On Splines' Smoothness

Yu. K. Dem'yanovich
St. Petersburg State University
Department of Parallel Algorithms
7/9 Universitetskaya nab.
St. Petersburg, RUSSIA
y.demjanovich@spbu.ru

I. D. Miroshnichenko
St. Petersburg State University
Department of Parallel Algorithms
7/9 Universitetskaya nab.
St. Petersburg, RUSSIA
Yuri.Demjanovich@gmail.com

E. F. Musafarova
St. Petersburg State University
Department of Parallel Algorithms
7/9 Universitetskaya nab.
St. Petersburg, RUSSIA
e.muzafar@yandex.ru

Abstract: The aim of this article is to discuss the generalized smoothness for the splines on q-covered manifold, where q is the natural number. By using mentioned smoothness it is possible to consider the different types of smoothness, for example, the integral smoothness, the weight smoothness, the derivatives smoothness, etc. We find the necessary and sufficient conditions for calculation of basic splines with a'priori prescribed smoothness. The mentioned smoothness may contain no more than q (locally formulated) linearly independentconditions. If the number of the conditions is exactly q, then the discussed spline spaces on the embedded grids are also embedded.

Key–Words: approximation relations, embedded spaces, generalized smoothness, wavelet expansions

1 Introduction

Splines are a well-known processing apparatus forstreams of numerical information (see [1] - [5], [12], [14], [22] - [24]). Science researches are interested in such qualities as an approximation property, interpolation property, property of anaccuracy on certain linear subspaces, the property ofsmoothness in one sense or another (see [6] - [11], [13], [15]). The last property is important in the questions of refining approximation, in theconstruction of the finite element methods, in the wavelet decomposition, etc. (see [16] - [22]).

The history of the splines' development has not one decade (not later than the forties of the last century). Just now it is time to find the approach to the methods for construction of splines with a'priori prescribed properties (see [6] - [11], [16] - [22]).

A universal source of splines is approximation relations.

Consider the simplest case. On the real axis, consider the grid

$$X:\ \ldots < x_{-1} < x_0 < x_1 < \ldots, \tag{1}$$

$$\alpha = \lim_{i\to-\infty} x_i, \quad \beta = \lim_{i\to+\infty} x_i.$$

Discuss the approximation relations

$$\sum_j \mathbf{a}_j \omega_j(t) = \varphi(t), \tag{2}$$

$$\operatorname{supp}\omega_j \subset [x_j, x_{j+q}], \tag{3}$$

where $\mathbf{a}_j$ are q-component vectors (columns) with the property

$$\det(\mathbf{a}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_{j+q-1}) \neq 0 \qquad \forall j \in \mathbf{Z}, \tag{4}$$

$\varphi(t)$ is a q-component vector function.

The approximation relations (2) contain the next objects: the generating vector function $\varphi(t)$, the complete chain of vectors $\{\mathbf{a}_j\}_{j\in\mathbf{Z}}$, the location of the supports (3), the multiplicity q of the cover by the mentioned supports of the desired basic functions $\omega_j(t)$.

The generating vector function $\varphi(t)$ determines the structural characteristics of the spline (polynomial, trigonometric, exponential,mixed, etc.). The chain of vectors $\{\mathbf{a}_j\}_{j\in\mathbf{Z}}$ determines the degree and nature of spline smoothness (the number of available derivatives or integral smoothness,weight smoothness, etc.). In addition, the vector chain defines the embedding property for spline spaces. The mentioned chain also determines interpolation properties of the splines. The location (3) of the basic spline supports determines the spline type (splines of the Lagrangian type or the Hermitian type, splines of a mixed type etc.). Finally, the multiplicity $q$determines approximation properties (the order ofapproximation).

The aim of this article is to discuss the generalized smoothness for the splines on q-covered manifold, where q is the natural number. By using the mentioned smoothness it is possible to consider the different types of smoothness, for example, the integral smoothness, the weight smoothness, the derivatives smoothness, etc. We find the necessary and sufficient conditions for calculation of basic splines with a'priori prescribed smoothness.

The mentioned smoothness may contain no more than q (locally formulated) linearly independentconditions. If the number of the conditions is exactly q, then the discussed spline spaces on the embedded grids are also embedded.

As discussed before, the property of the embedding spaces on the subdivisions is very important for the finite element method, multigrid method, and wavelet decompositions. However, this property is not always fulfilled. Let's give a simple example of the breaking of this property. Consider the case of $q = 3$.

Relations (2) – (4) define the functions $\omega_j(t)$ on the set $(\alpha, \beta)\backslash X$. By (2) we have

$$\omega_j(t) = \frac{\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \varphi(t))}{\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j)} \tag{5}$$

$$\text{for} \quad t \in (x_j, x_{j+1}),$$

$$\omega_j(t) = \frac{\det(\mathbf{a}_{j-1}, \varphi(t), \mathbf{a}_{j+1})}{\det(\mathbf{a}_{j-1}, \mathbf{a}_j, \mathbf{a}_{j+1})} \tag{6}$$

$$\text{for} \quad t \in (x_{j+1}, x_{j+2}),$$

$$\omega_j(t) = \frac{\det(\varphi(t), \mathbf{a}_{j+1}, \mathbf{a}_{j+2})}{\det(\mathbf{a}_j, \mathbf{a}_{j+1}, \mathbf{a}_{j+2})} \tag{7}$$

$$\text{for} \quad t \in (x_{j+2}, x_{j+3}).$$

Now we discuss the enlarged grid $\widehat{X} = X \backslash x_{k+1}$ obtained from (1) by deleting the knot x_{k+1},

