Interval Estimation Using Integro-Differential Splines of the Third Order of Approximation

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Abstract: Interpolation by local splines in some cases gives a better result than other splines or interpolation by classical interpolation polynomials. Integro-differential splines are one of the types of local splines that use, in addition to the values of the functions at the nodes of the grid, integrals over grid intervals. To construct an approximation on a finite interval, in order to improve the approximation quality, we use left or right integro-differential splines near the ends of this interval. At some distance from the ends, besides the left or right splines, we can also use the middle integro-differential splines. Sometimes it is not necessary to calculate the approximation of the function at intermediate points in $[x_j, x_{j+1}]$. Instead of calculating the approximation in many points in $[x_j, x_{j+1}]$ it is sufficient to estimate only the upper and lower boundaries of the variety of the approximation on this interval. The paper discusses the estimation of the boundaries of approximation of functions using left, right and middle trigonometrical integro-differential splines of the third order of approximation. The process of constructing the basic splines is discussed. The approximation theorems are given. Unimprovable constants in the approximation inequalities are given. Numerical examples of construction of the approximations and interval estimation are given.

Key-Words: Interpolation, interval estimation, trigonometrical integro-differential splines

1 Introduction

In some cases, it is required not to calculate an approximate solution of the problem, but to quickly obtain estimates of the upper and lower boundaries of the solution. Interval analysis methods make it easy to solve this problem. Nowadays, there are many well-known monographs on interval analysis (see [1]-[9]). Of particular interest is the representation of the behaviour of a function based on the values of this function in the grid nodes. In [10] for the quadratic Lagrange interpolation function, an algorithm is proposed to provide explicit and verified bound for the interpolation error constant that appears in the interpolation error estimation. Paper [11] shows the results of using interval analysis for interval estimation of approximations with polynomial splines of the fifth order. It is also important to have information about the uncertainty of the forecast. In paper [12] estimation of optimal prediction intervals is discussed.

Paper [13] discusses the use of integro-differential splines for solving statistical problems. This paper is a continuation of the cycle of papers (see [14]-[18]) devoted to the construction and study of the properties of polynomial and nonpolynomial integro-differential splines.

2 Third-order Spline Approximation

2.1 Left and Right Splines

Let function $u(x)$ be such that $u \in C^3[\alpha,\beta]$. The set of nodes $\{x_k\} \in [\alpha,\beta]$. The formulas of the right local integro-differential splines $w_j(x)$, $w_{j+1}(x), w_j^{<1>}(x)$ on an interval $[x_j, x_{j+1}]$ are obtained by solving the following system of equations:

$$\varphi_i(x_j)w_j(x) + \varphi_i(x_{j+1})w_{j+1}(x) + \int_{x_{j+1}}^{x_{j+2}} \varphi_i(t)dt\, w_j^{<1>}(x) = \varphi_i(x),$$

$$i = 0, 1, 2.$$

The formulae of the left local integro-differential splines $v_j(x)$, $v_{j+1}(x)$, $v_j^{<-1>}(x)$ on the interval $[x_j, x_{j+1}]$ are obtained by solving the following system of equations:

$$\varphi_i(x_j)v_j(x) + \varphi_i(x_{j+1})v_{j+1}(x) + \int_{x_{j-1}}^{x_j} \varphi_i(t)dt\, v_j^{<-1>}(x) = \varphi_i(x),$$
$$i = 0, 1, 2.$$

As it was shown in previous papers, the system of functions $\{\varphi_i\}$ should be the Chebyshov system on the interval $[\alpha, \beta]$.

Based on different systems $\{\varphi_i\}$ we get different basis functions $w_j(x)$, $w_{j+1}(x)$, $w_j^{<1>}(x)$ or $v_j(x)$, $v_{j+1}(x)$, $v_j^{<-1>}(x)$. We construct the approximation of function $u(x)$ with the right integro-differential splines on the interval $[x_j, x_{j+1}]$ in the form:

$$U(x) = u(x_j)w_j(x) + u(x_{j+1})w_{j+1}(x) + \int_{x_{j+1}}^{x_{j+2}} u(t)dt\, w_j^{<1>}(x),$$

as well as we construct the approximation of function $u(x)$ with the left integro-differential splines $v_j(x)$, $v_{j+1}(x)$, $v_j^{<-1>}(x)$ on the interval $[x_j, x_{j+1}]$ in the form:

$$V(x) = u(x_j)v_j(x) + u(x_{j+1})v_{j+1}(x) + \int_{x_{j-1}}^{x_j} u(t)dt\, v_j^{<-1>}(x).$$

Using the results of paper [14] it is easy to formulate the theorem of approximation with this splines.

