On the Logarithmic Prime Geodesic Theorem

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Abstract: In this paper we refine the error term in the prime geodesic theorem for compact, even-dimensional locally symmetric Riemannian manifolds with strictly negative sectional curvature. The ingredients for the starting prime geodesic theorem come from our most recent research of the zeta functions of Selberg and Ruelle associated with locally homogeneous bundles over compact locally symmetric spaces of rank one. In this paper, we shall restrict our investigations to compact, even-dimensional, locally symmetric spaces. For this class of spaces, we prove that there exists a set ∇ of positive real numbers, which is relatively small in the sense that its logarithmic measure is finite, such that the error term of the aforementioned prime geodesic theorem is improved outside ∇. The derived prime geodesic theorem generalizes the corresponding prime number theorem, where it is proved that the error term under Riemann hypothesis assumption can be further reduced except on a set of finite logarithmic measure.

Key–Words: Prime geodesic theorem, locally symmetric spaces, logarithmic measure

1 Introduction and preliminaries

In this paper we consider a logarithmic prime geodesic theorem over compact n-dimensional (n even) locally symmetric Riemannian manifold Y of strictly negative sectional curvature.

Our aim is to prove that the prime geodesic theorem

$$\psi_0(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1 \times \sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$$

as $x\to\infty$, which is derived in [1, p. 317, (39)], can be further improved.

More precisely, the prime geodesic theorem given above does not depend on the choice of x, $x\to\infty$.

In this paper we prove that its error term can be further refined if $x\to\infty$ is taken outside some explicitly known set of finite logarithmic measure $\mu^{\times}$.

Note that $\psi_0(x)$ is given in terms of singularities $s^{p,\tau,\lambda}$ of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$ (such zeta functions are described in detail in [2], and satisfy the equation (4) in [1, p. 305]).

As it is usual, $\psi_j(x)$, $j = 0, 1, ...$ are introduced recursively (see, [1, p. 312]).

Let

$$S(v) = \sum_{\nu\in A} c(\nu)\, e^{2\pi \mathrm{i}\,\nu v}$$

be an absolutely convergent exponential sum, where the frequencies ν run over arbitrary strictly increasing sequence A of real numbers, and the coefficients $c(\nu)$ are complex.

Let $\delta = \frac{\theta}{V}$, with $0<\theta<1$. Then (see, [4, p. 330, Lemma 1]),

$$\int_{-V}^{V} |S(v)|^2\, dv \ll_\theta \int_{-\infty}^{+\infty} \left|\delta^{-1}\sum_{t}^{t+\delta} c(\nu)\right|^2 dt. \tag{1}$$

Here, according to Vinogradov's notation, $A \ll_\theta B$ means that $|A|\le C\,B$, where $C>0$ is an unspecified constant that depends on θ, i.e., $C=C(\theta)$.

In this paper, we shall apply (1) in the following form

$$\int_{-V}^{V} \left|\sum_{\nu\in A} c(\nu)\, e^{2\pi \mathrm{i}\,\nu v}\right|^2 dv \le \tag{2}$$

$$\left(\frac{\pi\theta}{\sin(\pi\theta)}\right)^2 \int_{-\infty}^{+\infty} \left| \frac{V}{\theta} \sum_{t\leq\nu\leq t+\frac{\theta}{V}} c(\nu) \right|^2 dt.$$

2 Main Result

Theorem 1. *Let Y be a compact, n-dimensional (n even) locally symmetric Riemannian manifold with strictly negative sectional curvature. There exists a set ∇ of finite logarithmic measure, such that*

$$\psi_0(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1 \times \times \sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}, 2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}} \times \times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}} (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big)$$

as $x \to +\infty$, $x \notin \nabla$, where $\varepsilon > 0$ is arbitrarily small.

Proof. Recall the equation (12) in [1], We have,

$$\psi_{2n}(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{z\in A^{p,\tau,\lambda}} c_z(p,\tau,\lambda),$$

$x > 1$. Here, $A^{p,\tau,\lambda}$ denotes the set of poles of

$$\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)},$$

and

$$c_z(p,\tau,\lambda) = \mathrm{Res}_{s=z}\left(\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)}\right).$$

Let $x > 1$.

We may assume that $x \gg 1$ when necessary.

The residues $c_z(p,\tau,\lambda)$, $z \in A^{p,\tau,\lambda}$ are given by the equations (13), (14), and (15) in [1].

Define the following sets:

$$I_{-2n} = \{0, -1, ..., -2n\},$$

$$B_{p,\tau,\lambda} = \left\{ -j \in I_{-2n} : c_{-j}(p,\tau,\lambda) = O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right) \right\},$$

$$B'_{p,\tau,\lambda} = I_{-2n} \backslash B_{p,\tau,\lambda},$$

$$S_{\mathbb{R}}^{p,\tau,\lambda} = S^{p,\tau,\lambda} \cap \mathbb{R},$$

$$S_{-\rho+\lambda}^{p,\tau,\lambda} = S^{p,\tau,\lambda} \backslash S_{\mathbb{R}}^{p,\tau,\lambda},$$

$$C^1_{p,\tau,\lambda} = \left\{ s^{p,\tau,\lambda} \in S_{\mathbb{R}}^{p,\tau,\lambda} : s^{p,\tau,\lambda} \leq -2n-1 \right\},$$

$$C^2_{p,\tau,\lambda} = \left\{ s^{p,\tau,\lambda} \in S_{\mathbb{R}}^{p,\tau,\lambda} : -2n-1 < s^{p,\tau,\lambda} \leq -2n + \rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} \right\},$$

$$C^3_{p,\tau,\lambda} = \left\{ s^{p,\tau,\lambda} \in S_{\mathbb{R}}^{p,\tau,\lambda} : -2n + \rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} < s^{p,\tau,\lambda} \leq \rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} \right\},$$

$$C^4_{p,\tau,\lambda} = \left\{ s^{p,\tau,\lambda} \in S_{\mathbb{R}}^{p,\tau,\lambda} : \rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} < s^{p,\tau,\lambda} \leq 2\rho \right\}.$$