$$\widehat{X}: \ \ldots < \widehat{x}_{-1} < \widehat{x}_0 < \widehat{x}_1 < \ldots, \tag{8}$$

where $\widehat{x}_j = x_j$ with $j \leq k$, $\widehat{x}_{j-1} = x_j$ with $j > k+1$. We define the coordinate functions $\widehat{\omega}_j$ by the approximation relation

$$\sum_j \widehat{\mathbf{a}}_j \widehat{\omega}_j(t) = \varphi(t), \qquad \mathrm{supp}\widehat{\omega}_j \subset [\widehat{x}_j, \widehat{x}_{j+3}], \tag{9}$$

where $\widehat{\mathbf{a}}_j$ are three-component vectors (columns) with the property

$$\det(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{j+1}, \widehat{\mathbf{a}}_{j+2}) \neq 0 \qquad \forall j \in \mathbf{Z},$$

and $\varphi(t)$ is the former three-component vector function.

Analogously by (9) we obtain

$$\widehat{\omega}_j(t) = \frac{\det(\widehat{\mathbf{a}}_{j-2}, \widehat{\mathbf{a}}_{j-1}, \varphi(t))}{\det(\widehat{\mathbf{a}}_{j-2}, \widehat{\mathbf{a}}_{j-1}, \widehat{\mathbf{a}}_j)} \tag{10}$$

$$\text{for} \quad t \in (\widehat{x}_j, \widehat{x}_{j+1}),$$

$$\widehat{\omega}_j(t) = \frac{\det(\widehat{\mathbf{a}}_{j-1}, \varphi(t), \widehat{\mathbf{a}}_{j+1})}{\det(\widehat{\mathbf{a}}_{j-1}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{j+1})} \tag{11}$$

$$\text{for} \quad t \in (\widehat{x}_{j+1}, \widehat{x}_{j+2}),$$

$$\widehat{\omega}_j(t) = \frac{\det(\varphi(t), \widehat{\mathbf{a}}_{j+1}, \widehat{\mathbf{a}}_{j+2})}{\det(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{j+1}, \widehat{\mathbf{a}}_{j+2})} \tag{12}$$

$$\text{for} \quad t \in (\widehat{x}_{j+2}, \widehat{x}_{j+3}).$$

There are many options for choosing vectors $\widehat{\mathbf{a}}_i$ and $\mathbf{a}_j$. We are interested in the cases for which each function $\widehat{\omega}_i$ can be represented as a finite linear combination of functions ω_j. Such representations are called *calibration relations* (see [20]).

Let us proceed to the presentation of the example when the calibration relations are violated.

Suppose the vector function $\varphi(t)$ satisfy the condition

$$\det(\varphi(t_0), \varphi(t_1), \varphi(t_2)) \neq 0 \tag{13}$$

$$\text{for any different} \quad t_0, t_1, t_2 \in \mathbf{R}^1.$$

We introduce the notation $\varphi_s = \varphi(x_s)$, $\widehat{\varphi}_s = \varphi(\widehat{x}_s)$ $\forall s \in \mathbf{Z}$.

By definition put $\mathbf{a}_j = \varphi_{j+1}$ and $\widehat{\mathbf{a}}_j = \widehat{\varphi}_{j+1}$. Taking into account formulas (5) – (7) and (10) – (13), we obtain functions ω_j and $\widehat{\omega}_j$. The last one can be prolonged on interval (α, β) continuously (see [11]). In what follows we suppose that such prolongation is fulfilled.

At first we give a negative example of an algorithm that shows the case for which the mentioned representation are absent.

As it is said above the functions ω_j and $\widehat{\omega}_j$ are continuous on the interval (α, β), but it is easy to see that the first derivative of these functions has discontinuities of the first kind in the nodes. Each of the systems of functions $\{\omega_j\}_{j \in \mathbf{Z}}$ and $\{\widehat{\omega}_j\}_{j \in \mathbf{Z}}$ is a linear independent system.

Let us show that the function $\widehat{\omega}_{k-2}$ cannot be represented by a finite linear combination of functions ω_i. Suppose the contrary, i.e. that with some constants c_{-2}, c_{-1} true ratio

$$\widehat{\omega}_{k-2} = c_{-2}\omega_{k-2} + c_{-1}\omega_{k-1} \tag{14}$$

(it is easy to see that the use of other functions ω_s is not necessary due to the location of their supports).

Consider the relation (14) for $t = x_{k+1}$. Because $x_{k+1} \in (\widehat{x}_k, \widehat{x}_{k+1})$, by (12) for $j = k-2$ we have

$$\widehat{\omega}_{k-2}(x_{k+1}) = \frac{\det(\varphi(x_{k+1}), \varphi(\widehat{x}_k), \varphi(\widehat{x}_{k+1}))}{\det(\varphi(\widehat{x}_{k-1}), \varphi(\widehat{x}_k), \varphi(\widehat{x}_{k+1}))} =$$

$$= \frac{\det(\varphi(x_{k+1}), \varphi(x_k), \varphi(x_{k+2}))}{\det(\varphi(x_{k-1}), \varphi(x_k), \varphi(x_{k+2}))}.$$

Thus $\widehat{\omega}_{k-2}(x_{k+1}) \neq 0$. By (6) for $j = k-1$ and by (7) for $j = k-2$ it is clear to see that $\omega_{k-2}(x_{k+1}) = \omega_{k-1}(x_{k+1}) = 0$. This contradiction concludes the proof. Thus we see the relation (14) is impossible.

