Iterative Methods for the Solutions of a Predator-Prey Model

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Abstract: - This study considers the Predator-Prey model taking the predator and the prey to be custom officers and vehicle smugglers respectively. For ease of computation, the numerical methods applied are the Adomian Decomposition Method (ADM) and the Picard Iteration Method (PIM). The results obtained via the ADM are compared with those from PIM. The comparison shows that both methods approximate the solutions effectively. Although, Adomian polynomials are required in the case of ADM unlike the PIM.

Key-Words: -
Predator-Prey model; Adomian Decomposition Method; Picard Iteration; Iteration; Approximate solution; species interaction

1 Introduction

The Predator-Prey model represents a pair of first-order, non-linear, differential equations mainly used for the description of dynamical systems in biological sciences in which two species interact: one as a predator and the other as a prey. It has a wide range of applications in Production Engineering, Economics, and so on. The equations were initially proposed by Lotka in the theory of autocatalytic reactions in 1910 [1, 2]. Lotka continued his research through the use of a Kolmogorov model, a more generalized model of organic systems where he used herbivore animal species and a plant species. In 1925, he used the equation to resolve predator-prey interactions which resulted into the predator-prey equations [3, 4].

Many researchers have considered the applications of the model to other areas such as social and applied sciences. In Economics, the predator-prey equations were used by Goodwin [5] in relation to theory of business economic growth while explaining the interactions between various industrial sectors.

In terms of solvability, efficient solution methods need to be considered for solving the differential equations associated with real life problems. Historically, the Adomian Decomposition method (ADM) was introduced by Adomian in 1994 where it was used to solve linear and nonlinear functional equations [6]. The ADM has since been used by various researchers in many fields to solve problems that involve differential equations [7-13].

The Picard Iteration Method (PIM) was used by Saeed *et al.* [14] for linearization of system of differential equations. Different types of non-linear equations can easily be handled by the Haar wavelet Picard technique. Yin, Han, Song, Cao in [15], combined the Picard Iterative method with the Legendre wavelets method in order to solve Non-linear Initial Value Problem, some computational work was done, while keeping the accuracy in check. Bobkov *et al*. [16] used the generalized form of the Picard Iterative method (PIM) to solve stiff problems after which they were compared to the conventional Picard Iterative Method. Other reviews on PIM include [17-21].

2 The Methods of Solution

For simplicity and ease of computation, this section considers the basic concepts of the ADM [6-12] and the PIM [18-21].

2.1 Adomian Decomposition Method (ADM)

Let us consider the differential equation with F as a differential operator, of the form:

$$F_y = g. \tag{1}$$

Suppose F is decomposed as: $F = L + R + N$ then (2.1) becomes:

$$L_y + R_y + N_y = g \tag{2}$$

where L represents an easily invertible linear differential operator, R represents the remaining part of the linear operator, N represents a non-linear operator, and g a source term (not necessarily a function of the dependent variable).

Generally, we choose $L_{\cdot} = \frac{d^n}{dx^n}(\cdot)$, to be the nth-order differential operator and thus its inverse L^{-1} follows as the nth-order integral operator.

Therefore, applying the inverse linear operator L^{-1} to both sides of (2), we have:

$$L^{-1}L_y = L^{-1}g - \left(L^{-1}R_y + L^{-1}N_y\right) \tag{3}$$

where

$$L^{-1}L_y = y - \phi \tag{4}$$

and ϕ signifies the initial value.

Therefore, (3) becomes:

$$y = m(x) - \left[L^{-1}Ry + L^{-1}Ny\right] \tag{5}$$

where

$$m(x) = L^{-1}g + \phi \ , \tag{6}$$

which signifies a function obtained by integrating the source term $g(x)$ with respect to the initial condition(s).