Note that

$$\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} \leq 2\rho\frac{n+\rho-1}{n+2\rho-1}$$

since the corresponding equivalent inequality $n \geq 1$ is always valid.

Also, note that

$$A^{p,\tau,\lambda} = B_{p,\tau,\lambda} \cup B'_{p,\tau,\lambda} \cup \left(\bigcup_{k=1}^{4} C^k_{p,\tau,\lambda}\right) \cup S_{-\rho+\lambda}^{p,\tau,\lambda}.$$

Reasoning in the same way as in [1, p. 315], we obtain that

$$\sum_{z\in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{-1}\right).$$

Furthermore,

$$\sum_{z\in C^2_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = \sum_{z\in C^2_{p,\tau,\lambda}} \frac{o_z^{p,\tau,\lambda} x^{z+2n}}{\prod_{k=0}^{2n}(z+k)} = O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right),$$

where $o_z^{p,\tau,\lambda}$ denotes the order of the singularity z of $Z_S(s+\rho-\lambda,\tau)$.

Finally, by the very definition of the set $B_{p,\tau,\lambda}$, we have that

$$\sum_{z\in B_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right).$$

Consequently,

$$\begin{aligned}&\psi_{2n}(x)\\ =&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}} c_z(p,\tau,\lambda)+\\ =&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}} c_z(p,\tau,\lambda)+\\ =&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{k=3}^{4}\sum_{z\in C^k_{p,\tau,\lambda}} c_z(p,\tau,\lambda)+\\ &O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right),\end{aligned}$$

$x>1$.

As it is usual, we introduce the differential operator

$$\Delta^+_{2n} f(x) = \sum_{i=0}^{2n}(-1)^i\binom{2n}{i} f(x+(2n-i)h)$$

for appropriately chosen h.

Let $\varepsilon>0$. Put $\varepsilon' = \frac{n+\rho}{\rho}\varepsilon$, $\varepsilon'' = \frac{1}{n-1}\varepsilon$.

We shall take

$$\begin{aligned}h=&x^{\frac{1}{2}\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log x)^{\frac{1}{2}\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log\log x)^{\frac{1}{2}\frac{n-1}{2n^2-2n+4n\rho+\rho}+\frac{\varepsilon}{2\rho}}.\end{aligned}$$

Obviously, $1<h<x$.

For the sake of simplicity, we shall keep h in our notation until further notice.

Since $n-\rho\geq\frac{1}{2}$, we deduce

$$\begin{aligned}&h^{-2n}\Delta^+_{2n} O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right)\\ =&h^{-2n}O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right)\\ =&O\left(h^{-2n}x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right)\\ =&O\Big(x^{(-n+\rho)\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log x)^{-n\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log\log x)^{-n\frac{n-1}{2n^2-2n+4n\rho+\rho}-\frac{n}{\rho}\varepsilon}\Big)\\ =&O\Big(x^{-\frac{1}{2}\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log x)^{-n\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log\log x)^{-n\frac{n-1}{2n^2-2n+4n\rho+\rho}-\frac{n}{\rho}\varepsilon}\Big)\\ =&O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&h^{-2n}\Delta^+_{2n}\psi_{2n}(x)\\ =&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}}1\times\\ &h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda)+\\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}1\times\\ &\times h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda)+\\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{k=3}^{4}\sum_{z\in C^k_{p,\tau,\lambda}}1\times\\ &\times h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda)+\\ &O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times\\ &\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big),\end{aligned}$$

$x>1$.

Note that the operator Δ^+_{2n} may be defined by [1, (22)], and satisfies [1, (23)] in the case when f is at least $2n$ times differentiable function.

Reasoning in exactly the same way as in [1, p. 316], we obtain that

$$\sum_{z\in B'_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda) = O(\log x).$$

Furthermore, if $z \in C^3_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda) = h^{-2n}o_z^{p,\tau,\lambda}\frac{\Delta^+_{2n}x^{z+2n}}{\prod_{k=0}^{2n}(z+k)} = \frac{o_z^{p,\tau,\lambda}}{z}\tilde{x}^z_{p,\tau,\lambda,z},$$

where $\tilde{x}_{p,\tau,\lambda,z} \in [x, x+2nh]$.

We conclude,

$$h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda) = O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right).$$

Therefore,

$$\sum_{z\in C^3_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda) = O\left(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\right).$$

Similarly, if $s^{p,\tau,\lambda} \in C^4_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta^+_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda) = \frac{o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}\tilde{x}^{s^{p,\tau,\lambda}}_{s^{p,\tau,\lambda}},$$

where $\tilde{x}_{s^{p,\tau,\lambda}} \in [x, x+2nh]$.