There are many ways to build sequences of vectors $\mathbf{a}_j$ and $\widehat{\mathbf{a}}_j$ for which the resulting functions $\widehat{\omega}_j$ can be expressedas a finite linear combination of functions ω_j. Briefly we discuss one such method.

Let $\varphi \in C^1(\alpha, \beta)$. Consider the vectors $\mathbf{a}^*_j$ and $\widehat{\mathbf{a}}^*_j$, defined using vector product

$$\mathbf{a}^*_j = c_j(\varphi_{j+1} \times \varphi'_{j+1}) \times (\varphi_{j+2} \times \varphi'_{j+2}),$$

$$\widehat{\mathbf{a}}^*_j = \widehat{c}_j(\widehat{\varphi}_{j+1} \times \widehat{\varphi}'_{j+1}) \times (\widehat{\varphi}_{j+2} \times \widehat{\varphi}'_{j+2}); \tag{15}$$

here $\varphi'_s = \varphi'(x_s)$ and $\widehat{\varphi}'_s = \varphi'(\widehat{x}_s)$.

Assuming $\mathbf{a}_s = \mathbf{a}^*_s$ and $\widehat{\mathbf{a}}_s = \widehat{\mathbf{a}}^*_s \ \forall s \in \mathbf{Z}$ in approximation relations (2) and (9), respectively, we obtain the functions $\widehat{\omega}^*_i$ and$\omega^*_j \ \forall i, j \in \mathbf{Z}$. For these functions the next calibration ratios

$$\widehat{\omega}^*_j(t) \equiv \omega^*_j(t) \quad \forall j \le k-3;$$

$$\widehat{\omega}^*_j(t) \equiv \omega^*_{j+1}(t) \quad \forall j \ge k+1,$$

$$\widehat{\omega}^*_i = c^*_{i,0}\omega^*_i + c^*_{i,1}\omega^*_{i+1} \tag{16}$$

$$\text{for} \quad i = k-2, k-1, k,$$

are valid. Here $c^*_{i,0}$ and $c^*_{i,1}$ are some numeric constants(see [11]).

The spaces of the aforementioned splines, built on embedded grids, are emdedded in each other.

In particular, if $\varphi(t) = (1, t, t^2)^T$, then by (15) we find

$$\mathbf{a}^*_j = 2(x_{j+1} - x_j)(1, (x_j + x_{j+1})/2, x_j x_{j+1})^T,$$

$$\widehat{\mathbf{a}}^*_j = 2(\widehat{x}_{j+1} - \widehat{x}_j)(1, (\widehat{x}_j + \widehat{x}_{j+1})/2, \widehat{x}_j \widehat{x}_{j+1})^T.$$

In this case we get the continuously differentiable quadratic splines (see [1]).

The introduction of generalized smoothness is allowed to diversify types of spline spaces and take into account the peculiarities of the approximated functions (for example, breaks of the function itself or its derivatives).In this case, the generalized smoothness still leads to embedded spaces and calibration relations (see [16], [17], [19]).

In the multidimensional case, continuity, and even more so, smoothnessis the exception rather than the rule. For example, the requirement of continuity of coordinate functions of the Courant type can lead to the need to build acurvilinear grid (see [21]), which is accompanied by conditions that are difficult to implement on practice.

Wavelet decomposition for information flows, which emanate from complex-shaped bodies, wasdiscussed in [20] –[21]. But the embedding conditions, formulated there, are not convenient for practical use.

Enlargement of the cover and use of similar approximation relations, associated with the new cover, leads to calibration relations and tothe corresponding embedded space. As a result of the projection of the original space onto the embedded space we getspline-wavelet decomposition. Notice that the proposed method is associated with a specific class oflocal enlargements of the manifold covers. There are a number of other classes of local enlargements (see [12], [14]); they are not considered in this paper.

2 Notation and auxiliary statements. Manifold cover and its equipment

Consider a smooth n-dimensional (generally speaking, non-compact) manifold $\mathcal{M}$ (i.e. a topological space in which each point possesses a neighborhood that is diffeomorphic to an open n -dimensional ball of the Euclidean space $\mathbf{R}^n$).

Let q be a natural integer, $q \ge 1$, and $\mathcal{J}$ be an ordered set of indices, no more than countable. Let $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathcal{J}}$ be a family of subsets $\mathcal{S}_j \subset \mathcal{M}$, each of which is homeomorphic to an open n-dimensional ball. Suppose that $Cl\left(\bigcup_{j\in\mathcal{J}} \mathcal{S}_j\right) = \mathcal{M},$ where Cl is closure.

Let $\partial\mathcal{S}_j$ be the boundarysets of $\mathcal{S}_j$. If each point t of the set $\mathcal{M}\backslash(\bigcup_{|\in\mathcal{J}} \partial\mathcal{S}_|)$ belongs toexactly q subsets of $\mathcal{S}_j$, then $\mathcal{S}$ is called *q-cover* ofthe manifold $\mathcal{M}$. In what follows we discuss only q-cover $\mathcal{S}$ ofthe manifold $\mathcal{M}$. $\mathcal{S}_j$ is called a covering set of the cover $\mathcal{S}$.