The ADM expresses the solution $y(t)$ in series form:

$$y = \sum_{n=0}^{\infty} y_n \ . \tag{7}$$

Also, the non-linear term can be expressed as Adomian polynomials:

$$Ny = \sum_{n=0}^{\infty} A_n. \tag{8}$$

The Adomian polynomials, A_n are dependent on the values of $y_0, y_1, y_2, ..., y_n$ and are obtained for the nonlinearity $Ny = f(y)$ by the formula:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,... \tag{9}$$

Thus,

$$\sum_{n=0}^{\infty} y_n = \varphi(x) - \left[L^{-1}R\sum_{n=0}^{\infty} y_n + L^{-1}\sum_{n=0}^{\infty} A_n\right] \tag{10}$$

The associated recursive equation is:

$$\begin{cases} y_0(x) = \varphi(x) \\ y_{n+1} = -L^{-1}\left[Ry_n + A_n\right]. \end{cases} \tag{11}$$

After several iterations, the nth term series approximation of the differential equation is

$$y = \lim_{N\to\infty}\left(\sum_{k=0}^{N} y_k(x)\right). \tag{12}$$

2.2 Picard Iteration Method

The PIM is an integral method used for differential equations with emphasis on the existence and uniqueness of solutions of the differential equations, hence, an equation to be solved by the PIM must satisfy the Lipchitz continuity condition.

2.3 Lipschitz Continuity Condition

A function $f(x, y)$ is said to satisfy the Lipchitz condition with respect to y in a region D in the XY-plane, if there exists a positive constant L such that

$$\left|f(x, y_a) - f(x, y_b)\right| \le L\left|y_a - y_b\right|$$

whenever (x, y_a) and (x, y_b) are in D, L is called the Lipchitz constant. The PIM associated with the IVP:

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{13}$$

is given as follows [20, 21]:

$$\phi_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(t, \phi_n(t))dt \tag{14}$$

where $y(x) = \phi_{n+1}(x)$ and $y(t) = \phi_n(t)$.

3 Formulation of Model Equation

Global criminal activities such as smuggling exercise in Nigeria is gradually becoming deadlier and more puzzling as the smugglers are formulating new methods of bringing contraband into the country. This needs to be frequently checked by legal authorities [22]. Here, the predator-prey model is considered based on the following assumptions (A1-A4).

Let $N(t)$ be the number of vehicle smugglers at time, t and $P(t)$ the number (population) size of custom officers at time t , then the total number of population at time, t is:

$$M(t) = N(t) + P(t). \tag{15}$$

Consider the following:

A1: The population of vehicle smugglers $(N(t))$ grows exponentially in the absence of custom officers. This implies that:

$$\frac{dN}{dt} = \alpha N \quad (16)$$

A2: In the absence of vehicle smugglers, the size of custom officers decrease exponentially, hence:

$$\frac{dP}{dt} = -\gamma P \quad (17)$$

A3: Growth in population size of custom officers is proportional to the number of death among vehicle smugglers. This implies that:

$$\frac{dP}{dt} = \delta NP \quad (18)$$

A4: A reduction in the population size of vehicle smugglers is proportional to the meeting between the vehicle smugglers and the custom officers. This implies that:

$$\frac{dN}{dt} = -\beta NP \quad (19)$$

So combining equations (16-19), we have:

$$\begin{cases} \dfrac{dP}{dt} = -\gamma P + \delta NP \\ \dfrac{dN}{dt} = \alpha N - \beta NP \end{cases} \quad (20)$$

where the constants $\alpha, \beta, \gamma, \delta \geq 0$. are such that α and β are prey's birth and death rates respectively, δ and γ represent the predator's growth and death rates respectively. Solutions of differential models of the forms (20) can be approximated by numerical, and/or semi-analytical methods [23-35].

4 Solution to the Model

In this subsection, the two methods of solutions (ADM & PIM) are used for numerical solutions of the Predator-prey model (20).

