On the Length Spectrum for Compact, Odd-dimensional, Real Hyperbolic Spaces

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Abstract: We derive a prime geodesic theorem for compact, odd-dimensional, real hyperbolic spaces. The obtained result corresponds to the best known result obtained in the compact, even-dimensional case, as well as to the best known result obtained in the case of non-compact, real hyperbolic manifolds with cusps. The result derived in this paper follows from the fact that the prime geodesic theorem gives a growth asymptotic for the number of closed geodesics counted by their lengths, and the fact that free homotopy classes of closed paths on compact locally symmetric Riemannian manifold with negative sectional curvature are in natural one-to-one correspondence with the set of conjugacy classes of the corresponding discrete, co-compact, torsion-free group. The current article is dedicated to quotients of the real hyperbolic space.

Key–Words: Length spectrum, compact, odd-dimensional, real hyperbolic spaces, counting functions.

1 Introduction

In [3], the authors improved the error term in DeGeorge's prime geodesic theorem [7], for compact, locally symmetric spaces of real rank one.

In this paper we pay our particular attention to compact, odd-dimensional, real hyperbolic spaces.

While the main result in [3] is achieved by application of the method analogous to the method developed by Randol [29], the main result of this paper will follow from our recent research on the logarithmic derivative of the corresponding zeta functions of Selberg and Ruelle [17], and the techniques applied by Park [25], Fried [10], and Hejhal [18], [19].

For the corresponding result in the compact, even-dimensional case, we refer to [16].

Note that Gangolli [14] and DeGeorge [7] proved the same result independently. Moreover, an analogous result was proved by Gangolli-Warner [15], when locally symmetric space has a finite volume.

In [18] and [19], Hejhal extensively studied the Selberg zeta function over a hyperbolic Riemann surface. There, he derived a prime geodesic theorem with error terms (see also, [20], [21] and [29]).

There have been many works, for instance the work of Iwaniec [23] and Luo-Sarnak [24], to obtain the optimal size of the error term for a specific arithmetic discrete subgroup $\Gamma \subset PSL\left(2,\mathbb{R}\right)$.

We also refer to the work of Parry-Pollicott [26], where they used the Ruelle zeta function for an axiom A flow to derive a prime geodesic theorem.

In [25], Park proved Theorem 1.1 for the Ruelle zeta function twisted by a special unitary representation χ of Γ. According to Park, this can be used for prime geodesic theorem in a fixed homology class, which would be a refinement of [1], [9], [28] for real hyperbolic manifolds with cusps.

The structure of the paper is as follows. Section 2 is devoted to harmonic analysis on compact symmetric spaces. We introduce symmetric and locally symmetric spaces of rank 1, give the corresponding restriction map, introduce the real hyperbolic space, and define Casimir and Dirac operators. We adopt some necessary notation related to the admissible lifts and the trace formula (the hyperbolic contribution). Finally, we introduce the zeta functions and the geodesic flow, and give the corresponding singularity pattern. In Section 3 we consider the Selberg zeta functions and the corresponding differential forms on d-dimensional real hyperbolic manifolds. We provide an adopted singularity pattern expressed in terms of the corresponding form Laplacians. Section 4 is devoted to preliminary results. There, we assemble those theorems we will need. In Section 5 we introduce various counting functions and give the Weyl asymptotic law in the form sufficient to derive desired results. In Section 6 we state and prove the main result of the paper. Finally, Section 7 is devoted to concluding remarks.

2 Harmonic analysis on compact symmetric spaces

We introduce the notation following [5].

Let Y be a compact locally symmetric Riemannian manifold with negative sectional curvature.

Denote by X the universal covering of Y.

Since X is a Riemannian symmetric space of rank one, it is either a real, or a complex, or a quaternionic hyperbolic space, or the hyperbolic Cayley plane.

As it is well known, we may write $Y = \Gamma \setminus G \,/\, K$, and $X = G \,/\, K$, where G is a connected semi-simple Lie group of real rank one, K is a maximal compact subgroup of G, and Γ is a discrete, co-compact torsion-free subgroup of G.

We require G to be linear in order to have the possibility of complexification.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and M the centralizer of $\mathfrak{a}$ in K with Lie algebra $\mathfrak{m}$.

We normalize the $\mathrm{Ad}\,(G)$-invariant inner product $(.,.)$ on $\mathfrak{g}$ to restrict to the metric on $\mathfrak{p}$.

Let $\Phi\,(\mathfrak{g}, \mathfrak{a})$ be the root system determined by the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$.

Let $W = W\,(\mathfrak{g}, \mathfrak{a})$ be its Weyl group.

Fix a system of positive roots $\Phi^+\,(\mathfrak{g}, \mathfrak{a}) \subset \Phi\,(\mathfrak{g}, \mathfrak{a})$, and let $\mathfrak{n} = \sum\limits_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a})} \mathfrak{n}_\alpha$ be the sum of the root spaces corresponding to the elements of $\Phi^+\,(\mathfrak{g}, \mathfrak{a})$.

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ corresponds to the Iwasawa decomposition $G = KAN$ of the group G.

Define $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a})} \dim\,(\mathfrak{n}_\alpha)\,\alpha$, and the positive Weyl chamber $\mathfrak{a}^+$ (as the half line in $\mathfrak{a}$ on which the positive roots take positive values).

Let $A^+ = \exp\,(\mathfrak{a}^+) \subset A$.

Let $i^* : R\,(K) \to R\,(M)$ be the restriction map induced by the embedding $i : M \hookrightarrow K$, where $R\,(K)$ and $R\,(M)$ are the representation rings ever $\mathbb{Z}$ of K and M, respectively.

Suppose that $\dim\,(Y) = d \geq 3$.

It follows that X is the real hyperbolic space $H\mathbb{R}^d$, where $K = Spin\,(d)$, $M = Spin\,(d-1)$ or $K = SO\,(d)$, $M = SO\,(d-1)$.