Consequently (see also, [9], [8]),

$$\sum_{z\in C^4_{p,\tau,\lambda}} h^{-2n}\Delta^+_{2n}c_z(p,\tau,\lambda)$$
$$= \sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + O\left(h^{2\rho}\right)$$
$$= \sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}} \times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}} \times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big),$$

where $s^{p,\tau,\lambda}$ is counted $o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}$ times in the corresponding sum.

We shall use this practice in the sequel.

Combining the obtained estimates, we end up with

$$h^{-2n}\Delta^+_{2n}\psi_{2n}(x) = \sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times \sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + \sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}} 1\times h^{-2n}\Delta^+_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda) + O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}} \times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}} \times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big), \tag{3}$$

$x > 1$.

For any $x > 1$, we may define $m = \lfloor \log x \rfloor$.

Now, $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ yields that

$$m = \lfloor \log x \rfloor \le \log x < \lfloor \log x \rfloor + 1 = m + 1.$$

Hence,

$$e^m \le x < e^{m+1}.$$

Thus, taking any $x > 1$, we obtain that $x \in [e^m, e^{m+1})$.

Define,

$$\alpha = (n+\rho)\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},$$
$$\beta = \gamma = (n+\rho)\frac{n-1}{2n^2-2n+4n\rho+\rho}.$$

Furthermore, let $T = e^m$, and

$$Y = e^{\frac{m(4n+2\rho-2\alpha)}{2n+3}} m^{\frac{1-2\beta}{2n+3}} (\log m)^{\frac{1-2\gamma}{2n+3}+\varepsilon''}.$$

It is easily seen that $Y < T$.

We shall write $s^{p,\tau,\lambda} = -\rho + \lambda + \mathrm{i}\,\gamma^{p,\tau,\lambda}$ if $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{-\rho+\lambda}$.

Now,

$$\sum_{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}} h^{-2n}\Delta^+_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda) = \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ |\gamma^{p,\tau,\lambda}|\le Y}} h^{-2n}\Delta^+_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda) + h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<|\gamma^{p,\tau,\lambda}|\le T}} \Delta^+_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda) + h^{-2n}\Delta^+_{2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ |\gamma^{p,\tau,\lambda}|>T}} c_{s^{p,\tau,\lambda}}(p,\tau,\lambda). \tag{4}$$

Recall the equation (34) in [1]. We deduce,

$$\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|\leq Y}} h^{-2n}\Delta^{+}_{2n}c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right)$$
$$=O\left(\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|\leq Y}}\left|s^{p,\tau,\lambda}\right|^{-1}x^{-\rho+\lambda}\right)$$
$$=O\left(x^{-\rho+\lambda}\int_{0}^{Y}t^{-1}dN_{p,\tau,\lambda}\left(t\right)\right) \tag{5}$$
$$=O\left(x^{-\rho+\lambda}\int_{0}^{Y}t^{n-2}dt\right)=O\left(x^{-\rho+\lambda}Y^{n-1}\right)$$
$$=O\Big(x^{-\rho+\lambda}e^{\frac{m(4n+2\rho-2\alpha)}{2n+3}(n-1)}\times$$
$$\times\, m^{\frac{1-2\beta}{2n+3}(n-1)}\left(\log m\right)^{\left(\frac{1-2\gamma}{2n+3}+\varepsilon''\right)(n-1)}\Big)$$

$$=O\Big(x^{\rho}x^{\frac{4n+2\rho-2\alpha}{2n+3}(n-1)}\times$$
$$\times\left(\log x\right)^{\frac{1-2\beta}{2n+3}(n-1)}\left(\log\log x\right)^{\left(\frac{1-2\gamma}{2n+3}+\varepsilon''\right)(n-1)}\Big)$$
$$=O\Big(x^{\rho\frac{4n^{2}-4n+4n\rho+\rho}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}+\varepsilon}\Big).$$

Here, $N_{p,\tau,\lambda}\left(t\right)=O\left(t^{n}\right)$ denotes the number of singularities of $Z_{S}\left(s+\rho-\lambda,\tau\right)$ on the interval $-\rho+\lambda + \mathrm{i}\,x$, $0<x\leq t$.

Furthermore,

$$h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq T}}\Delta^{+}_{2n}c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right)$$
$$=h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq T}}\frac{o_{s^{p,\tau,\lambda}}\Delta^{+}_{2n}\left(x^{s^{p,\tau,\lambda}+2n}\right)}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)} \tag{6}$$
$$=h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq T}}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times$$
$$\sum_{i=0}^{2n}\left(-1\right)^{i}\binom{2n}{i}\left(x+\left(2n-i\right)h\right)^{s^{p,\tau,\lambda}+2n}.$$

Finally, $T>e^{-1}x$ yields that

$$\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|>T}} c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right)$$
$$=\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|>T}}\frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}$$
$$O\left(x^{-\rho+\lambda+2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|>T}}\left|s^{p,\tau,\lambda}\right|^{-2n-1}\right)$$
$$=O\left(x^{\rho+2n}\int_{T}^{+\infty}t^{-2n-1}dN_{p,\tau,\lambda}\left(t\right)\right)$$
$$=O\left(x^{\rho+2n}\int_{T}^{+\infty}t^{-n-2}dt\right)=O\left(x^{\rho+2n}T^{-n-1}\right)$$
$$=O\left(x^{\rho+2n}x^{-n-1}\right)=O\left(x^{\rho+n-1}\right).$$