Case 1:

PIM -Taking the integral of both sides of equations (20) gives:

$$\begin{cases} N(t) = N(0) + \int_0^t \left(\alpha N(s) - \beta N(s) P(s)\right) ds \\ P(t) = P(0) + \int_0^t \left(-\gamma P(s) + \delta N(s) P(s)\right) ds \end{cases} \quad (21)$$

$$\Rightarrow \begin{cases} N_{k+1} = N(0) + \int_0^t \left(\alpha N_k - \beta N_k P_k\right) ds \\ P_{k+1} = P(0) + \int_0^t \left(-\gamma P_k + \delta N_k P_k\right) ds. \end{cases} \quad (22)$$

For $k = 0, 1, 2, \cdots,$ and by using the following parameters in *Case 1*:

$$\begin{pmatrix} N(0) = N_0 = 4,\ P(0) = 9,\ \alpha = 0.1, \\ \beta = 0.0014,\ \gamma = 0.0012,\ \delta = 0.08,\ h = 0.1 \end{pmatrix},$$

we have the following:

$$N_0 = 4,\ P_0 = 9$$

$$N_1 = 4 + 0.34960t,\ P_1 = 9 + 2.8692t$$

$$\begin{cases} N_2 = 4 + 0.34960t + 0.014488t^2 - 0.0014043t^3 \\ P_2 = 9 + 2.8692t + 1.1664t^2 + 0.080246t^3 \end{cases}$$

$$N_3 = \begin{cases} 4 + 0.34960t + 0.014488t^2 - 0.0066700t^3 \\ -0.0012012t^4 - 0.000057292t^5 \\ +6.6561 \times 10^{-7} t^6 + 1.5776 \times 10^{-7} t^7 \end{cases}$$

$$P_3 = \begin{cases} 9 + 2.8692t + 1.1664t^2 - 0.46253t^3 - 0.060519t^4 \\ +0.0032739t^5 - 0.000038035t^6 - 0.0000090151t^7 \end{cases}$$

$$N(t)^{PIM-C1} = \begin{cases} 4 + 0.34960t + 0.014488t^2 \\ -0.0066700t^3 - 0.0012012t^4 \\ -0.000057292t^5 + 6.6561 \times 10^{-7} t^6 \\ +1.5776 \times 10^{-7} t^7 \end{cases} \quad (22)$$

$$P(t)^{PIM-C1} = \begin{cases} 9 + 2.8692t + 1.1664t^2 - 0.46253t^3 \\ -0.060519t^4 + 0.0032739t^5 \\ -0.000038035t^6 - 0.0000090151t^7. \end{cases} \quad (23)$$

Similarly, the ADM is applied as follows:

Suppose $L(\bullet) = \dfrac{d}{dt}$ then, equation (20) becomes:

$$\begin{cases} LN = \alpha N - \beta NP \\ LP = -\gamma P + \delta NP \end{cases}. \quad (24)$$

According to ADM, with the series solutions expressed as $N(t) = \sum_{k=0}^{\infty} N_k$ and $P(t) = \sum_{k=0}^{\infty} P_k$, equation (24) becomes:

$$\begin{cases} \sum_{k=0}^{\infty} N_k = N_*(0) + \alpha L^{-1}\left(\sum_{k=0}^{\infty} N_k\right) - \beta L^{-1}\left(\sum_{k=0}^{\infty} A_k\right) \\ \sum_{k=0}^{\infty} P_k = P_*(0) - \gamma L^{-1}\left(\sum_{k=0}^{\infty} P_k\right) + \delta L^{-1}\left(\sum_{k=0}^{\infty} B_k\right) \end{cases} \tag{25}$$

where $A_k = B_k$ are Adomian Polynomials. Thus, simplifying (25) gives the recursive relations:

$$\begin{cases} N_0 = N_*(0) \\ N_{k+1} = \alpha L^{-1}(N_k) - \beta L^{-1}(A_k) \end{cases} \tag{26}$$

and

$$\begin{cases} P_0 = P_*(0) \\ P_{k+1} = -\gamma L^{-1}(P_k) + \delta L^{-1}(B_k) \end{cases}. \tag{27}$$

Therefore, for $k \in \mathbb{N}$ and using the parameters as in *Case 1*, we obtain the following:

$N_0 = 4,\ P_0 = 9$

$N_1 = 0.3496t,\ P_1 = 2.8692t$

$N_2 = 0.01448752t^2,\ P_2 = 1.166413t^2$, and so on.