Moreover, the fact that d is odd, yields that we have to distinguish between two cases:

Case (a): $\sigma \in \hat{M}$ is invariant under the action of the Weyl group W.

Case (b): $\sigma \in \hat{M}$ is not invariant under the action of the Weyl group.

Here, $\sigma \in \hat{M}$ is a representation of M.

First, we consider the case (a).

By Proposition 1.1 and Proposition 1.2 in [5, pp. 20-23], there exists an element $\gamma \in R\,(K)$ such that $i^*\,(\gamma) = \sigma$.

Note that γ is uniquely determined by this condition.

To γ, we join $\mathbb{Z}_2$-graded homogeneous vector bundle $V\,(\gamma) = V\,(\gamma)^+ \oplus V\,(\gamma)^-$ as follows.

We represent γ as

$$\gamma = \bigoplus a_i \gamma_i,$$

where $a_i \in \mathbb{Z}$, $\gamma_i \in \hat{K}$, and put

$$V_\gamma^\pm = \bigoplus_{\mathrm{sign}(a_i)=\pm 1} \bigoplus_{m=1}^{|a_i|} V_{\gamma_i},$$

where V_{γ_i} is the representation space of γ_i.

Then, we define

$$V\,(\gamma)^\pm = G \times_K V_\gamma^\pm.$$

If (χ, V_χ) is a finite dimensional unitary representation of Γ, then we define

$$V_{Y,\chi}\,(\gamma) = \Gamma \backslash \left(V_\gamma \otimes V\,(\gamma)\right).$$

We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$, and a system of positive roots $\Phi^+\,(\mathfrak{m}_\mathbb{C}, \mathfrak{t})$.

Let

$$c\,(\sigma) = |\rho|^2 + |\rho_\mathfrak{m}|^2 - |\mu_\sigma + \rho_\mathfrak{m}|^2,$$

where the norms are induced by the complex bilinear extension to $\mathfrak{g}_\mathbb{C}$ of the inner product $(.,.)$, μ_σ is the highest weight of σ, and $\rho_\mathfrak{m} = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+(\mathfrak{m}_\mathbb{C},\mathfrak{t})} \alpha$.

The inner product $(.,.)$ also fixes the Casimir element Ω of the complex universal enveloping algebra $\mathcal{U}\,(\mathfrak{g})$.

We define the operators

$$A\,(\gamma,\sigma)^2 : C^\infty\,(X, V\,(\gamma)) \to C^\infty\,(X, V\,(\gamma)),$$
$$A_{Y,\chi}\,(\gamma,\sigma)^2 : C^\infty\,(Y, V_{Y,\chi}\,(\gamma)) \to C^\infty\,(Y, V_{Y,\chi}\,(\gamma))$$

by

$$A\,(\gamma,\sigma)^2 = -\Omega - c\,(\sigma),$$
$$A_{Y,\chi}\,(\gamma,\sigma)^2 = -\Omega - c\,(\sigma).$$

Second, we consider the case (b).

By Proposition 1.1 in [5, p. 20], there is a unique element $\gamma^{'} \in \hat{Spin}(d)$, and a splitting $s \otimes \gamma^{'} = \gamma^{+} \oplus \gamma^{-}$, where s is the spin representation of $Spin(d)$ and $\gamma^{\pm}$ are representations of K, such that for the non-trivial element $w \in W$

$$\sigma - w\sigma = \operatorname{sign}(\nu_k)\left(s^{+} - s^{-}\right) i^{*}\left(\gamma^{'}\right),$$
$$\sigma + w\sigma = i^{*}\left(\gamma^{+} - \gamma^{-}\right),$$

where ν_k is the last coordinate of the highest weight of σ, and $s^{\pm}$ are the half-spin representations of $Spin(d-1)$.

We put $\gamma = \gamma^{+} - \gamma^{-} \in R(K)$ and $\gamma^{\mathbf{s}} = \gamma^{+} + \gamma^{-} \in R(K)$.

Now, we define the bundles $V(\gamma)$, $V_{Y,\chi}(\gamma)$, $V(\gamma^{\mathbf{s}})$, $V_{Y,\chi}(\gamma^{\mathbf{s}})$, and the operators $A(\gamma,\sigma)$, $A_{Y,\chi}(\gamma,\sigma)$, $A(\gamma^{\mathbf{s}},\sigma)$, $A_{Y,\chi}(\gamma^{\mathbf{s}},\sigma)$ in the same way as in the case (a).

Note that $V(\gamma^{\mathbf{s}})$, $V_{Y,\chi}(\gamma^{\mathbf{s}})$ are Clifford bundles. Hence, they carry Dirac operators $D(\sigma)$, $D_{Y,\chi}(\sigma)$.

In order to make these Dirac operators unique, we proceed in exactly the same way as in [5, pp. 39-30].

We obtain, $D(\sigma)^2 = A(\gamma^{\mathbf{s}},\sigma)^2$, and since the Dirac operators are self-adjoint, we have $A(\gamma^{\mathbf{s}},\sigma) = |D(\sigma)|$ and $A_{Y,\chi}(\gamma^{\mathbf{s}},\sigma) = |D_{Y,\chi}(\sigma)|$.

Let $E_A(.)$ be the family of spectral projections of a normal operator A. We define for $s \in \mathbb{C}$

$$m_{\chi}(s,\gamma,\sigma) = \operatorname{Tr} E_{A_{Y,\chi}(\gamma,\sigma)}(\{s\}),$$
$$m_{\chi}^{\mathbf{s}}(s,\sigma) = \operatorname{Tr}\Big(E_{D_{Y,\chi}(\sigma)}(\{s\}) - E_{D_{Y,\chi}(\sigma)}(\{-s\})\Big).$$

Since d is odd, these multiplicities do not depend on the choice of the representation $\gamma = \bigoplus a_i\gamma_i$, $a_i \in \mathbb{Z}$, $\gamma_i \in \hat{K}$ given above.