Hence,

$$h^{-2n}\Delta^{+}_{2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ \left|\gamma^{p,\tau,\lambda}\right|>T}}c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right)$$
$$=h^{-2n}\Delta^{+}_{2n}O\left(x^{\rho+n-1}\right)=O\left(h^{-2n}x^{\rho+n-1}\right)$$
$$=O\Big(x^{\rho+n-1-n\frac{4n^{2}-4n+4n\rho+\rho}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}-\frac{n}{\rho}\varepsilon}\Big)$$
$$=O\Big(x^{\rho+n-1-n}\left(\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}-\frac{n}{\rho}\varepsilon}\Big) \tag{7}$$
$$=O\Big(x^{\rho-1}\left(\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{-n\frac{n-1}{2n^{2}-2n+4n\rho+\rho}-\frac{n}{\rho}\varepsilon}\Big)$$
$$=O\Big(x^{\rho}\left(\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}+\varepsilon}\Big)$$
$$=O\Big(x^{\rho\frac{4n^{2}-4n+4n\rho+\rho}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}}\times$$
$$\times\left(\log\log x\right)^{\rho\frac{n-1}{2n^{2}-2n+4n\rho+\rho}+\varepsilon}\Big).$$

Combining (4), (5), (6) and (7), we obtain

$$\sum_{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}} h^{-2n}\Delta^{+}_{2n}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$$
$$=h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times$$
$$\times\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\left(x+(2n-i)h\right)^{s^{p,\tau,\lambda}+2n}$$
$$+O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

This equality and the equality (3), imply that

$$h^{-2n}\Delta^{+}_{2n}\psi_{2n}(x)$$
$$=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times$$
$$\times\sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}+$$
$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$
$$\times\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times \tag{8}$$
$$\times\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\left(x+(2n-i)h\right)^{s^{p,\tau,\lambda}+2n}$$
$$+O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big),$$

$x>1$.

Consider the sum on the right hand side of (8) that corresponds to $i=2n$, i.e., the sum

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$
$$\times\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}x^{s^{p,\tau,\lambda}+2n}.$$

As we already noted, $x\in\left[e^m,e^{m+1}\right)$.
Define

$$\Delta_{m,1}=\bigcup_{p=0}^{n-1}\bigcup_{(\tau,\lambda)\in I_p}\Delta^{p,\tau,\lambda}_{m,1},$$

where $\Delta^{p,\tau,\lambda}_{m,1}$ is the set of all $x\in\left[e^m,e^{m+1}\right)$ such that

$$\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|$$
$$>x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma+\varepsilon'}.$$

Thus, if $x\in\Delta^{p,\tau,\lambda}_{m,1}$, then

$$1<\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\times$$
$$\times\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\gamma+2\varepsilon'}}.$$

We estimate,

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}=\int_{\Delta^{p,\tau,\lambda}_{m,1}}\frac{dx}{x}$$
$$=O\Bigg(\int_{\Delta^{p,\tau,\lambda}_{m,1}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\times$$
$$\times\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\gamma+2\varepsilon'}}\frac{dx}{x}\Bigg)$$
$$=O\Bigg(\int_{e^m}^{e^{m+1}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{-\rho+\lambda+\mathrm{i}\,\gamma^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\times$$
$$\times\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\gamma+2\varepsilon'}}\frac{dx}{x}\Bigg)$$

$$=O\left(\frac{e^{2(m+1)(-\rho+\lambda+2n)}}{e^{2m\alpha}m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\times\right.$$
$$\left.\times\int_{e^m}^{e^{m+1}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{\mathrm{i}\,\gamma^{p,\tau,\lambda}}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\frac{dx}{x}\right)$$
$$=O\left(\frac{e^{2m(-\rho+\lambda+2n)}}{e^{2m\alpha}m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\times\right.$$
$$\left.\times\int_{e^m}^{e^{m+1}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{\mathrm{i}\,\gamma^{p,\tau,\lambda}}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\frac{dx}{x}\right)$$
$$=O\left(\frac{e^{2m(\rho+2n)}}{e^{2m\alpha}m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\times\right.$$
$$\left.\times\int_{e^m}^{e^{m+1}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{x^{\mathrm{i}\,\gamma^{p,\tau,\lambda}}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2\frac{dx}{x}\right).$$

Putting $x=e^{m+2\pi u}$, we obtain

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}$$
$$=O\left(\frac{e^{2m(\rho+2n)}}{e^{2m\alpha}m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\times\right.$$
$$\left.\times\int_{0}^{\frac{1}{2\pi}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{e^{\mathrm{i}\,m\gamma^{p,\tau,\lambda}}e^{\mathrm{i}\,2\pi u\gamma^{p,\tau,\lambda}}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|^2du\right).$$

Now, putting $v=u-\frac{1}{4\pi}$, it follows that

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}$$
$$=O\left(\frac{e^{2m(\rho+2n)}}{e^{2m\alpha}m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\times\right.$$
$$\times\int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{e^{\mathrm{i}\,\gamma^{p,\tau,\lambda}\left(m+\frac{1}{2}\right)}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times\right. \quad (9)$$
$$\left.\left.\times\, e^{2\pi\,\mathrm{i}\,\gamma^{p,\tau,\lambda}v}\right|^2dv\right).$$

Let $V=\theta=\frac{1}{4\pi}$.