Theorem 1. Let $\varphi_0 = 1, \varphi_1 = \sin(x), \varphi_2 = \cos(x)$, and function $u(x)$ be such that $u \in C^3[\alpha, \beta]$. Suppose $\{x_k\} \in [\alpha, \beta]$ is the set of nodes, such that $x_{j+1} - x_j = h$, $h < 1$. Then for $x \in [x_j, x_{j+1}]$ we have:

$$|u(x) - U(x)| \le K_1 h^3 \parallel u''' + u' \parallel_{[x_j, x_{j+2}]},$$

$$|u(x) - V(x)| \le K_1 h^3 \parallel u''' + u' \parallel_{[x_{j-1}, x_{j+1}]},$$

where $K_i = 0.056$.

Proof. Using representation (see [14])

$$u(x) = 2\int_{x_j}^{x} (u'''(x) + u'(x))\sin^2\frac{x-t}{2}dt + c_1\sin(x) + c_2\cos(x) + c_3,$$

(c_i are arbitrary constants), and the formulae for the left or right integro-differential splines we receive the estimation of the error of approximation.

The factual and theoretical errors of the approximation, with the left integro-differential splines, are presented in Table 1. In the second column of Table 1 the maximums of the factual errors of approximations in absolute values are given. In the third column of Table 1 the maximums of the theoretical errors of approximations in absolute values are done. Calculations were made in Maple with Digits=15, $[a, b] = [-1,1], h = 0.1$.

Table 1. Factual and theoretical errors of the approximation with the left integro-differential splines

$u(x)$	Factual error	Theoretical error
x^3	0.00046	0.00050
$\sin(3x)$	0.0012	0.0013
$\sin(x) - \cos(x) + x$	0.000052	0.000056
$\sin(7x) - \cos(9x)$	0.053	0.058
$\sin(2x) - \cos(x)$	0.00031	0.00034

In the next subsection we consider interval estimation using trigonometrical integro-differential splines of the third order of approximation. As it is known, the goal of estimating by means of interval arithmetic is to obtain a narrower estimating interval. Our goal is to compare the widths of the resulting evaluation intervals. First, consider the interval estimation using the left and the right trigonometrical splines.

2.2 Interval Estimation Using the Left and the Right Trigonometrical Splines

We first write out the approximation formulae by the left and the right interpolation polynomial integro-differential splines. For this we take $\varphi_0 = 1, \varphi_1 = \sin(x), \varphi_2 = \cos(x)$. Formulae of the right polynomial integro-differential splines $w_j(x)$, $w_{j+1}(x)$, $w_j^{<1>}(x)$ have the form:

$$w_j(x) = (-1 + \cos(x_{j+1} - x_{j+2}) + +(x_{j+2} - x_{j+1})\sin(x - x_{j+1}) + \cos(x - -x_{j+1}) - \cos(x - x_{j+2}))/(-1 + \cos(x_{j+1} - -x_{j+2}) + (x_{j+2} - x_{j+1})\sin(x_j - x_{j+1}) + +\cos(x_j - x_{j+1}) - \cos(x_j - x_{j+2})),$$

$$w_{j+1}(x) = (-\cos(x - x_{j+1}) + \cos(x - x_{j+2}) + \\ +(x_{j+1}-x_{j+2})\sin(x - x_j) + \cos(x_j - x_{j+1}) - \\ -\cos(x_j - x_{j+2}))/(-1 + \cos(x_{j+1} - x_{j+2}) + \\ +(x_{j+2} - x_{j+1})\sin(x_j - x_{j+1}) + \cos(x_j - x_{j+1}) - \\ -\cos(x_j - x_{j+2})),$$

$$w_j^{<1>}(x) = (-\sin(x - x_{j+1}) + \sin(x_j - x_{j+1}) + \\ +\sin(x - x_j))/(-1 + \cos(x_{j+1} - x_{j+2}) + \\ +(x_{j+2} - x_{j+1})\sin(x_j - x_{j+1}) + \cos(x_j - x_{j+1}) - \\ -\cos(x_j - x_{j+2})).$$