Thus,

$$\begin{cases} N(t)^{ADM} = N_0 + N_1 + N_2 + N_3 + \cdots \\ P(t)^{ADM} = P_0 + P_1 + P_2 + P_3 + \cdots \end{cases} \tag{28}$$

$$\begin{cases} N(t)^{ADM-C1} = \begin{pmatrix} 4 + 0.3496t + 0.01448752000t^2 \\ -0.006670004576t^3 \\ -0.003802205997t^4 \end{pmatrix} \\ P(t)^{ADM-C1} = \begin{pmatrix} 9 + 2.8692t + 1.166412960t^2 \\ +0.462592516t^3 + 0.1785995671t^4 \end{pmatrix}. \end{cases} \tag{29}$$

Case 2

Using PIM with the following parameters, we have:

$N(0) = N_0 = 24,\ P(0) = 11,\ \alpha = 0.05,\ \beta = 0.0032,\ \gamma = 0.2,\ \delta = 0.08,\ h = 1$

$$\begin{cases} N(t)^{PIM-C2} = \begin{cases} 24 + 0.35520t - 1.4478t^2 \\ -2.5662t^3 + 0.0087029t^4 \\ +0.15291t^5 + 0.0047518t^6 \\ +0.000036998t^7 \end{cases} \\ P(t)^{PIM-C2} = \begin{cases} 11 + 18.920t + 32.855t^2 \\ +55.774t^3 - 0.35198t^4 \\ -3.8227t^5 - 0.11879t^6 \\ -0.00092495t^7. \end{cases} \end{cases} \tag{30}$$

Similarly, using ADM with the same parameter in *Case 2,* we have the solutions as follows:

$$\begin{cases} N(t)^{ADM-C2} = \begin{pmatrix} 24 + 0.3552t \\ -1.447799040t^2 \\ -2.566194812t^3 + \cdots \end{pmatrix}, \\ P(t)^{ADM-C2} = \begin{pmatrix} 11 + 18.92t + 32.85497600t^2 \\ +55.77412629t^3 + \cdots \end{pmatrix}. \end{cases} \tag{31}$$

The approximate results for both cases are presented graphically in Fig. 1-Fig.4 as follows.

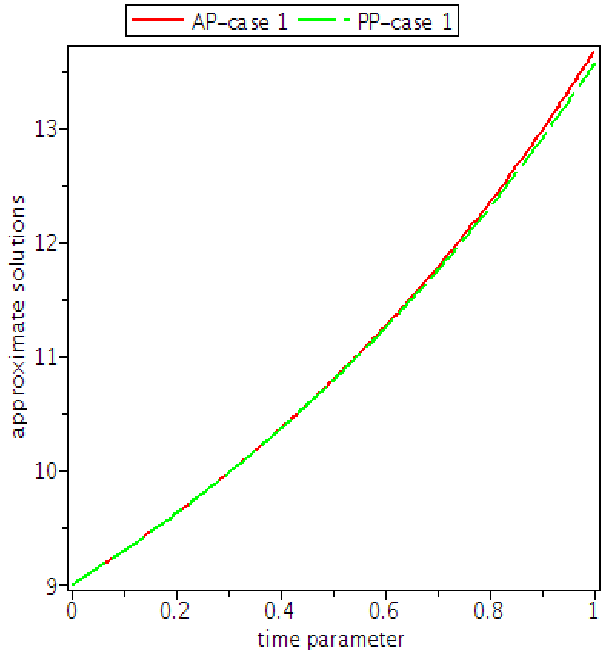

Fig 1: ADM vs.PIM (P-Values-Case 1)