For each point$t \in \mathcal{M}\backslash\bigcup_{j\in\mathcal{J}} \partial\mathcal{S}_j$ consider the collection of the sets containing it and discuss the intersection $\mathcal{C}_{(t)}$ of mentioned sets, $\mathcal{C}_{(t)} = \bigcap_{j\in\mathcal{J},\mathcal{S}_j\ni t} \mathcal{S}_j$. It is obvious that if $t' \in \mathcal{C}_{(t)}$ then $\mathcal{C}_{(t')} = \mathcal{C}_{(t)}$. We suppose that collection $\mathcal{C}$ of different sets $\mathcal{C}_{(t)}$ for the mentioned t, no more than countable. Further we denote them by $\mathcal{C}_i$, $i \in \mathcal{K}$ (here $\mathcal{K}$ is an ordered set of indices). So, $\mathcal{C} = \{\mathcal{C}_i \mid i \in \mathcal{K}\}$. Thus, cover $\mathcal{S}$ matches the set $\mathcal{C}$. The mapping rule described above isdenoted by $\mathcal{F}$, $\mathcal{C} = \mathcal{F}(\mathcal{S})$. Aggregate $\mathcal{C}$ is called *splitting of cover $\mathcal{S}$*.

We suppose that each set $\mathcal{C}_i$ from the splitting $\mathcal{F}(\mathcal{S})$ is homeomorphic to an open ball. In this case the set $\mathcal{C}_i$ is called a cell.

Example 1. As an illustration, consider the sphere S (centered at the origin), divided into eight identical spherical triangles obtained by the coordinate planes which go through the center of the sphere. We remove the boundary of these triangles, so that we consider them as open sets. The resulting triangulation contains six vertices. With each vertices we associate the corresponding barycentric star. The body of each barycentric star is the corresponding hemi-

sphere, which is obtained by closing the set of points of triangles ofthis barycentric stars. By removing the boundary of each such hemisphere (i.e., removing corresponding circumference of large circle), we obtain the covering set $\widehat{\mathcal{S}}_j$, $j = 1, 2, 3, 4, 5, 6$. So, the "open" hemispheres are covering sets $\widehat{\mathcal{S}}_j$, $j \in \widehat{\mathcal{J}}$, $\widehat{\mathcal{J}} = \{1, 2, 3, 4, 5, 6\}$, and the open triangles are cells $\widehat{\mathcal{C}}_i$, $i \in \widehat{\mathcal{K}}$, $\widehat{\mathcal{K}} = \{1, 2, \dots, 8\}$. The multiplicity of the resulting cover is equal to three.

Definition 1 *Suppose that $i \neq i'$, $i, i' \in \mathcal{K}$, and a point t belongs to the boundary $\partial \mathcal{C}_i$ of the cell $\mathcal{C}_i$. Let $\mathcal{C}_{i'}$ be a cell, and let some neighborhood of the point t be in the union $\mathcal{C}_{i'} \bigcup Cl\ (\mathcal{C}_i)$. Then the cell $\mathcal{C}_{i'}$ is called adjacent to cell $\mathcal{C}_i$ $(i, i' \in \mathcal{K})$ in splitting of cover $\mathcal{S}$.*

Obviously, if the cell $\mathcal{C}_{i'}$ is adjacent to $\mathcal{C}_i$, then $\mathcal{C}_i$ is adjacent to $\mathcal{C}_{i'}$; $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ cells are called *the adjacent cells* (in the splitting of cover $\mathcal{S}$). If $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ are adjacent cells we write $\mathcal{C}_i \cong \mathcal{C}_{i'}$.

Definition 2 *If $\mathcal{C}_i \cong \mathcal{C}_{i'}$ and the difference $\{j \mid \mathcal{S}_j \supset \mathcal{C}_i\} \backslash \{j' \mid \mathcal{S}_{j'} \supset \mathcal{C}_{i'}\}$ contains exactly p elements (where $p \geq 1$), then the family $\mathcal{S}$ is called a p-step q-cover for manifold $\mathcal{M}$.*

Example 2. As an illustration of the last definition, consider situation arising in the construction of the Hermitian type splines (see [19]). Let X be a grid on the interval $(\alpha, \beta) \in \mathbf{R}^1$ (see formula (1)) . We assume that $\mathcal{M} = (\alpha, \beta)$, $J = \mathbf{Z}$, $\mathcal{K} = \mathbf{Z}$. Let$\mathcal{S}_{2j-1} = \mathcal{S}_{2j} = (x_j, x_{j+2})$ be cover sets, $j \in J$, and $\mathcal{S} = \{\mathcal{S}_{2j-1}, \mathcal{S}_{2j} \mid j \in J\}$ be the cover of (α, β). Then intervals $\mathcal{C}_i = (x_i, x_{i+1})$ be the cells, $i \in \mathcal{K}$, and the set $\mathcal{C} = \{\mathcal{C}_i \mid i \in \mathcal{K}\}$ be the splitting of the covering $\mathcal{S}$. Obviously, in this case $q = 4$, $p = 2$.

With each set $\mathcal{S}_j$ of family $\mathcal{S}$ we associate a vector $\mathbf{a}_j$ from a q-dimensional Euclidean space $\mathbf{R}^q$, $j \in \mathcal{J}$. Set $A = \{\mathbf{a}_j \mid j \in \mathcal{J}, \quad \mathbf{a}_j \in \mathbf{R}^q\}$ is called an equipment of the family $\mathcal{S}$.

Definition 3 *It is said that the q-cover $\mathcal{S} = \{\mathcal{S}_j\}_{j \in \mathcal{J}}$ manifold $\mathcal{M}$ is equipped with a complete vector system $A = \{\mathbf{a}_j \mid j \in \mathcal{J}, \quad \mathbf{a}_j \in \mathbf{R}^q\}$ if for points $t \in \mathcal{M} \backslash \bigcup_{j \in \mathcal{J}} \partial \mathcal{S}_j$ vector system*

$$A_{\langle t \rangle} = \{\mathbf{a}_j \mid \forall j \in \mathcal{J}, \mathcal{S}_j \ni t\} \tag{17}$$

is the basis of space $\mathbf{R}^q$. In this case the vector system A is called the complete equipment of the family $\mathcal{S}$.