The root system $\Phi^{+}(\mathfrak{g},\mathfrak{a})$ is of the form $\Phi^{+}(\mathfrak{g},\mathfrak{a}) = \{\alpha\}$ or $\Phi^{+}(\mathfrak{g},\mathfrak{a}) = \left\{\frac{\alpha}{2},\alpha\right\}$, where α is the long root.

We set $T_1 = |\alpha|$, where α is the long root.

Since Γ is co-compact and torsion-free, there are only two types of conjugacy classes, the class of the identity $e \in \Gamma$ and classes of hyperbolic elements.

For $g \in \Gamma$ let $n_{\Gamma}(g) = \#(\Gamma_g/\langle g\rangle)$, where Γ_g is the centralizer of g in Γ, and $\langle g\rangle$ is the group generated by g.

By $C\Gamma$ we denote the set of conjugacy classes of Γ.

Let $g \in G$ be hyperbolic.

It is known (see, e.g., [13], [14], [15]) that g is conjugated to some element $a_g m_g \in A^{+}M$.

Thus, $g = \theta_g a_g m_g \theta_g^{-1}$ for some θ_g.

We define $l(g) = l\left(\theta_g a_g m_g \theta_g^{-1}\right) = |\log(a_g)|$.

Note that for $g \in \Gamma$, the number $l(g)$ is actually the length of the closed geodesic on Y defined by g.

Suppose that (σ, V_{σ}) and (χ, V_{χ}) are some finite-dimensional unitary representations of M and Γ, respectively.

For $s \in \mathbb{C}$, $\operatorname{Re}(s) > 2\rho$, the Ruelle zeta function $Z_{R,\chi}(s,\sigma)$ is given by

$$Z_{R,\chi}(s,\sigma) = \prod_{1\neq[g]\in C\Gamma} \det\Big(1 - (\sigma(m_g)\otimes\chi(g))\, e^{-sl(g)}\Big).$$

For $s \in \mathbb{C}$, $\operatorname{Re}(s) > \rho$, the Selberg zeta function $Z_{S,\chi}(s,\sigma)$ is given by

$$Z_{S,\chi}(s,\sigma) = \prod_{\substack{1\neq[g]\in C\Gamma \\ primitive}} \prod_{k=0}^{+\infty} 1 \times \det\Big(1 - \Big(\sigma(m_g)\otimes\chi(g)\otimes S^{k}\left(\operatorname{Ad}(m_g a_g)_{\bar{\mathfrak{n}}}\right)\Big) e^{-(s+\rho)l(g)}\Big),$$

where S^k denotes the k-th symmetric power of an endomorphism, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, θ is the Cartan involution of $\mathfrak{g}$, and $[g] \in C\Gamma$ is called primitive if $l(g)$ is the smallest time such that $\varphi(l(g), y) = y$, where

$$\varphi : \mathbb{R} \times (\Gamma\backslash G/M) \to \Gamma\backslash G/M,$$

$\varphi(t, \Gamma gM) = \Gamma g e^{-tH} M$ (H is the unit vector in $\mathfrak{a}^{+}$) is the geodesic flow determined by the metric of Y.

If $[g] \in C\Gamma$ is primitive, then $n_{\Gamma}(g) = 1$.

In the case (b) we also define

$$S_{\chi}(s,\sigma) = Z_{S,\chi}(s,\sigma)\, Z_{S,\chi}(s,w\sigma)$$

and the super zeta function

$$S_{\chi}^{\mathbf{s}}(s,\sigma) = \frac{Z_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,w\sigma)},$$

where $w \in W$ is the non-trivial element.

Let $\mathfrak{n}_{\mathbb{C}}$ be the complexification of $\mathfrak{n}$.

For $p \geq 0$, we consider $\wedge^{p}\mathfrak{n}_{\mathbb{C}}$ as a representation of MA.

For $\lambda \in \mathbb{C}$, let $\mathbb{C}_{\lambda}$ denote one-dimensional representation of A given by $a \to a^{\lambda}$.

There are sets $I_p = \left\{(\tau,\lambda) : \tau \in \hat{M}, \lambda \in \mathbb{R}\right\}$ such that $\wedge^{p}\mathfrak{n}_{\mathbb{C}}$ decomposes with respect to MA as $\wedge^{p}\mathfrak{n}_{\mathbb{C}} = \sum_{(\tau,\lambda)\in I_p} V_{\tau}\otimes\mathbb{C}_{\lambda}$, where V_{τ} is the space of the representation τ.

The Ruelle zeta function has the representation (see, e.g., [10], [11], [12])

$$Z_{R,\chi}(s,\sigma) = \prod_{p=0}^{d-1}\left(\prod_{(\tau,\lambda)\in I_p} Z_{S,\chi}(s+\rho-\lambda,\tau\otimes\sigma)\right)^{(-1)^p}.$$

By [5, p. 113, Th. 3.15], the zeta functions $Z_{S,\chi}(s,\sigma)$, $S_\chi(s,\sigma)$ and $S^{\mathbf{s}}_\chi(s,\sigma)$ have meromorphic continuations to all of $\mathbb{C}$.

In particular, the singularities of $Z_{S,\chi}(s,\sigma)$ (case (a)) and of $S_\chi(s,\sigma)$ (case (b)) are at $\pm\,\mathrm{i}\,s$ of order $m_\chi(s,\gamma,\sigma)$ if $s\neq 0$ is an eigenvalue of $A_{Y,\chi}(\gamma,\sigma)$, at $s=0$ of order $2m_\chi(0,\gamma,\sigma)$ if 0 is an eigenvalue of $A_{Y,\chi}(\gamma,\sigma)$.

In the case (b) the singularities of $S^{\mathbf{s}}_\chi(s,\sigma)$ are at $\mathrm{i}\,s$ and have order $m^{\mathbf{s}}_\chi(s,\sigma)$ if $s\in\mathbb{R}$ is an eigenvalue of $D_{Y,\chi}(\sigma)$.