For approximation, the form of the record is preferable under the following notation $x = x_j + th$, $t \in [0,1]$, $x_{j+1} - x_j = x_j - x_{j-1} = h$, thus we have

$$w_j(x_j + th) = (-1 + \cos(h) + h\sin(th - h) + \\ +\cos(th - h) - \cos(th - 2h))/(-1 + 2\cos(h) - \\ -h\sin(h) - \cos(2h)),$$

$$w_{j+1}(x_j + th) = -(-\cos(th)\cos(h) - \\ -\sin(th)\sin(h) + 2\cos(th)\cos^2(h) - \cos(th) + \\ +2\sin(th)\sin(h)\cos(h) + \cos(h) - 2\cos^2(h) + \\ +1 - h\sin(th))/(-2\cos(h) + h\sin(h) + \\ +2\cos^2(h)),$$

$$w_j^{<1>}(x_j + th) = (-\sin(th) + \sin(th)\cos(h) - \\ -\cos(th)\sin(h) + \sin(h))/(-2\cos(h) + \\ +h\sin(h) + 2\cos^2(h)).$$

Graphs of basic splines are shown in Fig. 1-3.

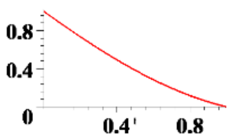

Fig. 1. Plot of the basic function $w_j(x)$ $(h = 1)$.

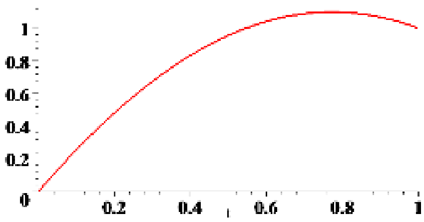

Fig. 2. Plot of the basic function $w_{j+1}(x)$ $(h = 1)$.

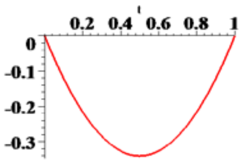

Fig. 3. Plot of the basic function $w_j^{<1>}(x)$ $(h = 1)$.

Fig. 4 shows the graph of the function $u(x) = \sin(7x) - \cos(9x)$, and its approximations with right trigonometrical splines on the interval $[0, \pi]$ with the number of nodes $n = 15$. Figs. 5, 6 show the graphs of the error of approximation of functions $u(x) = \sin(7x) - \cos(9x)$, $u(x) = \sin(2x) - \cos(x)$ on the interval $[0, \pi]$ with the number of nodes $n = 15$.

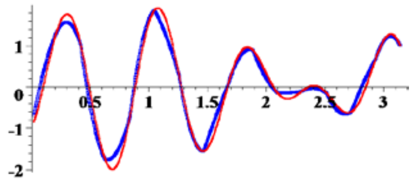

Fig. 4. Plot of the function $u(x) = \sin(7x) - \cos(9x)$ and its approximation with the right trigonometrical splines

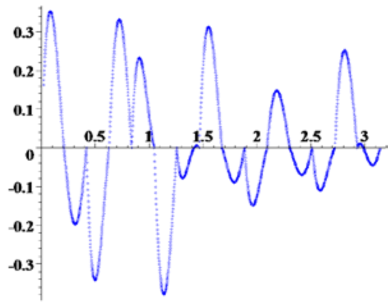

Fig. 5. Plot of the error of approximation with the right trigonometrical splines of $u(x) = \sin(7x) - \cos(9x)$

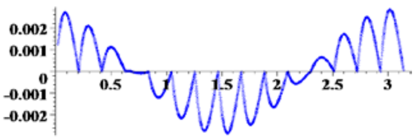

Fig. 6. Plot of the error of approximation with the right trigonometrical splines of function $u(x) = \sin(2x) - \cos(x)$

Formulae of the left trigonometrical integro-differential splines $v_j(x), v_{j+1}(x)$, $v_j^{<-1>}(x)$ have the form:

$v_j(x) = N_j(x)/Z_j(x)$, where

$$N_j(x) = \cos(x_{j-1} - x_{j+1}) - \cos(x_j - x_{j+1}) + \\ +(x_{j-1}-x_j)\sin(x - x_{j+1}) - \cos(x - x_{j-1}) + \\ +\cos(x - x_j),$$

$$Z_j(x) = \cos(x_{j-1} - x_{j+1}) - \cos(x_{j+1} - x_j) + \\ +(x_{j-1} - x_j)\sin(x_j - x_{j+1}) - \cos(x_j - x_{j-1}) + 1,$$

and $v_{j+1}(x) = N_{j+1}(x)/Z_{j+1}(x)$, where

$$N_{j+1}(x) = -\cos(x - x_{j-1}) + \cos(x - x_j) + +(x_{j-1} - x_j)\sin(x - x_j) + \cos(x_j - x_{j-1}) - 1,$$

$$Z_{j+1}(x) = -\cos(x_{j-1} - x_{j+1}) + \cos(x_j - x_{j+1}) + +(x_j - x_{j-1})\sin(x_j - x_{j+1}) + \cos(x_j - x_{j-1}) - 1,$$

and $v_j^{<-1>}(x) = N(x)/Z(x)$, where

$$N(x) = -\sin(x - x_{j+1}) + \sin(x_j - x_{j+1}) + \sin(x - x_j),$$

$$Z(x) = -\cos(x_{j-1} - x_{j+1}) + \cos(x_j - x_{j+1}) + +(x_j - x_{j-1})\sin(x_j - x_{j+1}) + \cos(x_j - x_{j-1}) - 1.$$

For approximation, the following form of the record is preferable under the notation $x = x_j + th$, $t \in [0,1]$, $x_{j+1} - x_j = x_j - x_{j-1} = h$:

$$v_j(x_j + th) = (\cos(2h) - \cos(h) - -h\sin(th - h) - \cos(th + h) + \cos(th))/ (\cos(2h) - 2\cos(h) + h\sin(h) + 1),$$

$$v_{j+1}(x_j + th) = -(-1 + \cos(h) - -\cos(th)\cos(h) + \sin(th)\sin(h) + \cos(th) - -h\sin(th))/(2(\cos^2(h) - 2\cos(h) + h\sin(h)),$$

$$v_j^{<-1>}(x_j + th) = (\sin(h) + \sin(th)\cos(h) - -\cos(th)\sin(h) - \sin(th))/(2(\cos^2(h) - -2\cos(h) + h\sin(h)).$$

Graphs of the basic splines are shown in Fig.7-9.

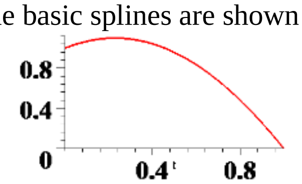

Fig. 7. Plot of the basic function $v_j(x)$ $(h = 1)$.

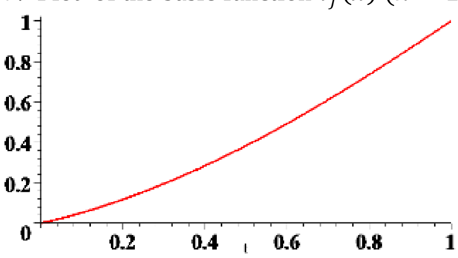

Fig. 8. Plot of the basic function $v_{j+1}(x)$ $(h = 1)$.

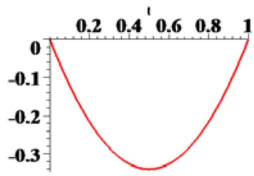

Fig. 9. Plot of the basic function $v_j^{<-1>}(x)$ $(h = 1)$.

Fig. 10 shows the graph of the function $u(x) = \sin(7x) - \cos(9x)$, and its approximations with left trigonometrical splines on the interval $[0, \pi]$ with the number of nodes $n = 15$. Figs.11, 12 show graphs of the error of approximation of functions $u(x) = \sin(7x) - \cos(9x)$, $u(x) = \sin(2x) - \cos(x)$ on the interval $[0, \pi]$ with the number of nodes $n = 15$.