By (17) it follows that if A is the complete equipment of $\mathcal{S}$, and $\mathcal{C} = \mathcal{F}(\mathcal{S})$, then for fixed $k \in \mathcal{K}$, $\mathcal{C}_k \in \mathcal{C}$, the ratios

$$A_{\langle t' \rangle} = A_{\langle t'' \rangle} \qquad \text{for} \quad \forall t', t'' \in \mathcal{C}_k \tag{18}$$

are correct. Now we can introduce a notion $A_k = A_{\langle t \rangle}$ for $t \in \mathcal{C}_k$.

Thus

$$A_k = \{\mathbf{a}_{j'} \mid \forall j' \in \mathcal{J},\ \mathcal{S}_{j'} \supset \mathcal{C}_k\}. \tag{19}$$

Using (17) – (19), we obtain the equivalences

$$\mathbf{a}_j \notin A_k \iff \mathcal{S}_j \cap \mathcal{C}_k = \emptyset. \tag{20}$$

It is also clear that if the cover is p-step one, and $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent, then the number vectors in the sets $A_k \backslash A_{k'}$ equals p.

Example 3. For an illustration we turn to example 1, in which the cover of the sphere S is considered. As equipment of this cover we take a system of six (nonzero) three-dimensional vectors that are directed along straight lines, outgoing from the center of the sphere and passing through vertices of the considered triangulation. As a result theequipment of each cell (in our case, that is the triangle) consists of three vectors corresponding to the vertices of the discussed triangle. Obviously, these vectors form alinearly independent system in $\mathbf{R}^3$. In this way, the resulting equipment is complete.

Example 4. Go back to thesituation arising in the construction of the Hermite type splines of thefirst height (see Example 2). Just mentioned splines aredetermined (see [19]) by the continuously differentiated four-component vector function $\varphi(t)$, which satisfying the condition

$$\det(\varphi(x), \varphi'(x), \varphi(y), \varphi'(y)) \neq 0 \tag{21}$$

$$\forall x, y \in (\alpha, \beta), x \neq y.$$

In this case, the vector $\varphi'(x_{j+1})$ is associated to the covering set $\mathcal{S}_{2j-1}$ and the vector $\varphi(x_{j+1})$ is associated to the covering set $\mathcal{S}_{2j}$. It is easy to see that the resulting equipment of the cover $\mathcal{S}$ is complete.

3 Minimal spline space

In what follows, the previously introduced sets$\mathcal{C}_k$ are consideredin the topology induced by the original atlas of the manifold $\mathcal{M}$.Let $\mathbf{U}$ be a linear space which is the directproduct of spaces $\mathbf{X}(\mathcal{C}_k)$,

$$\mathbf{U} = \bigotimes_{k \in \mathcal{K}} \mathbf{X}(\mathcal{C}_k).$$

We assume that the restriction of the function $u \in \mathbf{X}(\mathcal{M})$ on the cell $\mathcal{C}_k$ belongs to $\mathbf{X}(\mathcal{C}_{\|})$,thus the natural embedding $\mathbf{X}(\mathcal{M})$ in $\mathbf{U}$ is defined, $\mathbf{X}(\mathcal{M}) \subset \mathbf{U}$.We agree to consider the record $F_k \in \mathbf{X}^*(\mathcal{C}_k)$ as the equivalent of ratios $F_k \in U^*, \operatorname{supp} F_k \subset \mathcal{C}_k$.

Let m be a nonnegative integer, and $q = q$. Consider a vector function $\varphi : \mathcal{M} \to \mathbf{R}^{\text{II}}$ with components $[\varphi]_i(t)$ from space $\mathbf{X}(\mathcal{M})$, $i = 0, 1, 2, \ldots, m$. This fact is further expressed in the record $\varphi \in \mathbf{X}(\mathcal{M})$.

Hereafter, the notation A is also used for the matrix consisting of column vectors $\mathbf{a}_j$, $A = (\mathbf{a}_j)_{j\in\mathcal{J}}$.

Theorem 1 *Let $\mathcal{S}$ be a q-cover family (for $\mathcal{M}$), and the column vector system $A = \{\mathbf{a}_j\}_{j\in\mathcal{J}}$ be the full equipment of the family $\mathcal{S}$. Then a single (column) vector function $\omega(t) = (\omega_j(t))_{j\in\mathcal{J}}$, which satisfies the ratios*

$$A\omega(t) = \varphi(t) \quad \forall t \in \mathcal{M} \setminus \bigcup_{j\in\mathcal{J}} \partial\mathcal{S}_j, \qquad (22)$$

$$\omega_j(t') = 0 \quad \forall t' \notin \mathcal{S}_j$$

exists.

Proof: According to the definition of a set A_i (see also formulas (18) - (20) and (22)) we have

$$\sum_{\mathbf{a}_j\in A_i} \mathbf{a}_j\omega_j(t) = \varphi(t) \qquad \forall t \in \mathcal{C}_i \quad \forall i \in \mathcal{K}. \quad (23)$$

Since, by the definition of the full equipment, the set of $\{\mathbf{a}_j \mid \mathbf{a}_j \in A_i\}$ is a basis in $\mathbf{R}^q$, the matrix of the system (23) is non-singular, so the unknowns $\omega_j(t)$, considered for each fixed $t \in \mathcal{C}_i$ and for each $i \in \mathcal{K}$, are determined uniquely. This completes the proof.

