Smoothness and Embedding of Spaces in FEM

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Abstract: The smoothness of functions is absolutely essential in the case of space of functions in the finit element method (FEM): incompatible FEM slowly converges and has evaluations in nonstandard metrics. The interest in smooth approximate spaces is supported by the desire to have a coincidence of smoothness of an exact solution and an approximate one. The construction of smooth approximating spaces is the main problem of the finit element method. A lot of papers have been devoted to this problem. The embedding of FEM spaces is another important problem; the last one is extremely essential in different approaches to approximate problems, speeding up of convergence and wavelet decomposition. This paper is devoted to coordinate functions obtained with approximate relations which are a generalization of the Strang-Michlin's identities. The aim of this paper is to discuss the pseudo-smoothness of mentioned functions and embedding of relevant FEM spaces. Here we have the necessary and sufficien conditions for the pseudo-smoothness, definitio of maximal pseudo-smoothness and conditions of the embedding for FEM spaces define on embedded subdivisions of smooth manifold. The relations mentioned above concern the cell decomposition of differentiable manifold. The smoothness of coordinate functions inside the cells coincides with the smoothness of the generating vector function of the right side of approximate relations so that the main question is the smoothness of the transition through the boundary of the adjacent cells. The smoothness in this case is the equality of values of functionals with supports in the adjacent cells. The obtained results give the opportunity to verify the smoothness on the boundary of support of basic functions and after that to assert that basic functions are smooth on the whole. In conclusion it is possible to say that this paper discusses the smoothness as the general case of equality of linear functionals with supports in adjacent cells of differentiable manifold. The results may be applied to different sorts of smoothness, for example, to mean smoothness and to weight smoothness. They can be used in different investigations of the approximate properties of FEM spaces, in multigrid methods and in the developing of wavelet decomposition.

Key–Words: finit element method, general smoothness, embedded spaces, minimal splines, approximation on manifold

1 Introduction

It is important to know about the smoothness of the discussed functions. In the particle it is absolutely essential in the case of the space of the functions in the finit element method (FEM). For example, in the simplest variant of FEM a construction of coordinate functions has to be in the energetic space of suitable self adjoint operator (see [1] – [8]). The usage of less smooth functions for the construction of FEM complicates the situation significantl , so that the incompatible FEM slowly converges and has evaluations in nonstandard metrics.

On the other hand it is often needed to calculate some functionals on the solution (for example, the value of the solution or its derivatives in a point); for that sometimes it needs the additional smoothness of an approximate solution.

Interest in smooth approximate spaces is also supported by the circumstance. The circumstance that the exact solution is often so smooth that it appears to have the desire of coincidence of smoothness of exact solution and approximate one (see [9] – [27]).

In the paper [9] the cell-wise strain smoothing operations are incorporated into conventional finit elements and the smoothed finit element method for 2D elastic problems is proposed. The paper [10] examines the theoretical bases for the smoothed fi nite element method, which is formulated by incorporating the cell-wise strain smoothing operation into the standard compatible finit element method. The smoothed finit element method is discussed in [11]. An edge-based smoothed finit element method is implied to improve the accuracy and convergence rate of the standard finit element method for elastic solid mechanical problems and extended to more general cases (see [12]). In [14] the cell-based smoothed fi nite element method is used for the refinemen of the accuracy and stability of the standard finit element method.

Note also that under the condition of high velocity of convergence in the original space (for example, in the energetic space) it is possible to get an evaluation of convergence in the spaces of the highest smoothness (using, for example, an analog of Markov's inequality).

According to what has been said a certain investigation of smoothness of approximate solutions is required.

It is very important that the embedding property of the FEM spaces on the embedding subdivisions exists. This property is useful in the estimates of approximation for FEM, in the acceleration of convergence, in the wavelet decomposition and so on.

We note that such property isn't always right. Discuss a simple example of violation of this property.

Consider the grid

$$X: \ \ldots < x_{-1} < x_0 < x_1 < \ldots,$$

and approximate relations

$$\sum_j \mathbf{a}_j \omega_j(t) = \varphi(t), \qquad \mathrm{supp}\omega_j \subset [x_j, x_{j+3}],$$

where $\mathbf{a}_j$ are three-dimensional vectors

$$\det(\mathbf{a}_j, \mathbf{a}_{j+1}, \mathbf{a}_{j+2}) \neq 0 \qquad \forall j \in \mathcal{Z},$$

and $\varphi(t)$ is a three-dimensional vector function $\varphi(t) = (1, t, t^2)^T$.