Furthermore, in the case (b), the zeta function $Z_{S,\chi}(s,\sigma)$ has singularities at $\mathrm{i}\,s$, $\pm s\in\operatorname{spec}(A_{Y,\chi}(\gamma^{\mathbf{s}},\sigma))$ of order

$$\frac{1}{2}\left(m_\chi(|s|,\gamma,\sigma)+m^{\mathbf{s}}_\chi(s,\sigma)\right)$$

if $s\neq 0$ and $m_\chi(0,\gamma,\sigma)$ if $s=0$.

3 Real hyperbolic spaces

As noted in the previous section, in this paper we pay attention to the real hyperbolic space $X=H\mathbb{R}^d$, $d\geq 3$, d odd.

Thus, $K=Spin(d)$, $M=Spin(d-1)$ or $K=SO(d)$, $M=SO(d-1)$.

In particular, $\rho=\frac{d-1}{2}$.

We shall assume that the metric on Y is normalized to be of sectional curvature -1.

Consequently, $T_1=1$.

For the sake of simplicity, we fix some $\sigma\in\hat{M}$ and $\chi\in\hat{\Gamma}$.

Hence, we avoid to write σ and χ in the sequel (unless necessary).

It follows that

$$Z_R(s) = \prod_{p=0}^{d-1}\left(\prod_{(\tau,\lambda)\in I_p} Z_S\left(s+\frac{d-1}{2}-\lambda,\tau\right)\right)^{(-1)^p}.$$

Moreover, the Poincare duality

$$I_{d-1-p}=\{(\tau,d-1-\lambda)\,:\,(\tau,\lambda)\in I_p\},$$

where $p\in\left\{0,1,...,\frac{d-1}{2}-1\right\}$, yields that

$$Z_R(s) = \prod_{p=0}^{\frac{d-1}{2}-1}\Bigg(\prod_{(\tau,\lambda)\in I_p} Z_S\left(s+\frac{d-1}{2}-\lambda,\tau\right)\times \times Z_S\left(s-\frac{d-1}{2}+\lambda,\tau\right)\Bigg)^{(-1)^p}\times \times\left(\prod_{(\tau,\lambda)\in I_{\frac{d-1}{2}}} Z_S\left(s+\frac{d-1}{2}-\lambda,\tau\right)\right)^{(-1)^{\frac{d-1}{2}}}.$$

Finally, reasoning as in [6, pp. 40-45], we obtain that

$$Z_R(s) = \prod_{p=0}^{\frac{d-1}{2}-1}\left(Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)\times \times Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)\right)^{(-1)^p}\times \times\left(Z_S\left(s,\sigma_{\frac{d-1}{2}}\right)\right)^{(-1)^{\frac{d-1}{2}}}.$$

Thus, in this setting, we consider the Selberg zeta function $Z_S(s,\sigma_p)$, $p\in\left\{0,1,...,\frac{d-1}{2}\right\}$ for d-dimensional real hyperbolic manifold Y, where σ_p is the p-th exterior power of the standard representation of $SO(d-1)$.

Note that σ_p is irreducible unless $p=\frac{d-1}{2}$.

If $p=\frac{d-1}{2}$, then there exists a splitting $\sigma_{\frac{d-1}{2}}=\sigma^+\oplus\sigma^-$ into two irreducible components σ^+ and σ^- ($(1,1,...,1,\pm 1)$ is the highest weight of $\sigma^\pm$).

The singularities of $Z_S(s,\sigma_p)$ are expressed in terms of the form Laplacian Δ_p on Y.

Thus, the Selberg zeta function $Z_S(s,\sigma_p)$, $p\in\left\{0,1,...,\frac{d-1}{2}\right\}$, has a zero at

$$0\neq s=\mathrm{i}\,\lambda\in\mathrm{i}\,\mathbb{R}\cup\left(-\frac{d-1-2p}{2},\frac{d-1-2p}{2}\right)$$

of order

$$\dim\left\{\Delta_p\omega = \left(\lambda^2+\left(\frac{d-1-2p}{2}\right)^2\right)\omega, \delta\omega=0\right\},$$

a zero at $s=0$ of order

$$2\dim\left\{\Delta_p\omega=\left(\frac{d-1-2p}{2}\right)^2\omega,\delta\omega=0\right\},$$

if $p \neq \frac{d-1}{2}$, a singularity at $s = \frac{d-1-2p}{2}$ of order $\sum\limits_{k=0}^{p} (-1)^{p-k} b_k$, a singularity at $s = -\frac{d-1-2p}{2}$ of order $\sum\limits_{k=0}^{p} (-1)^{p-k} b_k$.

If $p = \frac{d-1}{2}$, the latter two singularities coincide, and the orders add up.

Here δ denotes the co-differential, and b_k is the k-th Betti number of Y.

4 Preliminary results

The following results will be applied in the sequel.

Theorem A. [22, p. 18, Th. A.] *Let λ_1, λ_2,... be a real sequence which increases (in the wide sense) and has the limit infinity, and let*

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the c_n may be real or complex, and the notation indicates a summation over the (finite) set of positive integers n for which $\lambda_n \leq x$. Then, if $X \geq \lambda_1$ and $\phi(x)$ has a continuous derivative, we have

$$\sum_{\lambda_n \leq X} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{X} C(x)\phi'(x)\,dx + C(X)\phi(X).$$

If further, $C(X)\phi(X) \to 0$ as $X \to \infty$, then

$$\sum_{1}^{\infty} c_n \phi(\lambda_n) = -\int_{1}^{\infty} C(x)\phi'(x)\,dx,$$

provided that either side is convergent.