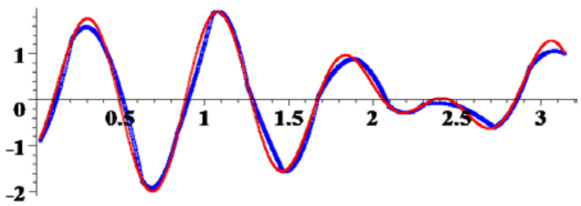

Fig. 10. Plot of the function $u(x) = \sin(7x) - \cos(9x)$ and its approximation with the left trigonometrical splines

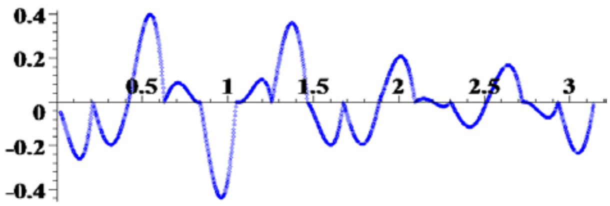

Fig. 11. The error of approximation with the left trigonometrical splines of function $u(x) = \sin(7x) - \cos(9x)$

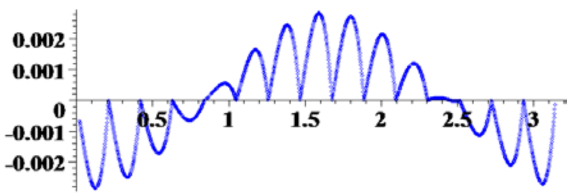

Fig. 12. The error of approximation of function $u(x) = \sin(2x) - \cos(x)$

In some cases, it is interesting not to calculate the approximate solution, but to estimate (possibly with a large margin) the range of variation of the solution. Knowing the values of the function at the nodes of the grid, as well as the minimum and maximum values of the basis functions on each grid interval, it is easy to construct the largest and smallest boundary of the function variation. Thus, for example, to obtain the upper boundary of a variation of a function using the right splines, it suffices to calculate the sum $1.0988u(x_j) + u(x_{j+1})$, because

$$\max_{t\in[0,1]} w_j\,(x_j + th) = 1.0988\ \forall h,$$

$$\max_{t\in[0,1]} w_{j+1}\,(x_j + th) = 1\ \forall h,$$

$$\max_{t\in[0,1]} w_j^{<1>}\,(x_j + th) = 0\ \forall h.$$

We can use suitable quadrature formulae. Applying the left splines on intervals of increasing the function and the right splines on intervals of decreasing the function, we obtain the area of variety of this function.

Fig. 13 shows the plot of the largest and smallest boundaries of function $u(x) = \sin(7x) - \cos(9x)$ and points of the function when $n = 15, a = 0, b = \pi$. Fig. 14 additionally presents the graph of the function $u(x)$.

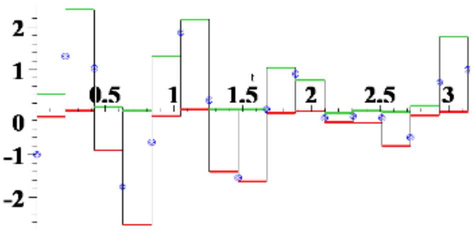

Fig. 13. Plot of the largest and smallest boundaries of function $u(x) = \sin(7x) - \cos(9x)$ and values of the function in nodes.

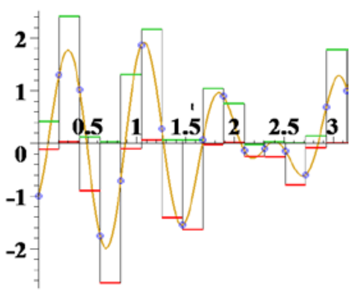

Fig. 14. Plot of the function $u(x) = \sin(7x) - \cos(9x)$ and the largest and smallest boundaries of this function.

Fig. 14 shows the plot of the largest and smallest boundaries of function $u(x) = \sin(2x) - \cos(x)$, values of the function in nodes and the graph of the function $u(x)$ when $n = 15, a = 0, b = \pi$.

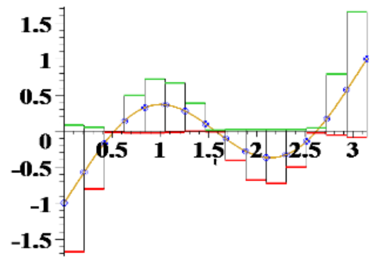

Fig. 15. The largest and smallest boundaries of function $u(x) = \sin(2x) - \cos(x)$ and values of the function in nodes.