Example 5. Again, we go back to examples 1 and 3, inwhich the sphere S with the center at the origin is discussed. The cells $\widehat{\mathcal{C}}_i$ of the considered cover $\widehat{\mathcal{S}}$ are spherical triangles. Let be$T = \widehat{\mathcal{C}}_i$ oneof them. Suppose that this triangle is the intersection of the covering sets $\widehat{\mathcal{S}}_j$ (bodies of barycentricstars), $j \in \{1, 2, 3\}$.Its equipment is a linearly independent system ofthree vectors $\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3$, which are outgoing from the origin to the vertex directions of this triangle. It is clear that the vectors $\widehat{\mathbf{a}}_j$, $j = 1, 2, 3$, are linearly independent. By definition put $A_i = \{\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3\}$. The ratio (23) takes the form

$$\widehat{\mathbf{a}}_1\widehat{\omega}_1(t) + \widehat{\mathbf{a}}_2\widehat{\omega}_2(t) + \widehat{\mathbf{a}}_3\widehat{\omega}_3(t) = \varphi(t) \qquad \forall t \in T.$$

By the last ratio we derive identities

$$\widehat{\omega}_1(t) = \frac{\det(\varphi(t), \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)}{\det(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)},\ \widehat{\omega}_2(t) = \frac{\det(\widehat{\mathbf{a}}_1, \varphi(t), \widehat{\mathbf{a}}_3)}{\det(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)},$$

$$\widehat{\omega}_3(t) = \frac{\det(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\varphi}(t))}{\det(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2, \widehat{\mathbf{a}}_3)},$$

or in brief

$$\widehat{\omega}_j(t) = \frac{\det\Big(\{\widehat{\mathbf{a}}_s \mid \widehat{\mathbf{a}}_s \in \widehat{A}_i,\ s \neq j\} \ ||'^j\ \varphi(t)\Big)}{\det\Big(\{\widehat{\mathbf{a}}_s \mid \widehat{\mathbf{a}}_s \in \widehat{A}_i\}\Big)}$$

for $\forall t \in T = \widehat{\mathcal{C}}_i \subset \widehat{\mathcal{S}}_j, \quad j = 1, 2, 3.$

If the family $\mathcal{S}$ is $r+1$ -step cover (r is nonnegative integer), then we say that $(\mathcal{S}, A, \varphi)$ -splines have height r. For $r = 0$ splines are called the splines of the Lagrange type, and for $r > 0$ are called the splines of the Hermite type). Otherwise case of talking about the splines of different height.

Example 6. Let us return to Example 4. Suppose thecondition (21) is fulfilled. In this case approximation relationstake the form

$$\sum_j (\varphi'_{j+1}\omega_{2j-1}(t) + \varphi_{j+1}\omega_{2j}(t)) = \varphi(t),$$

where

$$\operatorname{supp}\omega_{2j-1} \subset [x_j, x_{j+2}],$$

$$\operatorname{supp}\omega_{2j} \subset [x_j, x_{j+2}] \quad \forall j \in \mathbf{Z}.$$

For $\forall q \in \mathbf{Z}$ *we obtain coordinate splines (see [19])*

$$\omega_{2q-1}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi(t), \varphi_{q+1})}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}$$

for $\quad t \in (x_q, x_{q+1}),$

$$\omega_{2q-1}(t) = \frac{\det(\varphi(t), \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}{\det(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}$$

for $\quad t \in (x_{q+1}, x_{q+2}),$

$$\omega_{2q}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi(t))}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}$$

for $\quad t \in (x_q, x_{q+1}),$

$$\omega_{2q}(t) = \frac{\det(\varphi'_{q+1}, \varphi(t), \varphi'_{q+2}, \varphi_{q+2})}{\det(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}$$

for $\quad t \in (x_{q+1}, x_{q+2}).$

4 Pseudocontinuity of spline approximations

With each cell of $\mathcal{C}_k$ we associate a linear functional $F_k \in (\mathbf{X}(\mathcal{C}_k))^*$, $k \in \mathcal{K}$. If the cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent, then put $A_{k,k'} = \{\mathbf{a}_j \mid \mathbf{a}_j \in A_k \cap A_{k'}\}$. We introduce a condition

(A) a ratio

$$F_k\varphi = F_{k'}\varphi \qquad (24)$$

is true.

It is clear to see that the next assertion is correct.

Lemma 1 *Let for fixed $k, k' \in \mathcal{K}$ cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent, and linear functionals F_k, $F_{k'}$ have supports in cells of $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ respectively. At last suppose that the condition (A) is satisfied. If one of the relations*

$$F_k\,\omega_j = 0 \quad \textit{for} \quad \mathbf{a}_j \in A_k \backslash A_{k,k'}, \tag{25}$$

$$F_{k'}\,\omega_{j'} = 0 \quad \textit{for} \quad \mathbf{a}_{j'} \in A_{k'} \backslash A_{k,k'}, \tag{26}$$

is satisfied then the ratios

$$F_k \omega_j = F_{k'}\,\omega_j \qquad \forall j \in A_{k,k'} \tag{27}$$

are right. If in addition the vectors system $(A_k \cup A_{k'}) \backslash A_{k,k'}$ is linearly independent, and ratios (27) are fulfilled then relations (25) and (26) are right.