Theorem B. [22, p. 31, Th. B.] *If k is a positive integer, $c > 0$, $y > 0$, then*

$$\frac{1}{2\pi\,\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{y^s ds}{\prod\limits_{j=0}^{k}(s+j)} = \begin{cases} 0, & y \leq 1, \\ \frac{1}{k!}\left(1-\frac{1}{y}\right)^k, & y \geq 1. \end{cases}$$

Theorem C. [3, p. 307, Corollary 3.] *If $f(s) = Z_{R,\chi}(s,\sigma)$, then*

$$f(s) = \frac{Z_1(s)}{Z_2(s)},$$

where $Z_1(s)$, $Z_1(s)$ are entire functions of order at most d over $\mathbb{C}$.

Theorem D. [10, p. 509, Prop. 7] *Suppose $Z(s)$ is the ratio of two nonzero entire functions of order at most d. Then, there is a $D > 0$ such that for arbitrarily large choices of r*

$$\int_{r} \left|\frac{Z'(s)}{Z(s)}\right| |ds| \leq Dr^d \log r.$$

Theorem E. [17] *Let $\varepsilon > 0$ and $d-1 \geq \eta > 0$. Suppose that $t \gg 0$ is chosen so that $\mathrm{i}\,t$ is not a zero of $Z_S(s,\sigma_p)$, $p \in \left\{0,1,...,\frac{d-1}{2}\right\}$. Then,*

(i)

$$\frac{Z_R'(s)}{Z_R(s)} = O\left(t^{d-1+\varepsilon}\right) + \sum_{\left|t-\gamma_{S,0}\right| \leq 1} \frac{1}{s-\rho_{S,0}}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{d-1}{2} \leq \sigma^1 < \frac{1}{4}t - \frac{d-1}{2}$, where $\rho_{S,0} = \frac{d-1}{2} + \mathrm{i}\,\gamma_{S,0}$ is a zero of $Z_S\left(s - \frac{d-1}{2}, \sigma_0\right)$ on the line $\mathrm{Re}(s) = \frac{d-1}{2}$.

(ii)

$$\frac{Z_R'(s)}{Z_R(s)} = O\left(\frac{1}{\eta}t^{d-1+\varepsilon}\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{d-1}{2} + \eta \leq \sigma^1 < \frac{1}{4}t - \frac{d-1}{2}$.

5 Counting functions

Let Γ_{h} resp. $P\Gamma_{\mathrm{h}}$ denote the set of the Γ-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

It is well known that a prime geodesic over Y corresponds to the conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$.

We denote such prime geodesic by C_γ.

Let $\pi_\Gamma(x)$ be the number of prime geodesics C_γ over Y, whose length $l(\gamma)$ is not larger than $\log x$.

We may write

$$\pi_\Gamma(x) = \#\{C_\gamma : N(\gamma) \le x\},$$

where $N(\gamma) = e^{l(\gamma)}$.

It is also known that each $\gamma \in \Gamma_{\mathrm{h}}$ can be represented in the form $\gamma = \gamma_0^{n_\Gamma(\gamma)}$, where $\gamma_0 \in P\Gamma_{\mathrm{h}}$ is some element.

We introduce the following functions:

$$\Lambda(\gamma) = \Lambda\left(\gamma_0^{n_\Gamma(\gamma)}\right) = \log N(\gamma_0)$$

for $\gamma \in \Gamma_{\mathrm{h}}$,

$$\psi_0(x) = \sum_{\gamma\in\Gamma_{\mathrm{h}}, N(\gamma)\le x} \Lambda(\gamma),$$

$$\psi_j(x) = \int\limits_0^x \psi_{j-1}(t)\,dt,$$

$j \in \mathbb{N}$.

Let $N_{S,0}(y)$ be the number of zeros $\rho_{S,0} = \frac{d-1}{2} + \mathrm{i}\,\gamma_{S,0}$ of $Z_S\left(s - \frac{d-1}{2}, \sigma_0\right)$ on the interval $\frac{d-1}{2} + \mathrm{i}\,x$, $0 < x \le y$.

By [8, p. 89, Th. 9.1.],

$$N_{S,0}(y) = C_1 y^d + O\left(y^{d-1}(\log y)^{-1}\right)$$

for some explicitly known constant C_1.

Note that in [17], the estimate

$$N_{S,0}(y) = C_1 y^d + O\left(y^{d-1}\right)$$

was sufficient to derive the desired results (see also, [4] for the even-dimensional case).

In this paper, we shall apply the estimate

$$N_{S,0}(y) = O\left(y^d\right).$$

6 Prime geodesic theorem

Theorem 1. *(Prime Geodesic Theorem) Let X be the real hyperbolic space $H\mathbb{R}^d$, $d \ge 3$, d odd. Then,*

$$\pi_\Gamma(x) = \sum_{p=0}^{\frac{d-1}{2}-1} (-1)^p \sum_{s(p)\in\left(\frac{3}{4}(d-1),d-1\right]} \mathrm{li}\left(x^{s(p)}\right) O\left(x^{\frac{3}{4}(d-1)}(\log x)^{-1}\right)$$

as $x \to +\infty$, where $s(p)$ is a singularity of the Selberg zeta function $Z_S\left(s - \frac{d-1}{2} + p, \sigma_p\right)$.

Proof. Suppose that $k \ge 2d$ is an integer.

Furthermore, suppose that $x > 1$ and $c > d-1$.

By [5, p. 97, (3.4)],

$$\log Z_R(s) = -\sum_{\gamma\in\Gamma_{\mathrm{h}}} \frac{e^{-sl(\gamma)}}{n_\Gamma(\gamma)}$$

for $\mathrm{Re}(s) > d-1$.

Therefore,

$$\frac{Z_R'(s)}{Z_R(s)} = \sum_{\gamma\in\Gamma_{\mathrm{h}}} \frac{e^{-sl(\gamma)} l(\gamma)}{n_\Gamma(\gamma)} = \sum_{\gamma\in\Gamma_{\mathrm{h}}} \frac{l(\gamma)}{n_\Gamma(\gamma) N(\gamma)^s}$$

for $\mathrm{Re}(s) > d-1$.