If $\mathbf{a}_j = \varphi(x_{j+1})$, then

$$\omega_j(t) = \frac{(t - x_j)(t - x_{j-1})}{(x_{j+1} - x_j)(x_{j+1} - x_{j-1})}$$

$$\text{for} \quad t \in (x_j, x_{j+1}),$$

$$\omega_j(t) = \frac{(x_{j+2} - t)(t - x_j)}{(x_{j+2} - x_{j+1})(x_{j+1} - x_j)}$$

$$\text{for} \quad t \in (x_{j+1}, x_{j+2}),$$

$$\omega_j(t) = \frac{(x_{j+3} - t)(x_{j+2} - t)}{(x_{j+3} - x_{j+1})(x_{j+2} - x_{j+1})}$$

$$\text{for} \quad t \in (x_{j+2}, x_{j+3}).$$

Now discuss an enlarged grid

$$\widehat{X}: \ \ldots < \widehat{x}_{-1} < \widehat{x}_0 < \widehat{x}_1 < \ldots,$$

where $\widehat{x}_j = x_j$ for $j \leq k$, $\widehat{x}_{j-1} = x_j$ for $j > k + 1$. We defin the coordinate functions by approximate relations

$$\sum_j \widehat{\mathbf{a}}_j \widehat{\omega}_j(t) = \varphi(t), \qquad \mathrm{supp}\, \widehat{\omega}_j \subset [\widehat{x}_j, \widehat{x}_{j+3}],$$

where $\widehat{\mathbf{a}}_j$ are three-dimensional column vectors with property

$$\det(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{j+1}, \widehat{\mathbf{a}}_{j+2}) \neq 0 \qquad \forall j \in \mathcal{Z},$$

and $\varphi(t)$ is the previous three-dimensional vector function $\varphi(t) = (1, t, t^2)^T$.

If $\widehat{\mathbf{a}}_j = \varphi(\widehat{x}_{j+1})\ \forall j \in \mathcal{Z}$, then we obtain the function $\widehat{\omega}_j(t)$ by approximate relations; as a result the formulas for $\omega_j(t)$ are similar to formulas for $\omega_j(t)$ (in last one it needs to change x_s by $\widehat{x}_s$).

It is clear to see that the functions ω_j and $\widehat{\omega}_j$ are continuous on the interval (α, β), where $\alpha = \lim_{i\to-\infty} x_i$, $\beta = \lim_{i\to+\infty} x_i$.

Each system $\{\omega_j\}_{j\in\mathcal{Z}}$ and $\{\widehat{\omega}_j\}_{j\in\mathcal{Z}}$ is linear independent system. Consider functionals $g_i(u) = u(x_{i+1})$. The system of functionals $\{g_i\}_{i\in\mathcal{Z}}$ is biorthogonal to the system of functions $\{\omega_j\}_{j\in\mathcal{Z}}$.

By the definitio of grid $\widehat{X}$ we have

$$\widehat{\omega}_j(t) = \omega_j(t) \quad \text{for } j \leq k - 3,$$

$$\widehat{\omega}_j(t) = \omega_{j+1}(t) \quad \text{for } j \geq k + 2.$$

Now we demonstrate that the function $\widehat{\omega}_{k-2}$ can't be represented by linear combination of the functions ω_i.

Proof by contradiction. Suppose that constants c_{-2}, c_{-1}, c_0, c_1 exist such that the relation

$$\widehat{\omega}_{k-2} = c_{-2}\omega_{k-2} + c_{-1}\omega_{k-1} + c_0\omega_k + c_1\omega_{k+1} \quad (1)$$

is fulfilled It is clear that other functions ω_j don't need because of the disposal of their supports.

Applying the functionals g_i for $i = k - 2, k - 1, k, k + 1$, we obtain a false formula:

$$\widehat{\omega}_{k-2}(t) = \omega_{k-2}(t) + \widehat{\omega}_{k-2}(x_k)\omega_k(t).$$

Thus the supposition that the relation (1) is right isn't correct. This concludes the proof.

Now we note that if vectors $\mathbf{a}_j$ are define by the relations

$$\mathbf{a}_j = \mathbf{a}^*_j = (1, (x_j + x_{j+1})/2, x_j x_{j+1})^T,$$

then we obtain splines $\omega_j^*(t)$ with maximal smoothness (they are continuously differentiable quadratic splines: for splines of the second degree with support mentioned above such smoothness is maximal possible). The spaces of such splines are embedded in each other on embedded grids.

Here we consider the coordinate functions obtained with the approximate relations which are a generalization of the Strang-Michlin's identities. This paper is devoted to the pseudo-smoothness of the mentioned functions and the embedding of relevant FEM spaces. We formulate the necessary and sufficien conditions for the pseudo-smoothness, introduce maximal pseudo-smoothness and prove the embedding of FEM spaces define on embedded subdivisions of smooth manifold.

Next we briefl describe the obtained results. The support of the coordinate functions of FEM is the union of a certain number of elementary cells (for example, simplicial cells in the case of using of Courant's basis).

The smoothness of coordinate functions inside of the cells coincides with the smoothness of the generating vector function for the right side of the approximate relations so that the main question is the smoothness of transition through the boundary of adjacent cells.

For example, in the case of a smooth boundary between two cells it is possible to discuss the limit of derivatives in the direction which is orthogonal to the boundary in its fi ed point. The mentioned limit values could be discussed as results of action of two functionals: one of them with support in the firs cell, and another one with support in the second cell. The smoothness in this case is the equality of values of the functionals.