Suppose that A is an arbitrary, strictly increasing sequence of real numbers, such that $B\subset A$, where

$$B=\left\{\gamma^{p,\tau,\lambda}\,:\,Y<\left|\gamma^{p,\tau,\lambda}\right|\le T\right\}.$$

Define $c(\nu)=0$ if $\nu\in A\setminus B$, and

$$c\left(\gamma^{p,\tau,\lambda}\right)=\frac{e^{\mathrm{i}\,\gamma^{p,\tau,\lambda}\left(m+\frac{1}{2}\right)}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}$$

for $\gamma^{p,\tau,\lambda}\in B$.

Now,

$$\int_{-V}^{V}\left|\sum_{\nu\in A}c(\nu)\,e^{2\pi\,\mathrm{i}\,\nu v}\right|^2dv$$
$$=\int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}\frac{e^{\mathrm{i}\,\gamma^{p,\tau,\lambda}\left(m+\frac{1}{2}\right)}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times\right. \quad (10)$$
$$\left.\left.\times\, e^{2\pi\,\mathrm{i}\,\gamma^{p,\tau,\lambda}v}\right|^2dv\right).$$

Furthermore,

$$\left(\frac{\pi\theta}{\sin(\pi\theta)}\right)^2\int_{-\infty}^{+\infty}\left|\frac{V}{\theta}\sum_{t\le\nu\le t+\frac{\theta}{V}}c(\nu)\right|^2dt$$
$$=\left(\frac{\frac{1}{4}}{\sin\frac{1}{4}}\right)^2\int_{-\infty}^{+\infty}\left|\sum_{t\le\nu\le t+1}c(\nu)\right|^2dt.$$

Here, if $t=Y-2$, for example, we obtain that $Y-2\le\nu\le Y-1$. Thus, for any $\nu\in A$ such that $Y-2\le\nu\le Y-1$, we have that $\nu\in A\setminus B$, i.e., we have that $c(\nu)=0$.

Similarly, if $t=Y-1$, then $Y-1\le\nu\le Y$. Hence, $c(\nu)=0$ if $\nu\in A$.

However, if $t=Y-1+e$ for some $e>0$, then $Y-1+e\le\nu\le Y+e$. This means that $c(\nu)$ is not necessarily 0 if $\nu\in A$.

Similarly, if $t=T$, then $T\le\nu\le T+1$. Therefore, $c(\nu)$ is not necessarily 0 if $\nu\in A$.

We conclude,

$$\left(\frac{\pi\theta}{\sin(\pi\theta)}\right)^2\int_{-\infty}^{+\infty}\left|\frac{V}{\theta}\sum_{t\le\nu\le t+\frac{\theta}{V}}c(\nu)\right|^2dt$$
$$=\left(\frac{\frac{1}{4}}{\sin\frac{1}{4}}\right)^2\int_{-T-1}^{-Y}\left|\sum_{t\le\nu\le t+1}c(\nu)\right|^2dt+$$
$$\left(\frac{\frac{1}{4}}{\sin\frac{1}{4}}\right)^2\int_{Y-1}^{T}\left|\sum_{t\le\nu\le t+1}c(\nu)\right|^2dt$$

$$=O\Bigg(\int\limits_{-T-1}^{-Y}\Bigg(\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ t\le\gamma^{p,\tau,\lambda}\le t+1\\ -T\le\gamma^{p,\tau,\lambda}<-Y}}\frac{1}{|s^{p,\tau,\lambda}|^{2n+1}}\Bigg)^2dt\Bigg)+$$

$$O\Bigg(\int\limits_{Y-1}^{T}\Bigg(\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ t\le\gamma^{p,\tau,\lambda}\le t+1\\ Y<\gamma^{p,\tau,\lambda}\le T}}\frac{1}{|s^{p,\tau,\lambda}|^{2n+1}}\Bigg)^2dt\Bigg)$$

$$=O\Bigg(\int\limits_{Y-1}^{T+1}\Bigg(\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ t\le\gamma^{p,\tau,\lambda}\le t+1}}\frac{1}{|s^{p,\tau,\lambda}|^{2n+1}}\Bigg)^2dt\Bigg)$$

$$=O\Bigg(\int\limits_{Y-1}^{T+1}\frac{1}{t^{2(2n+1)}}\Bigg(\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ t\le\gamma^{p,\tau,\lambda}\le t+1}}1\Bigg)^2dt\Bigg)$$

$$=O\Bigg(\int\limits_{Y-1}^{T+1}\frac{1}{t^{4n+2}}\Big(N_{p,\tau,\lambda}(t+1)-N_{p,\tau,\lambda}(t)\Big)^2dt\Bigg).$$

As we already noted, $N_{p,\tau,\lambda}(t)=O(t^n)$.

More precisely,

$$N_{p,\tau,\lambda}(t)=D_1t^n+O\left(t^{n-1}(\log t)^{-1}\right)$$

for some explicitly known constant D_1 (see, [3, p. 89, Th. 9.1.]).

The estimate $N_{p,\tau,\lambda}(t)=D_1t^n+O\left(t^{n-1}\right)$, however, will be sufficient for our needs.