Since $\gamma \in \Gamma_{\mathrm{h}}$, it follows that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$ for some $\gamma_0 \in P\Gamma_{\mathrm{h}}$.

As noted earlier, we may write $\gamma_0 = \theta_{\gamma_0} a_{\gamma_0} m_{\gamma_0} \theta_{\gamma_0}^{-1}$ for some θ_{γ_0}, where $a_{\gamma_0} m_{\gamma_0} \in A^+M$.

Now,

$$\begin{aligned}\gamma =& \gamma_0^{n_\Gamma(\gamma)} = \left(\theta_{\gamma_0} a_{\gamma_0} m_{\gamma_0} \theta_{\gamma_0}^{-1}\right)^{n_\Gamma(\gamma)} \\ =& \theta_{\gamma_0} a_{\gamma_0} m_{\gamma_0} a_{\gamma_0} m_{\gamma_0} \ldots a_{\gamma_0} m_{\gamma_0} \theta_{\gamma_0}^{-1} \\ =& \theta_{\gamma_0} a_{\gamma_0}^{n_\Gamma(\gamma)} m_{\gamma_0}^{n_\Gamma(\gamma)} \theta_{\gamma_0}^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned} l(\gamma) =& l\left(\theta_{\gamma_0} a_{\gamma_0}^{n_\Gamma(\gamma)} m_{\gamma_0}^{n_\Gamma(\gamma)} \theta_{\gamma_0}^{-1}\right) \\ =& \left|\log\left(a_{\gamma_0}^{n_\Gamma(\gamma)}\right)\right| \\ =& n_\Gamma(\gamma) \left|\log(a_{\gamma_0})\right| = n_\Gamma(\gamma)\, l(\gamma_0) \\ =& n_\Gamma(\gamma) \log N(\gamma_0) = n_\Gamma(\gamma)\, \Lambda(\gamma).\end{aligned}$$

Consequently,

$$\frac{Z_R'(s)}{Z_R(s)} = \sum_{\gamma\in\Gamma_{\mathrm{h}}} \frac{\Lambda(\gamma)}{N(\gamma)^s}$$

for $\mathrm{Re}(s) > d-1$.

Hence, by Theorem A and Theorem B, we obtain that (see, e.g., [3, pp. 311-312])

$$\psi_k(x) = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty} \frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)} ds.$$

Let $A \gg 0$ be a number.

We consider the interval $\mathrm{i}\,t$, $A-1 < t \le A+1$.

It is not hard to apply the Dirichlet principle to conclude that there exists a point $\mathrm{i}\,\bar{A}$, $\bar{A}$

$\in (A-1, A+1]$, such that (see, e.g., [16], [3], [30], [27])

$$\left|\mathrm{i}\,\bar{A}-\alpha\right| > \frac{C}{\bar{A}^d},$$

where $C>0$ is fixed, and α is a zero of $Z_S(s,\sigma_p)$, $p \in \left\{0,1,...,\frac{d-1}{2}\right\}$.

Put

$$T=\sqrt{\bar{A}^2+\left(\frac{d-1}{2}\right)^2}.$$

Define

$$\begin{aligned} &C(T)\\ =&\left\{s\in\mathbb{C}\,:\,|s|\le T, \mathrm{Re}(s)\le\frac{d-1}{2}\right\}\\ &\cup\left\{s\in\mathbb{C}\,:\,\frac{d-1}{2}\le\mathrm{Re}(s)\le c,\right.\\ &\left.-\bar{A}\le\mathrm{Im}(s)\le\bar{A}\right\}. \end{aligned}$$

Since $\left|\mathrm{i}\,\bar{A}-\alpha\right| > \frac{C}{\bar{A}^d}$ for all α's and all $Z_S(s,\sigma_p)$'s, it immediately follows that no pole of $\frac{Z_R'(s)}{Z_R(s)}$ occurs on the boundary of the square part of $C(T)$ (note that $Z_S(s,\sigma_p)$, $p\in\left\{0,1,...,\frac{d-1}{2}\right\}$ has no singularities for $\mathrm{Re}(s)>\frac{d-1}{2}$).

Without loss of generality, we may also assume that no pole of

$$\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}$$

occurs on the boundary of the circular part of $C(T)$.

Now, we apply the Cauchy residue theorem to the function $\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}$ along the contour $C(T)$.

We obtain,

$$\begin{aligned} &\int\limits_{C_1(T)}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds\\ =&2\pi\,\mathrm{i}\sum_{z\in C(T)}\mathrm{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}\right), \end{aligned}$$

where $C_1(T)$ denotes the boundary of $C(T)$ taken with the anticlockwise orientation, and the sum along $C(T)$ is taken over singularities of $\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}$ inside $C(T)$.

Suppose that $0<\beta<c-\frac{d-1}{2}$.

Denote by $C^1(T)$ the boundary of the circular part of $C(T)$, taken with the anticlockwise orientation.

We have,

$$\begin{aligned} &\int\limits_{c-\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\int\limits_{c+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\int\limits_{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\int\limits_{C^1(T)}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\int\limits_{\frac{d-1}{2}-\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\int\limits_{\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A}}^{c-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ =&2\pi\,\mathrm{i}\sum_{z\in C(T)}\mathrm{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} &\psi_k(x)\\ =&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+\\ &\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\infty}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+ \end{aligned}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds.$$

Since

$$\frac{Z_R'(s)}{Z_R(s)}=\sum_{\gamma\in\Gamma_{\mathrm{h}}}\frac{\Lambda(\gamma)}{N(\gamma)^s}$$

for $\mathrm{Re}(s)>d-1$, we deduce

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\infty}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds$$
$$=O\left(x^{c+k}\int\limits_{c+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\infty}\frac{|ds|}{|s|^{k+1}}\right)$$
$$=O\left(x^{c+k}\bar{A}^{-k}\right).$$