Consider an example from statistics. Let the function have the form $F = f_1 + f_2$, where

$$f_i = \frac{1}{\sigma_i\sqrt{2\pi}}\, e^{-(x-\alpha_i)^2/2(\sigma_i)^2},\ \sigma_1 = 0.5, \sigma_2 = 0.8,$$

$$\alpha_1 = -0.8, \alpha_2 = 1.$$

The error of approximation of the function $F(x)$ with the left splines is given in Fig. 16. Fig. 17 shows plot of the largest and smallest boundaries of the function $F(x)$.

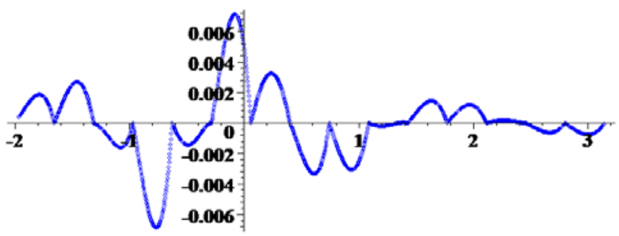

Fig. 16. The error of approximation of function $F(x)$.

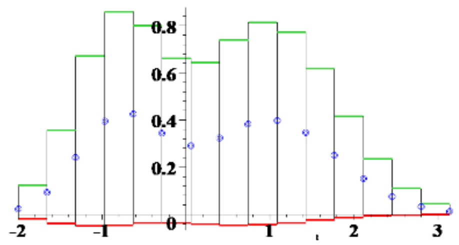

Fig. 17. The largest and smallest boundaries of the function $F(x)$ and values of the function in nodes.

3. Middle Splines

Now consider the middle variant of integro-differential trigonometrical splines:

$$S(x) = u(x_j)s_j(x) + u(x_{j+1})s_{j+1}(x) + \int_{x_j}^{x_{j+1}} u(t)dt\ s_j^{<0>}(x),$$

where the basic splines $s_j(x)$, $s_{j+1}(x), s_j^{<0>}(x)$ are the following:

$$s_j(x) = (-\cos(x_j - x_{j+1}) + 1 + (x_{j+1} - \\ -x_j)\sin(x - x_{j+1}) + \cos(x - x_j) - \cos(x - \\ -x_{j+1}))/(-2\cos(x_j - x_{j+1}) + 2 + x_{j+1}\sin(x_j - \\ -x_{j+1}) - x_j\sin(x_j - x_{j+1})),$$

$$s_{j+1}(x) = (-\cos(x - x_{j+1}) + \cos(x - x_{j+1}) + \\ +(x_j - x_{j+1})\sin(x - x_j) + 1 - \cos(x_j - \\ -x_{j+1}))/(-2\cos(x_j - x_{j+1}) + 2 + x_{j+1}\sin(x_j - \\ -x_{j+1}) - x_j\sin(x_j - x_{j+1})),$$

$$s_{j+1}^{<0>}(x) = (-\sin(x - x_{j+1}) + \sin(x_j - x_{j+1}) + \\ +\sin(x - x_j))/(2\cos(x_j - x_{j+1}) + 2 + \\ +x_{j+1}\sin(x_j - x_{j+1}) - x_j\sin(x_j - x_{j+1})).$$

For approximation, the form of the record is preferable under the following notation $x = x_j + th$, $t \in [0,1]$, $x_{j+1} - x_j = x_j - x_{j-1} = h$:

$$s_j(x_j + th) = (\cos(h) - 1 - h\sin(th - h) - \\ -\cos(th) + \cos(th - h))/(2\cos(h) - 2 + \\ +h\sin(h)),$$

$$s_{j+1}(x_j + th) = \\ -(1 - \cos(th) + \\ +\cos(th)\cos(h) + \sin(th)\sin(h) - \cos(h) - \\ -h\sin(th))/(2\cos(h) - 2 + h\sin(h)),$$

$$s_j^{<0>}(x_j + th) = \\ (\sin(th)\cos(h) - -\cos(th)\sin(h) - \sin(th) + \\ \sin(h))/(2\cos(h) - 2 + h\sin(h)).$$

These splines we get by solving the system of equations:

$$\varphi_i(x_j)s_j(x) + \varphi_i(x_{j+1})s_{j+1}(x) \\ + \int_{x_j}^{x_{j+1}} \varphi_i(t)dt\, s_j^{<0>}(x) = \varphi_i(x),$$

$$i = 0, 1, 2.$$

Theorem 2. Let $\varphi_0 = 1, \varphi_1 = \sin(x), \varphi_2 = \cos(x)$, and function $u(x)$ be such that $u \in C^3[\alpha,\beta]$. Suppose $\{x_k\} \in [\alpha,\beta]$ is the set of nodes, such that $x_{j+1} - x_j = h$, $(h < 1)$. Then for $x \in [x_j, x_{j+1}]$ we have:

$$|u(x) - S(x)| \leq K_2 h^3 \parallel u''' + u' \parallel_{[x_j, x_{j+1}]}$$

where $K_2 = 0.0082$.

