived from C by repeated applications of the inference rules VF1-VF4 and VM1-VM6, and the inference rules VF1-VF4 and VM1-VM6 are sound by [10, Th. 4] and [11, Th. 2], the fact that r satisfies all dependencies in C implies that r satisfies all dependencies in C^+.

Consequently, r satisfies c.

This is a contradiction.

We conclude, $c \notin C^+$.

Let r^* be a vague relation instance on $R(A_1, A_2, ..., A_n)$ joined to C^+ and c (in the way described in [13]).

By Theorem 1, there exists a two-element vague relation instance $s \subseteq r^*$ on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C^+, and violates c.

Since $C \subseteq C^+$, it follows that s satisfies all dependencies in C, and violates c.

This contradicts the fact that the assumption (b) holds true.

Hence, (a) holds true.

This completes the proof. □

The following theorem holds true.

Theorem 3. *Let* $R(A_1, A_2, ..., A_n)$ *be a relation scheme on domains* U_1, U_2,..., U_n, *where* A_i *is an attribute on the universe of discourse* U_i, $i \in I$. *Let* C *be some set of vague functional and vague multivalued dependencies on* $\{A_1, A_2, ..., A_n\}$. *Suppose that* $X \overset{\theta}{\to}_V Y$ *resp.* $X \overset{\theta}{\to\to}_V Y$ *is some vague functional resp. vague multivalued dependency on* $\{A_1, A_2, ..., A_n\}$. *The following two conditions are equivalent:*

(a) *Any two-element vague relation instance on* $R(A_1, A_2, ..., A_n)$ *which satisfies all dependencies in* C*, satisfies the dependency* $X \overset{\theta}{\to}_V Y$ *resp.* $X \overset{\theta}{\to\to}_V Y$.

(b) *Let* r *be any two-element vague relation instance on* $R(A_1, A_2, ..., A_n)$*, and* $\beta \in [0, 1]$. *Suppose that* $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ *for all* $\mathcal{K} \in C'$*, where* C' *is the set of fuzzy formulas with respect to* $i_{r,\beta}$*, joined to the elements of* C. *Then,*

$$i_{r,\beta}\left((\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)\right) > \frac{1}{2}$$

resp.

$$i_{r,\beta}\left((\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))\right) > \frac{1}{2},$$

where $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

9 Main result

Theorem 4. *Let* $R(A_1, A_2, ..., A_n)$ *be a relation scheme on domains* U_1, U_2,..., U_n, *where* A_i *is an attribute on the universe of discourse* U_i, $i \in I$. *Let* C *be some set of vague functional and vague multivalued dependencies on* $\{A_1, A_2, ..., A_n\}$. *Suppose that* $X \overset{\theta}{\to}_V Y$ *resp.* $X \overset{\theta}{\to\to}_V Y$ *is some vague functional resp. vague multivalued dependency on* $\{A_1, A_2, ..., A_n\}$. *The following two conditions are equivalent:*

(a) *Any vague relation instance on* $R\left(A_1, A_2, ..., A_n\right)$ *which satisfies all dependencies in* C*, satisfies the dependency* $X \overset{\theta}{\to}_V Y$ *resp.* $X \overset{\theta}{\to\to}_V Y$.

(b) *Let* r *be any two-element vague relation instance on* $R\left(A_1, A_2, ..., A_n\right)$*, and* $\beta \in [0,1]$*. Suppose that* $i_{r,\beta}\left(\mathcal{K}\right) > \frac{1}{2}$ *for all* $\mathcal{K} \in C^{'}$*, where* $C^{'}$ *is the set of fuzzy formulas with respect to* $i_{r,\beta}$*, joined to the elements of* C*. Then,*

$$i_{r,\beta}\left(\left(\wedge_{A \in X} A\right) \Rightarrow \left(\wedge_{B \in Y} B\right)\right) > \frac{1}{2}$$

resp.

$$i_{r,\beta}\left(\left(\wedge_{A \in X} A\right) \Rightarrow \left(\left(\wedge_{B \in Y} B\right) \vee \left(\wedge_{C \in Z} C\right)\right)\right) > \frac{1}{2},$$

where $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

Proof. Suppose that c that appears in Theorem 2 is given by $X \overset{\theta}{\to}_V Y$ resp. $X \overset{\theta}{\to\to}_V Y$.

Now, the assertion of the theorem is an immediate consequence of Theorem 2 and Theorem 3.

This completes the proof. □

10 Applications

Example 1. Let $R\left(A, B, ..., K\right)$ be a relation scheme on domains U_1, U_2,... U_{11}, where A is an attribute on the universe of discourse U_1, B is an attribute on the universe of discourse U_2,..., K is an attribute on the universe of discourse U_{11}. Suppose that the following vague functional and vague multivalued dependencies on $\{A, B, ..., K\}$ hold true:

$$\{A, B, C, D\} \overset{\theta_1}{\to\to}_V \{B, D, E, F, I, J\},$$
$$\{A, B, C, D\} \overset{\theta_2}{\to\to}_V \{C, D, F, G, H, I\},$$
$$\{B, C\} \overset{\theta_3}{\to}_V \{E, K\},$$
$$\{B, D, E, J\} \overset{\theta_4}{\to}_V \{G, E\}.$$

Then, the vague multivalued dependency

$$\{A, B, C, D\} \overset{\theta}{\to\to}_V \{E, G, K\}$$

on $\{A, B, ..., K\}$ holds also true where, $\theta = \min\{\theta_1, \theta_2, \theta_3, \theta_4\}$.

Proof. I One applies the inference rules VF1-VF7, VM1-VM10. □

Proof. II Follows from Theorem 4. □

11 Remarks

For analogous results in the case of fuzzy functional and fuzzy multivalued dependencies, we refer to [6], [7], [8], [9].

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