We have,

$$\left(\frac{\pi\theta}{\sin(\pi\theta)}\right)^2\int\limits_{-\infty}^{+\infty}\Bigg|\frac{V}{\theta}\sum_{t\le\nu\le t+\frac{\theta}{V}}c(\nu)\Bigg|^2dt$$

$$=O\Bigg(\int\limits_{Y-1}^{T+1}\frac{1}{t^{4n+2}}t^{2(n-1)}\Bigg)^2dt \tag{11}$$

$$=O\Bigg(\int\limits_{Y-1}^{T+1}\frac{dt}{t^{2n+4}}\Bigg)=O\left(\frac{1}{Y^{2n+3}}\right).$$

Applying (2) to (10) and (11), we obtain

$$\int\limits_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\Bigg|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<|\gamma^{p,\tau,\lambda}|\le T}}\frac{e^{\mathrm{i}\,\gamma^{p,\tau,\lambda}\left(m+\frac{1}{2}\right)}}{\prod\limits_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times$$

$$\times e^{2\pi\,\mathrm{i}\,\gamma^{p,\tau,\lambda}v}\Bigg|^2dv\Bigg)=O\left(\frac{1}{Y^{2n+3}}\right).$$

Thus, by (9)

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}$$

$$=O\left(\frac{e^{2m(\rho+2n-\alpha)}}{m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\frac{1}{Y^{2n+3}}\right)$$

$$=O\left(\frac{1}{m(\log m)^{1+2\varepsilon'+(2n+3)\varepsilon''}}\right).$$

Consequently,

$$\mu^{\times}\Delta_{m,1}=O\left(\sum_{p=0}^{n-1}\sum_{(\tau,\lambda)\in I_p}\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}\right)$$

$$=O\left(\frac{1}{m(\log m)^{1+2\varepsilon'+(2n+3)\varepsilon''}}\right),$$

$$\mu^{\times}\left(\bigcup_m\Delta_{m,1}\right)$$

$$=O\left(\sum_m\frac{1}{m(\log m)^{1+2\varepsilon'+(2n+3)\varepsilon''}}\right).$$

Therefore, $\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}<\infty$, $\mu^{\times}\Delta_{m,1}<\infty$, and $\mu^{\times}\left(\bigcup\limits_m\Delta_{m,1}\right)<\infty$.

Now, since $x\in\left[e^m,e^{m+1}\right)$, we can take x to be an element of the set

$$\left[e^m,e^{m+1}\right)\backslash\Delta_{m,1}$$

$$=\bigcap_{p=0}^{n-1}\bigcap_{(\tau,\lambda)\in I_p}\left[e^m,e^{m+1}\right)\backslash\Delta^{p,\tau,\lambda}_{m,1}.$$

Consequently, the sum on the right hand side of (8) that corresponds to $i=2n$, i.e., the sum

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<|\gamma^{p,\tau,\lambda}|\le T}}1\times$$

$$\times\frac{1}{\prod\limits_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}x^{s^{p,\tau,\lambda}+2n}$$

becomes

$$O\left(h^{-2n}x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma+\varepsilon'}\right)$$

$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$

$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$

$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

Now, $x \in [e^m, e^{m+1})$, and $h < x$ yield that

$$x + h < 2x < 2e^{m+1} < e^{m+2}.$$

In other words, $x + h \in [e^m, e^{m+1})$ or $x + h \in [e^{m+1}, e^{m+2})$.

If $x + h \in [e^m, e^{m+1})$, then, we may assume that $x \in [e^m, e^{m+1}) \setminus \Delta_{m,1}$ is selected so that $x + h \in [e^m, e^{m+1}) \setminus \Delta_{m,1}$. In this case, for the sum on the right hand side of (8) that corresponds to $i = 2n - 1$, we have that

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}} 1\times$$
$$\times -\binom{2n}{1} \frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)} (x+h)^{s^{p,\tau,\lambda}+2n}$$
$$=O\Big(h^{-2n}(x+h)^{\alpha}(\log(x+h))^{\beta}\times$$
$$\times(\log\log(x+h))^{\gamma+\varepsilon'}\Big)$$
$$=O\left(h^{-2n}x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma+\varepsilon'}\right)$$
$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

If $x + h \in [e^{m+1}, e^{m+2})$, we write

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}} 1\times$$
$$\times -\binom{2n}{1} \frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)} (x+h)^{s^{p,\tau,\lambda}+2n}$$
$$=\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le e^{m+1}}} 1\times$$
$$\times -\binom{2n}{1} \frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)} (x+h)^{s^{p,\tau,\lambda}+2n}+$$
$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ e^{m}<\left|\gamma^{p,\tau,\lambda}\right|\le e^{m+1}}} 1\times$$
$$\times\binom{2n}{1} \frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)} (x+h)^{s^{p,\tau,\lambda}+2n}.$$

Define

$$\Delta_{m+1,2} = \bigcup_{p=0}^{n-1} \bigcup_{(\tau,\lambda)\in I_p} \Delta^{p,\tau,\lambda}_{m+1,2},$$

where $\Delta^{p,\tau,\lambda}_{m+1,2}$ is the set of all $x \in [e^{m+1}, e^{m+2})$ such that

$$\left|\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ e^{m}<\left|\gamma^{p,\tau,\lambda}\right|\le e^{m+1}}} \frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\right|$$
$$>x^{\alpha}(\log x)^{\beta}(\log\log x)^{\gamma+\varepsilon'}.$$

Since

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m,1}$$
$$=O\left(\frac{e^{2m(\rho+2n-\alpha)}}{m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\frac{1}{Y^{2n+3}}\right),$$

if follows immediately that

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m+1,2}$$
$$=O\left(\frac{e^{2m(\rho+2n-\alpha)}}{m^{2\beta}(\log m)^{2\gamma+2\varepsilon'}}\frac{1}{e^{m(2n+3)}}\right)$$
$$=O\left(\frac{1}{e^{m(-2n+3-2\rho+2\alpha)}}\right).$$

Consequently,

$$\mu^{\times}\Delta_{m+1,2} =O\left(\frac{1}{e^{m(-2n+3-2\rho+2\alpha)}}\right),$$
$$\mu^{\times}\left(\bigcup_{m}\Delta_{m+1,2}\right) =O\left(\sum_{m}\frac{1}{e^{m(-2n+3-2\rho+2\alpha)}}\right).$$

Note that

$$\alpha > n+\rho > n+\rho-\frac{3}{2}.$$

Hence, $-2n + 3 - 2\rho + 2\alpha > 0$.