Proof. Using representation (see [14])

$$u(x) = 2\int_{x_j}^{x} (u'''(x) + u'(x))\sin^2\frac{x - t}{2}dt \\ + c_1\sin(x) + c_2\cos(x) + c_3,$$

and the formulae for the middle integro-differential splines we receive the estimation of the error of approximation.

The factual and theoretical errors of the approximation, with the middle integro-differential splines, are presented in Table 2. In the second column of Table 2 the maximums of the factual errors of approximations in absolute values are given. In the third column of Table 2 the maximums of the theoretical errors of approximations in absolute values are done. Calculations were made in Maple with $Digits = 15, [a, b] = [-1,1], h = 0.1$.

Table 2. Factual and theoretical errors of the approximation, with the middle integro-differential splines

$u(x)$	Factual error	Theoretical error
x^3	0.000070	0.000074
$\sin(3x)$	0.00019	0.000195
$\sin(x) - \cos(x) + x$	0.0000080	0.0000082
$\sin(7x) - \cos(9x)$	0.00814	0.00852
$\sin(2x) - \cos(x)$	0.000048	0.000049

The graphs of the middle basic splines are shown in Fig.18-20.

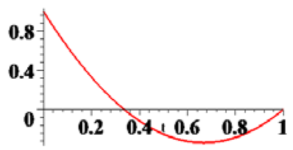

Fig. 18. Plot of the basic function $s_j(x)$ $(h = 1)$.

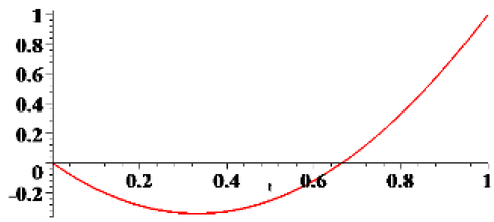

Fig. 19. Plot of the basic function $s_{j+1}(x)$ $(h = 1)$.

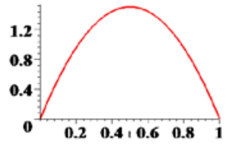

Fig. 20. Plot of the basic function $s_j^{<0>}(x)$ $(h = 1)$.

To construct interval estimates, we need the largest and smallest values of the basic functions. We have when $h = 1$ the values:

$$\max s_j = 1, \min s_j = -0.33,$$
$$\max s_{j+1} = 1, \min s_{j+1} = -0.33,$$
$$\max s_j^{<0>} = 1.50633, \min s_j^{<0>} = 0.$$

Fig. 21 shows the error of approximation the function $u(x) = \sin(2x) - \cos(x)$, with middle trigonometrical splines on the interval $[0, \pi]$ with the number of nodes $n = 15$.

Fig. 22 shows the plot of the largest and smallest boundaries of function $u(x) = \sin(7x) - \cos(9x)$ and points of the function when $n = 15, a = 0, b = \pi$.

Fig. 23 shows the error of approximation the function $u(x) = \sin(7x)$, with middle trigonometrical splines on the interval $[0, \pi]$ with the number of nodes $n = 15$. Fig. 24 shows the plot of the largest and smallest boundaries of function $u(x) = \sin(7x)$ and points of the function when $n = 15, a = 0, b = \pi$.

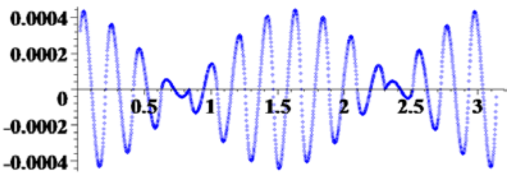

Fig. 21. Plot of the error of approximation of function $u(x) = \sin(2x) - \cos(x)$.