If the condition (24) is satisfied, then we put

$$F_{(k,k')}\varphi = F_k\varphi = F_{k'}\varphi. \tag{28}$$

Theorem 2 *Let $k, k' \in \mathcal{K}$ be fixed. Suppose the cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent, and linear functionals F_k, $F_{k'}$ have supports in cells of $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ respectively. In addition we assume the conditions (24) are valid. Then equalities' family*

$$F_k\,\omega_j = F_{k'}\,\omega_j \qquad \forall j \in J \tag{29}$$

and the relations

$$F_{(k,k')}\varphi \in \mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_{k,k'}\} \tag{30}$$

are equivalent.

Proof: If relations (29) are right then formulas (25) hold by Lemma 1. Applying the functional F_k to the relation (22) (see also (23)) and using the Cramer formula, by (25) we get

$$\det\Big(\{\mathbf{a}_s \,|; \mathbf{a}_s \in A_k,\, s \neq j\}\,||'^j\; F_k\varphi\Big) = 0 \tag{31}$$

$$\text{for} \quad \mathbf{a}_j \in A_k \backslash A_{k,k'}.$$

By relation (28) and formula (31) we see that the vector $F_{(k,k')}\varphi$ belongs to the linear hull $\mathcal{L}_j = \mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_k, s \neq j\}$, where j is such that $\mathbf{a}_j \in A_k \backslash A_{k,k'}$. So the vector $F_{(k,k')}\varphi$ pertains to the intersection of the mentioned linear hulls, and this is equivalent to the formula (30).

Otherwise, if (30) is fulfilled, then (31) holds, and we have relations (25) – (26) and also (27) (see Lemma 1).

Thus, the relations (25) and (30) are equivalent. This concludes the proof.

5 Maximum pseudo-smooth coordinate functions

Consider some linear subspace $\mathbf{U}_0$ spaces $\mathbf{U}$, containing space $\mathbf{X}(\mathcal{M})$. Let $\mathcal{F}_k$ be the set of linear functionals from $\mathbf{U}^*$ with supports in $\mathcal{C}_k$ $\forall k \in \mathcal{K}$. Among them, we single out the (possibly empty) set $\mathcal{F}_k^0$ those functionals $F_{k,i}$, for each of which there is a functional $F_{k',l}$ s the support in the next cell $\mathcal{C}_{k'}$, $k' = k'(k,i)$, $l = l(k,i)$ such that

$$F_{k,i}u = F_{k'(k,i),\,l(k,i)}u \qquad \forall u \in \mathbf{U}_0. \tag{32}$$

By definition, put $\mathbf{F} = \bigcup_{k\in\mathcal{K}} \mathcal{F}_k^0$. If the property (32) holds for all $F_{k,i} \in \mathcal{F}_k^0$ and all considered $k \in \mathcal{K}$ (those k for which $\mathcal{F}_k^0 = \emptyset$ are excluded), then the function u is called**F**-*smooth*. The set of **F**-smooth functions u, $u \in U$, is denoted by $\mathbf{U_F}$. It's clear that $\mathbf{U_F}$ is a linear space and $\mathbf{U}_0 \subset \mathbf{U_F}$. Next, we assume that $\mathcal{F}_k^0$ are non-empty sets.

Suppose now that the condition is satisfied

(B) The vector function $\varphi(t)$ is **F**-smooth (i.e., its components are **F** -smooth functions).

The condition (B) means that

$$F_{k,i}\varphi = F_{k'(k,i),\,l(k,i)}\varphi \qquad \forall F_{k,i} \in \mathcal{F}_k^0 \quad \forall k \in \mathcal{K}.$$

Thus, for a pair of neighboring cells $(\mathcal{C}_k, \mathcal{C}_{k'})$ and pairs of functionals $F_k = F_{k,i}$ and $F_{k'} = F_{k'(k,i),\,l(k,i)}$ the condition (A) is satisfied.

Theorem 3. *Let condition (B) be fulfilled. For* **F**-*smoothness of coordinate functions ω_j, $j \in \mathcal{J}$, it is necessary and sufficient that for each $k \in \mathcal{K}$ the vectors $F_{k,i}\varphi$ lay in a linear hull $\mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_{k,k'}\}$ $\forall F_{k,i} \in \mathcal{F}_k^0$, where $k' = k'(k,i)$.*

Proof: The formulated assertion follows from Theorem 2 (see also formulas (28) – (30)).

Let $\mathcal{L}\{\mathcal{F}_k^0\}$ be the linear hull of the set $\mathcal{F}_k^0$.

Definition 4 *If the ratios $\dim \mathcal{L}\{\mathcal{F}_k^0\} = q \quad \forall k \in \mathcal{K}$ are correct, then the* **F**-*smoothness is called maximal pseudo-smoothness.*

Note, that maximal pseudo-smoothness is not unique.

Theorem 4. *If $\varphi \in \mathbf{U_F}$, and* **F**-*smoothness is maximal pseudo-smoothness, then functions $\omega_j(t)$ are determined by the restriction of vector functions $\varphi(t)$ on the set $\mathrm{supp}\omega_j$.*

Proof: Let $\mathcal{C}_k$ be a subset of $\mathrm{supp}\omega_j$. Under the condition (B), all the vectors $\mathbf{a}_s$ from set A_k can be represented as linear combinations of vectors $F\varphi$ with functionals F from the set $\mathcal{F}_k^0$. Since the functionals of the set $\mathcal{F}_k^0$ have a support in the cell $\mathcal{C}_k$, then by formula (23) it follows that the function ω_j on the cell $\mathcal{C}_k$

is determined by the values of vector functions $\varphi(t)$ on this cell. A view of all cells in this set $\mathcal{S}_j$ allows us to conclude that the theorem have been proved.