Therefore, $\mu^{\times}\Delta^{p,\tau,\lambda}_{m+1,2} < \infty$, $\mu^{\times}\Delta_{m+1,2} < \infty$, and $\mu^{\times}\left(\bigcup_{m}\Delta_{m+1,2}\right) < \infty$.

Now for $x + h \in [e^{m+1}, e^{m+2})$, we may assume that $x \in [e^m, e^{m+1}) \setminus \Delta_{m,1}$ is selected so that $x+h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,1}$ and $x+h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,2}$.

We obtain,

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$
$$\times-\binom{2n}{1}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}(x+h)^{s^{p,\tau,\lambda}+2n}$$
$$=O\Big(h^{-2n}(x+h)^{\alpha}(\log(x+h))^{\beta}\times$$
$$\times(\log\log(x+h))^{\gamma+\varepsilon'}\Big)+$$
$$O\Big(h^{-2n}(x+h)^{\alpha}(\log(x+h))^{\beta}\times$$
$$\times(\log\log(x+h))^{\gamma+\varepsilon'}\Big)$$
$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

Consider the sum on the right hand side of (8) that corresponds to $i=2n-2$.

Since $x\in\left[e^{m},e^{m+1}\right)$, and $h<x$ we have that

$$x+2h<3x<e^2e^{m+1}=e^{m+3}.$$

Thus, $x+2h\in\left[e^{m},e^{m+1}\right)$ or $x+2h\in\left[e^{m+1},e^{m+2}\right)$ or $x+2h\in\left[e^{m+2},e^{m+3}\right)$.

If $x+2h\in\left[e^{m},e^{m+1}\right)$, then, we may assume that $x\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$ is selected so that $x+h\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$ and $x+2h\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$. Then,

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$
$$\times\binom{2n}{2}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}(x+2h)^{s^{p,\tau,\lambda}+2n}$$
$$=O\Big(h^{-2n}(x+2h)^{\alpha}(\log(x+2h))^{\beta}\times$$
$$\times(\log\log(x+2h))^{\gamma+\varepsilon'}\Big)$$
$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

If $x+2h\in\left[e^{m+1},e^{m+2}\right)$, then, we may assume that $x\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$ is selected so that $x+h\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$ (if $x+h\in\left[e^{m},e^{m+1}\right)$) and $x+2h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,1}$, $x+2h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,2}$. If $x+h\in\left[e^{m+1},e^{m+2}\right)$, then, we may assume that $x\in\left[e^{m},e^{m+1}\right)\setminus\Delta_{m,1}$ is selected so that $x+h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,1}$, $x+h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,2}$ and $x+2h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,1}$, $x+2h\in\left[e^{m+1},e^{m+2}\right)\setminus\Delta_{m+1,2}$.

In any case,

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$
$$\times\binom{2n}{2}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}(x+2h)^{s^{p,\tau,\lambda}+2n}$$
$$=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le e^{m+1}}}1\times$$
$$\times\binom{2n}{2}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}(x+2h)^{s^{p,\tau,\lambda}+2n}-$$
$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ e^{m}<\left|\gamma^{p,\tau,\lambda}\right|\le e^{m+1}}}1\times$$
$$\times\binom{2n}{2}\frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}(x+2h)^{s^{p,\tau,\lambda}+2n}$$
$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times(\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

If $x+2h\in\left[e^{m+2},e^{m+3}\right)$, then, we write

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}h^{-2n}\sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}}1\times$$

$$\times \binom{2n}{2} \frac{1}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} (x+2h)^{s^{p,\tau,\lambda}+2n}$$

$$= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq e^{m+2}}} 1\times$$

$$\times \binom{2n}{2} \frac{1}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} (x+2h)^{s^{p,\tau,\lambda}+2n} -$$

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ e^m<\left|\gamma^{p,\tau,\lambda}\right|\leq e^{m+2}}} 1\times$$

$$\times \binom{2n}{2} \frac{1}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} (x+2h)^{s^{p,\tau,\lambda}+2n}.$$