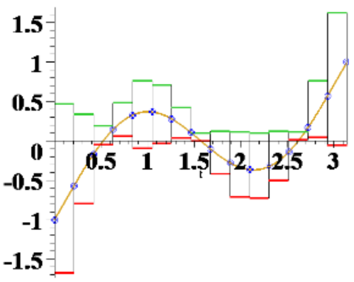

Fig. 22. Plot of the function $u(x) = \sin(2x) - \cos(x)$ and the largest and smallest boundaries of this function.

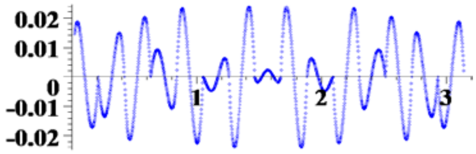

Fig. 23. Plot of the error of approximation of function $u(x) = \sin(7x)$.

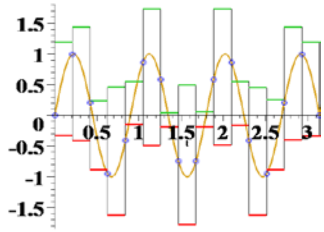

Fig. 24. Plot of the function $u(x) = \sin(7x)$ and the largest and smallest boundaries of this function.

References:

[1] R.E. Moore, *Methods and Applications of Interval Analysis*, Philadelphia: SIAM, 1979.

[2] R.E. Moore, R.B. Kearfott, *Cloud Introduction to Interval Analysis*, Philadelphia: SIAM, 2009.

[3] A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge: Cambridge University Press, 1990.

[4] H. Ratschek, J. Rokne, *Computer Methods for the Range of Functions*, New York – Chichester: Ellis Horwood – John Wiley, 1984.

[5] H. Ratschek, J. Rokne, *New Computer Methods for Global Optimization*, New York – Chichester: Ellis Horwood, 1988.

[6] J. Rohn, *A handbook of results on interval linear problems*, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, Technical report No.V-1163, 2005-2012.

[7] H. Ratschek, J. Rokne, *New Computer Methods for Global Optimization*, New York – Chichester: Ellis Horwood, 1988.

[8] J.Stolfi, L.H.de Figueiredo, *Self-Validated Numerical Methods and Applications*, Rio de Janeiro: IMPA, Brazilian Mathematics Colloquium monograph, 1997.

[9] R. Hammer, M. Hocks, U. Kulisch, D. Ratz, *Numerical Toolbox for Verified Computing*, I: Basic numerical problems. Berlin-Heidelberg: Springer, 1993.

[10] X. Liu, C. You, Explicit bound for quadratic Lagrange interpolation constant on triangular finite elements, *Applied Mathematics and Computation*, Vol. 319, 2018, pp. 693-701.

[11] I.G. Burova, A.A. Vartanova, Interval Estimation of Polynomial Splines of the Fifth Order, in *Proc. 4th Int. Conf. Math. Comput. Sci. Ind., MCSI*- 2017, Corfu Island, Greece, Jan. 2018, pp. 293-297.

[12] I.M. Galván, J.M. Valls, A.Cervantes, R Aler, Multi-objective evolutionary optimization of prediction intervals for solar energy forecasting with neural networks, *Information Sciences*, Vol. 418-419, 2017, pp. 363-382.

[13] I.G. Burova, O.V. Rodnikova, Application of integrodifferential splines to solving an interpolation problem, *Computational Mathematics and Mathematical Physics*, Vol. 54, No.12, 2014, pp. 1903-1914.

[14] I.G. Burova, On left integro-differential splines and Cauchy problem, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol. 9, 2015, pp. 683-690.

[15] I.G. Burova, O.V. Rodnikova, Integro-differential polynomial and trigonometrical splines and quadrature formulae, *WSEAS Transactions on Mathematics*, Vol. 16, 2017, pp. 11-18.

[16] I.G. Burova, A.G. Doronina, I.D. Miroshnichenko, A Comparison of Approximations with left, right and middle Integro-Differential Polynomial Splines of the Fifth Order, *WSEAS Transactions on Mathematics*, Vol.16, 2017, pp. 339-349.

[17] I.G. Burova, S.V. Poluyanov, On approximations by polynomial and trigonometrical integro-differential splines, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.10, 2016, pp. 190-199.

[18] I.G. Burova, A.G. Doronina, On approximations by polynomial and nonpolynomial integro-differential splines, *Applied Mathematical Sciences*, Vol.10, No.13-16, 2016, pp. 735-745.