Similarly,

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds$$
$$=O\left(x^{c+k}\bar{A}^{-k}\right).$$

Therefore,

$$\psi_k(x)$$
$$=\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds+$$
$$O\left(x^{c+k}\bar{A}^{-k}\right),$$

i.e.,

$$\int\limits_{c-\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds$$
$$=2\pi\,\mathrm{i}\,\psi_k(x)-O\left(x^{c+k}\bar{A}^{-k}\right).$$

Furthermore, we estimate the integrals along $\left[c+\mathrm{i}\,\bar{A},\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}\right]$, $\left[\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A},\frac{d-1}{2}+\mathrm{i}\,\bar{A}\right]$, $C^1(T)$, $\left[\frac{d-1}{2}-\mathrm{i}\,\bar{A},\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A}\right]$ and $\left[\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A},c-\mathrm{i}\,\bar{A}\right]$.

In order to estimate the integral over $C^1(T)$, we apply Theorem C and Theorem D.

We obtain,

$$\int\limits_{C^1(T)}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds$$
$$=O\left(x^{\frac{d-1}{2}+k}T^{-k-1}\int\limits_{C^1(T)}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)$$
$$=O\left(x^{\frac{d-1}{2}+k}T^{-k-1}\int\limits_{|s|=T}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)$$
$$=O\left(x^{\frac{d-1}{2}+k}T^{-k-1+d}\log T\right).$$

In order to estimate the remaining integrals, we apply Theorem E and the fact that $\left|\mathrm{i}\,\bar{A}-\alpha\right|>\frac{C}{\bar{A}^d}$ for all α's.

Fix some $\varepsilon>0$.

Obviously,

$$\int\limits_{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds$$
$$=O\left(x^{\frac{d-1}{2}+\beta+k}T^{-k-1}\int\limits_{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\mathrm{i}\,\bar{A}}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right).$$

By (i) of Theorem E,

$$\frac{Z_R'(s)}{Z_R(s)}$$
$$=O\left(\bar{A}^{d-1+\varepsilon}\right)+\sum_{\left|\bar{A}-\gamma_{S,0}\right|\leq1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{d-1}{2}\leq\sigma^1<\frac{1}{4}\bar{A}-\frac{d-1}{2}$.

In particular,

$$\frac{Z_R'(s)}{Z_R(s)}$$
$$=O\left(\bar{A}^{d-1+\varepsilon}\right)+\sum_{\left|\bar{A}-\gamma_{S,0}\right|\leq1}\frac{1}{s-\rho_{S,0}}$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{d-1}{2}\leq\sigma^1\leq\frac{d-1}{2}+\beta$.

Therefore,

$$\begin{aligned}
&\frac{Z_R'(s)}{Z_R(s)}\\
=&O\left(\bar{A}^{d-1+\varepsilon}\right)+O\left(\bar{A}^{d}\sum_{\left|\bar{A}-\gamma_{S,0}\right|\leq 1}1\right)\\
=&O\left(\bar{A}^{d-1+\varepsilon}\right)+O\left(\bar{A}^{2d}\right)\\
=&O\left(\bar{A}^{2d}\right)=O\left(T^{2d}\right)
\end{aligned}$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{d-1}{2}\leq\sigma^1\leq\frac{d-1}{2}+\beta$.

Consequently,

$$\begin{aligned}
&\int\limits_{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds\\
=&O\left(x^{\frac{d-1}{2}+\beta+k}T^{-k-1+2d}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int\limits_{\frac{d-1}{2}-\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds\\
=&O\left(x^{\frac{d-1}{2}+\beta+k}T^{-k-1+2d}\right).
\end{aligned}$$

Finally, by (ii) of Theorem E,

$$\frac{Z_R'(s)}{Z_R(s)}=O\left(\frac{1}{\beta}\bar{A}^{d-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{d-1}{2}+\beta\leq\sigma^1<\frac{1}{4}\bar{A}-\frac{d-1}{2}$.

In particular,

$$\frac{Z_R'(s)}{Z_R(s)}=O\left(\frac{1}{\beta}\bar{A}^{d-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{d-1}{2}+\beta\leq\sigma^1\leq c$.

We obtain,

$$\begin{aligned}
&\int\limits_{c+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds\\
=&O\left(x^{c+k}T^{-k-1}\int\limits_{c+\mathrm{i}\,\bar{A}}^{\frac{d-1}{2}+\beta+\mathrm{i}\,\bar{A}}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)\\
=&O\left(\frac{1}{\beta}x^{c+k}T^{-k-2+d+\varepsilon}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int\limits_{\frac{d-1}{2}+\beta-\mathrm{i}\,\bar{A}}^{c-\mathrm{i}\,\bar{A}}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}ds\\
=&O\left(\frac{1}{\beta}x^{c+k}T^{-k-2+d+\varepsilon}\right).
\end{aligned}$$

Combining the estimates derived above, we end up with

$$\begin{aligned}
&2\pi\,\mathrm{i}\,\psi_k(x)-O\left(x^{c+k}\bar{A}^{-k}\right)+\\
&O\left(\frac{1}{\beta}x^{c+k}T^{-k-2+d+\varepsilon}\right)+\\
&O\left(x^{\frac{d-1}{2}+\beta+k}T^{-k-1+2d}\right)+\\
&O\left(x^{\frac{d-1}{2}+k}T^{-k-1+d}\log T\right)\\
=&2\pi\,\mathrm{i}\sum_{z\in C(T)}\mathrm{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}\right).
\end{aligned}$$

Letting $T\to+\infty$ (then $\bar{A}\to+\infty$ as well), and taking into account that $k\geq 2d$, we conclude that

$$\begin{aligned}
&\psi_k(x)\\
=&\sum_{z\in P_R^k}\mathrm{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}\right),
\end{aligned}$$

where P_R^k denotes the set of singularities (poles) of $\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}$.