Define

$$\Delta_{m+2,3} = \bigcup_{p=0}^{n-1} \bigcup_{(\tau,\lambda)\in I_p} \Delta^{p,\tau,\lambda}_{m+2,3},$$

where $\Delta^{p,\tau,\lambda}_{m+2,3}$ is the set of all $x \in \left[e^{m+3}, e^{m+3}\right)$ such that

$$\left| \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ e^m<\left|\gamma^{p,\tau,\lambda}\right|\leq e^{m+2}}} \frac{x^{s^{p,\tau,\lambda}+2n}}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} \right|$$

$$> x^{\alpha} (\log x)^{\beta} (\log\log x)^{\gamma+\varepsilon'}.$$

Reasoning as before, we obtain that

$$\mu^{\times}\Delta^{p,\tau,\lambda}_{m+2,3}$$

$$= O\left(\frac{e^{2m(\rho+2n-\alpha)}}{m^{2\beta} (\log m)^{2\gamma+2\varepsilon'}} \frac{1}{e^{m(2n+3)}} \right)$$

$$= O\left(\frac{1}{e^{m(-2n+3-2\rho+2\alpha)}} \right).$$

Hence, $\mu^{\times}\Delta^{p,\tau,\lambda}_{m+2,3} < \infty$, $\mu^{\times}\Delta_{m+2,3} < \infty$, and $\mu^{\times}\left(\bigcup_m \Delta_{m+2,3}\right) < \infty$, where

$$\mu^{\times}\left(\bigcup_m \Delta_{m+2,3}\right) = O\left(\sum_m \frac{1}{e^{m(-2n+3-2\rho+2\alpha)}} \right).$$

Now, for $x + 2h \in \left[e^{m+2}, e^{m+3}\right)$, we may assume that $x \in \left[e^m, e^{m+1}\right) \setminus \Delta_{m,1}$ is selected so that $x + h \in \left[e^m, e^{m+1}\right) \setminus \Delta_{m,1}$ (if $x + h \in \left[e^m, e^{m+1}\right)$) and $x + 2h \in \left[e^{m+2}, e^{m+3}\right) \setminus \Delta_{m+2,1}$, $x + 2h \in \left[e^{m+2}, e^{m+3}\right) \setminus \Delta_{m+2,3}$. If $x + h \in \left[e^{m+1}, e^{m+2}\right)$, then, we may assume that $x \in \left[e^m, e^{m+1}\right) \setminus \Delta_{m,1}$ is selected so that $x + h \in \left[e^{m+1}, e^{m+2}\right) \setminus \Delta_{m+1,1}$, $x + h \in \left[e^{m+1}, e^{m+2}\right) \setminus \Delta_{m+1,2}$ and $x + 2h \in \left[e^{m+2}, e^{m+3}\right) \setminus \Delta_{m+2,1}$, $x + 2h \in \left[e^{m+2}, e^{m+3}\right) \setminus \Delta_{m+2,3}$.

In any case,

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq T}} 1\times$$

$$\times \binom{2n}{2} \frac{1}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} (x+2h)^{s^{p,\tau,\lambda}+2n}$$

$$= O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}} \times$$

$$\times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}} \times$$

$$\times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon} \Big).$$

Summarizing what we already derived, we are in position to conclude that $x \in \left[e^m, e^{m+1}\right) \setminus \Delta_{m,1}$ can be chosen so that the sum on the right hand side of (8) that corresponds to i, $i \in \{2n-2, 2n-1, 2n\}$, i.e., the sum

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\leq T}} 1\times$$

$$\times \frac{1}{\prod_{k=0}^{2n} \left(s^{p,\tau,\lambda}+k\right)} \times$$

$$\times \sum_{i=2n-2}^{2n} (-1)^i \binom{2n}{i} (x+(2n-i)h)^{s^{p,\tau,\lambda}+2n}$$

is

$$+ O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}} \times$$

$$\times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}} \times$$

$$\times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon} \Big).$$

Proceeding with the same reasoning as in the case i, $i \in \{2n-2, 2n-1, 2n\}$, we conclude that $x \in$

$\left[e^m, e^{m+1}\right) \setminus \Delta_{m,1}$ can be chosen so that

$$\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} h^{-2n} \sum_{\substack{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{-\rho+\lambda} \\ Y<\left|\gamma^{p,\tau,\lambda}\right|\le T}} 1\times$$
$$\times \frac{1}{\prod_{k=0}^{2n}\left(s^{p,\tau,\lambda}+k\right)}\times$$
$$\times \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \left(x+(2n-i)\,h\right)^{s^{p,\tau,\lambda}+2n}$$
$$=O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big).$$

Put $\nabla = \bigcup_m \Delta_{m,1}$.

As we noted earlier, $\mu^{\times}(\nabla) < \infty$.

Also, note that $\psi_0(x) \le h^{-2n}\Delta^{+}_{2n}\psi_{2n}(x)$.

Taking into account these facts, and letting $x \to \infty$ in (8) along $x \notin \nabla$ (where we assume that $x \notin \nabla$ is selected in the way described above), we finally obtain that

$$\psi_0(x)$$
$$=\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1\times$$
$$\times \sum_{s^{p,\tau,\lambda}\in\left(\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho},2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}+$$
$$O\Big(x^{\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times (\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}}\times$$
$$\times (\log\log x)^{\rho\frac{n-1}{2n^2-2n+4n\rho+\rho}+\varepsilon}\Big)$$

as $x \to +\infty$, $x \notin \nabla$.

This completes the proof. □

3 Final remarks

As it is noted in the proof of Theorem 1,

$$\rho\frac{4n^2-4n+4n\rho+\rho}{2n^2-2n+4n\rho+\rho} \le 2\rho\frac{n+\rho-1}{n+2\rho-1}.$$

Therefore, Theorem 1 refines the corresponding result (39) in [1, p. 317].

For yet another proof of the result (39), see [6].

Note that the inequality (1) is applied in [5]. There, it is proved that the error term in prime number theorem (under Riemann hypothesis assumption), can be reduced except on a set of finite logarithmic measure.

Also, note that an analogue of Theorem 1 is derived in [7], where the author considered the case of compact Riemann surfaces.

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