6 Conclusion

The use of approximation relations does not guarantee the embedding of the resulting spaces. However, if the coordinate functions are smooth, then the spaces are embedded on embedded grids. In the one-dimensional casethis is true for spline spaces consisting of smooth (generally speaking, non-polynomial) splines of both the Lagrangian and Hermitian types.

In this paper the embedded spaces are built for functions defined on a differentiable manifold. The source objects are the q-cover manifold and associated approximation relations.

In the widened version of this work it will be represented the simple verifiable conditions for the spline-wavelet decomposition in the multidimensional case.

Acknowledgements: The authors would like to thank RFBR for partly support by Grant N 15-01-008817.

References:

[1] I.J. Schoenberg, Contribution to the problem of approximation of equidistant data by analytic functions, *Quart. Appl. Math.* 4, 1945, pp. 45-99.

[2] R. Courant, Variational methods for the solution of problems of equilibrium and vibrations, *Bulletin of the American Mathematical Society* 49, 1943, 1-23.

[3] G. Strang , G. Fix, Fourier Analysis of the finite element method in Ritz - Galerkin Theory, *Stud. Appl. Math.* 48, 1969, pp. 265-273.

[4] S.G. Michlin, *Approximation auf dem Kubischen Gitter* Berlin, 1970.

[5] Phillipe Ciarle. *The Finite Element Method for Elliptic Problems* North-Golland Publishing Company, 1978.

[6] K.Y. Dai, G.R. Liu, Free and forced vibration analysis using the smoothed finite element method (SFEM), *Journal of Sound and Vibration* 301, 2007, pp. 803-820.

[7] G.R. Liu, G.R. Zhang, Edge-based Smoothed Point Interpolation Methods," *International Journal of Computational Methods* 5, 2008, pp. 621-646.

[8] G.R. Liu, T. Nguyen-Thoi, K.Y. Lam, An edge-based smoothed finite element method (ES-FEM) for static, free and forced vibration analyses in solids, *Journal of Sound and Vibration* 320, 2009, pp. 1100-1130.

[9] Z.Q. Zhang, G.R. Liu, Upper and lower bounds for natural frequencies: A property of the smoothed finite element methods, *International Journal for Numerical Methods in Engineering* 84, 2010, pp. 149-178.

[10] T. Nguyen-Thoi, G.R. Liu, H. Nguyen-Xuan, et al., Adaptive analysis using the node-based smoothed finite element method (NS-FEM), *International Journal for Numerical Methods in Biomedical Engineering* 27, 2011, pp. 198-218.

[11] Y.K.Dem'yanovich, O.M.Kosogorov, Splines and Biorthogonal Systems," *Journal of Mathematical Siences* 165, 2010, pp. 501-510.

[12] Y.K.Dem'yanovich, L.M.Romanovskii, Spline-Wavelet Coarsening of Courant-Type Approximations, *Journal of Mathematical Sciences* United States, 199, 2014, pp. 414-431.

[13] C. Jiang, Z.Q. Zhang, G.R. Liu, X. Han, W. Zeng, An edge-based/node-based selective smoothed finite element method using tetrahedrons for cardiovascular tissues, *Engineering Analysis with Boundary Elements* 59, 2015, pp. 62-77.

[14] Y.K.Dem'yanovich, I.V.Gerasimov, Local Coarening of Simplicial Subdivisions, *J.of Math. Sci.ł* 216, 2016, pp.219-236.

[15] W. Zeng, G.R. Liu, Smoothed finite element methods (S-FEM): An overview and recent developments. Archives of Computational, *Methods in Engineering* 2016, doi: 10.1007/s11831-016-9202-3.

[16] Yu.K.Dem'yanovich, E.S.Kovtunenko, T.A.Safonova, Existence and uniqueness of spaces of splines of maximal pseudosmoothness, *J. of Math. Sci.* 224, 2017, pp.647-660.

[17] Yu.K.Dem'yanovich, On embedding and extended smoothness of spline spaces, *Far East Journal of Mathematical Sciences (FJMS)* 102, 2017, pp. 2025-2052.

[18] Yu.K.Demjanovich, Spline-wavelet decompositions on manifolds, *Journal of Mathematical Sciences* 150, 2008, pp.1787-1798.

[19] Yu.K.Demyanovich, O.V.Beliakova, Le Thi Ni Bich, Generalized Smoothness of Hermite Type Splines, *WSEAS Transactions on Mathematics* 17, 2018, pp. 359-368.

[20] Yu.K.Demyanovich, Spline Approximations on Manifolds, *International Journal of Wavelets. Multiresolution and Information Processing,* 4, 2006, pp.383-403.

[21] Yu.K.Demyanovich, A.V.Zimin, Wavelet decompositions on a manifold, *Journal of Mathematical Sciences* 150, 2008, pp.1929-1936.

[22] I.G. Burova, On left integro-differential splines and Cauchy problem, *International Journal of Mathematical Models and Methods in Applied Sciences* 9, 2015, pp. 683-690.

[23] I.G. Burova, O.V.Rodnikova, Application of integrodifferential splines to solving an interpolation problem, *Computational Mathematics and Mathematical Physics* 54, 2014, pp. 1903-1914.

[24] I.G. Burova, T.O.Evdokimova, On construction third order approximation using values of integrals, *WSEAS Transactions on Mathematics* 13, 2014, pp. 676-683.