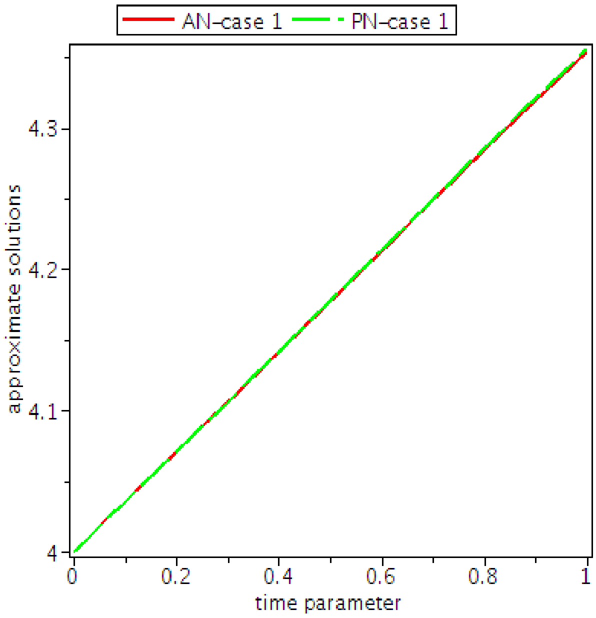

Fig 2: ADM vs. PIM (N-Values- Case 1)

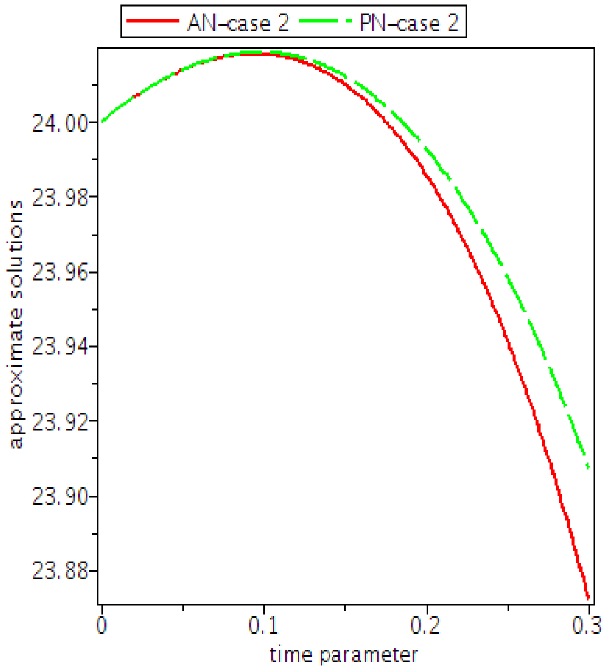

Fig. 4: ADM vs. PIM (N-Values Case 2)

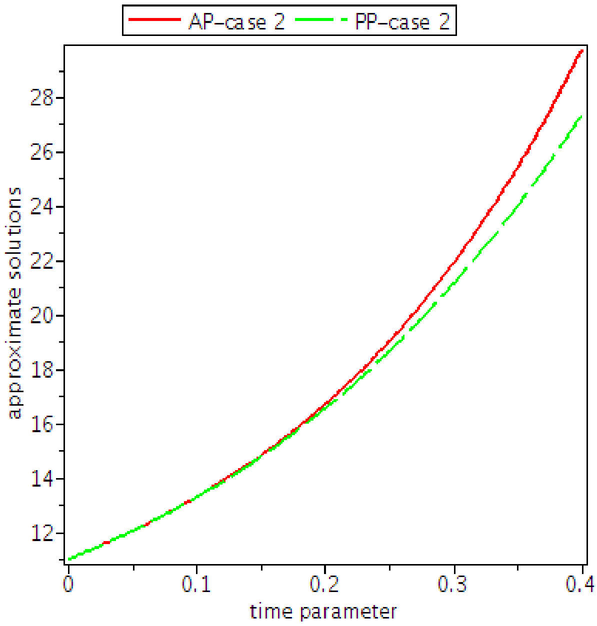

Fig. 3: ADM vs. PIM (P-Values-Case 2)

4 Conclusion

The dynamics of predator-prey model are in a constant cycle of growth and decline. The existence and growth of the predators are depended on the availability of the number of preys existing in the population and vice versa. The ADM and the PIM have been applied successfully in solving the Predator-prey model which is a system of nonlinear differential equation. Both methods yield good approximation; though, the PIM transforms the differential equation to its equivalent in integral form provided the Lipschitz continuity condition is satisfied. The methods can also be extended to nonlinear models of higher order.

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