Recall the equation (ii) in [3, p. 313].

There ($n=\dim(Y)$, n is odd now)

$$\begin{aligned}
&\psi_k(x)\\
=&\sum_{p=0}^{n-1}(-1)^p\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A_k^{p,\tau,\lambda}}1\times\\
&\times\mathrm{Res}_{s=z}\left(\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{\prod\limits_{j=0}^{k}(s+j)}\right),
\end{aligned}$$

where $A_k^{p,\tau,\lambda}$ is the set of poles

$$\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)},$$

and $k\geq 2n$.

Since

$$\frac{Z_R'(s)}{Z_R(s)} = \sum_{p=0}^{n-1}(-1)^p\sum_{(\tau,\lambda)\in I_p}\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)},$$

it follows that we can write

$$\psi_k(x) = \sum_{z\in A_R^k}\operatorname{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}\right),$$

where A_R^k is the set of poles

$$\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}.$$

The main result in [3, p. 311] states that

$$\begin{aligned}\pi_\Gamma(x) &= \sum_{p=0}^{n-1}(-1)^p\sum_{(\tau,\lambda)\in I_p}1\times \\ &\times\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\operatorname{li}\left(x^{s^{p,\tau,\lambda}}\right) \\ &+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where $s^{p,\tau,\lambda}$ is a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$.

Reasoning as above (see also, [16, p. 192, (12)]), we may write

$$\begin{aligned}\pi_\Gamma(x) &= \sum_{s_R\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\operatorname{li}(x^{s_R}) \\ &+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where s_R is a singularity of the Ruelle zeta function $Z_R(s)$.

In short, the equation

$$\psi_k(x) = \sum_{z\in A_R^k}\operatorname{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}\right)$$

yields that

$$\begin{aligned}\pi_\Gamma(x) &= \sum_{s_R\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\operatorname{li}(x^{s_R}) \\ &+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$.

In the present setting, $n=d$, $\rho=\frac{d-1}{2}$.

Thus, the equation

$$\psi_k(x) = \sum_{z\in P_R^k}\operatorname{Res}_{s=z}\left(\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{\prod_{j=0}^{k}(s+j)}\right)$$

yields that

$$\begin{aligned}\pi_\Gamma(x) &= \sum_{s_R\in\left(\frac{3}{4}(d-1),d-1\right]}\operatorname{li}(x^{s_R}) \\ &+O\left(x^{\frac{3}{4}(d-1)}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$.

Now, the fact that

$$\begin{aligned}Z_R(s) &= \prod_{p=0}^{\frac{d-1}{2}-1}\left(Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)\times\right. \\ &\left.\times Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)\right)^{(-1)^p}\times \\ &\times\left(Z_S\left(s,\sigma_{\frac{d-1}{2}}\right)\right)^{(-1)^{\frac{d-1}{2}}},\end{aligned}$$

implies that

$$\begin{aligned}&\frac{Z_R'(s)}{Z_R(s)}\\&=\sum_{p=0}^{\frac{d-1}{2}-1}(-1)^p\frac{Z_S'\left(s+\frac{d-1}{2}-p,\sigma_p\right)}{Z_S\left(s+\frac{d-1}{2}-p,\sigma_p\right)}+\\&\sum_{p=0}^{\frac{d-1}{2}-1}(-1)^p\frac{Z_S'\left(s-\frac{d-1}{2}+p,\sigma_p\right)}{Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)}+\\&(-1)^{\frac{d-1}{2}}\frac{Z_S'\left(s,\sigma_{\frac{d-1}{2}}\right)}{Z_S\left(s,\sigma_{\frac{d-1}{2}}\right)}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{p=0}^{\frac{d-1}{2}-1}(-1)^p\sum_{s(p)\in\left(\frac{3}{4}(d-1),d-1\right]}\mathrm{li}\left(x^{s(p)}\right)\\&+O\left(x^{\frac{3}{4}(d-1)}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where $s(p)$ is a singularity of the Selberg zeta function $Z_S\left(s-\frac{d-1}{2}+p,\sigma_p\right)$.

This completes the proof. □

7 Conclusion

Note that the result given by Theorem 1 agrees with the corresponding result in the compact, even-dimensional case (see, [16, p. 192, (13)]).

In [2], the authors proved that (see, [25] for somewhat weaker error term)

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{s_n(k)\in\left(\frac{3}{4}(d-1),d-1\right]}(-1)^k\,\mathrm{li}\left(x^{s_n(k)}\right)\\&+O\left(x^{\frac{3}{4}(d-1)}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where $(s_k-k)(d-1-k-s_n(k))$ is a small eigenvalue in $\left[0,\frac{3}{4}\left(\frac{d-1}{2}\right)^2\right]$ of Δ_k on $\pi_{\sigma_k,\lambda_n(k)}$ with $s_n(k)=\frac{d-1}{2}+\mathrm{i}\,\lambda_n(k)$ or $s_n(k)=\frac{d-1}{2}-\mathrm{i}\,\lambda_n(k)$ in $\left(\frac{3}{4}(d-1),d-1\right]$, Δ_k is the Laplacian acting on the space of k-forms over X_Γ, $\pi_{\sigma_k,\lambda_n(k)}$ is the principal series representation, and X_Γ is a d-dimensional real hyperbolic manifold with cusps.

Obviously, Theorem 1 is in line with this result.

In particular, Randol [29] proved that (see, [20], [21], [18] for a weaker form of the error term)

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{s_n\in\left(\frac{3}{4},1\right]}\mathrm{li}\left(x^{s_n(k)}\right)+O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where $\lambda_n=s_n(1-s_n)$ is a small eigenvalue in $\left[0,\frac{3}{16}\right]$ of the Laplacian Δ_0 acting on $L^2(R)$, and R is a compact Riemann surface of genus $g\geq 2$.

Thus, Theorem 1 is in line with this result as well.

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