This paper discusses the general case of linear functionals with support in adjacent cells. Therefore it discusses the essential generalization of smoothness. The obtained results give the opportunity to verify the smoothness on the boundary of support of the basic functions and after that to assert that the basic functions are smooth on the whole.

2 Notion and auxiliary assertions

Consider a smooth n-dimensional (generally speaking, noncomact) manifold $\mathcal{M}$ (i.e. topological space where each point has a neighborhood which is diffeomorphic to the open n-dimensional ball of Euclidean space $\mathbf{R}^n$).

Let $\{U_\zeta\}_{\zeta\in\mathcal{Z}}$ be a family of opened sets covering $\mathcal{M}$, and such homeomorphisms ψ_ζ, $\psi_\zeta : E_\zeta \mapsto U_\zeta$ opened balls E_ζ of the space $\mathbf{R}^n$ that the maps

$$\psi_\zeta^{-1}\psi_{\zeta'} : \psi_{\zeta'}^{-1}(U_\zeta \cap U_{\zeta'}) \mapsto \psi_\zeta^{-1}(U_\zeta \cap U_{\zeta'})$$

(for all $\zeta, \zeta' \in \mathcal{Z}$, for which the map $U_\zeta \cap U_{\zeta'} \neq \emptyset$) are continuously differential (needed a number of times); here $\mathcal{Z}$ is a set of indices.

We discuss a map $\psi_\zeta : E_\zeta \mapsto U_\zeta$ and a set $\{\psi_\zeta : E_\zeta \mapsto U_\zeta \mid \zeta \in \mathcal{Z}\}$; the last one, called atlas, define the manifold $\mathcal{M}$.

Let $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathcal{J}}$ be a covering family for manifold $\mathcal{M}$ where subsets $\mathcal{S}_j$ are homeomorphic to opened n-dimensional ball; thus

$$\bigcup_{j\in\mathcal{J}} \mathcal{S}_j = \mathcal{M},$$

where $\mathcal{J}$ is an ordered set of indices. The sets $\mathcal{S}_j$ are called the elements of cover $\mathcal{S}$; the boundary of the set $\mathcal{S}_j$ is denoted $\partial\mathcal{S}_j$.

Consider set

$$\mathcal{C}_{(t)} = \bigcap_{j\in\mathcal{J},\, \mathcal{S}_j\ni t} \mathcal{S}_j.$$

for each point $t \in \mathcal{M}\setminus\bigcup_{j\in\mathcal{J}}\partial\mathcal{S}_j$. Collection $\{\mathcal{C}_{(t)}\}$ at most countable; later on we denote mentioned sets by $\mathcal{C}_i$, $i \in \mathcal{K}$, where $\mathcal{K}$ is an ordered set of indices.

We have $\mathcal{C} = \{\mathcal{C}_i \mid i \in \mathcal{K}\}$, and the next relations are right:

$$\mathcal{C}_{i'} \cap \mathcal{C}_{i''} = \emptyset \quad \text{for} \quad i' \neq i'',\ i', i'' \in \mathcal{K},$$

$$Cl\ (\mathcal{S}_j) = Cl\left(\bigcup_{\mathcal{C}_i\subseteq\mathcal{S}_j} \mathcal{C}_i\right),$$

$$Cl\left(\bigcup_{i\in\mathcal{K}} \mathcal{C}_i\right) = Cl\ (\mathcal{M}); \tag{2}$$

here Cl is closure in topology of manifold $\mathcal{M}$.

Thus, the aggregates $\mathcal{M}$ and $\mathcal{S}_j$ are split into sets $\mathcal{C}_i$, so that the cover $\mathcal{S}$ is associated with the collection $\mathcal{C}$; the rule of association described above is denoted by $\mathcal{F}$: $\mathcal{C} = \mathcal{F}(\mathcal{S})$. The collection $\mathcal{C}$ is called *the subdivision of the cover* $\mathcal{S}$.

Definitio 1. *If all sets $\mathcal{C}_i$ from $\mathcal{F}(\mathcal{S})$ are homeomorphic to an open ball then $\mathcal{S}$ is called a cover of a simple structure; in this case set $\mathcal{C}_i$ is named a cell.*

Later on we discuss the cover of a simple structure.

Definitio 2. *Let $t \in \mathcal{M}$ be a fixed point; a number $\kappa_t(\mathcal{S})$ of elements of the collection $\{j \mid t \in \mathcal{S}_j\}$ is called a multiplicity of cover of point t by the family $\mathcal{S}$.*

Definitio 3. *If there exists natural number q, such that an identity*

$$\kappa_t(\mathcal{S}) = q, \qquad (3)$$

is right almost everywhere for $t \in \mathcal{M}$ then $\mathcal{S}$ is called q-covering family (for $\mathcal{M}$), and the number q is named a multiplicity of cover of manifold $\mathcal{M}$ by the family $\mathcal{S}$.

Definitio 4. *A cell $\mathcal{C}_{i'}$ is named a neighboring cell to the cell $\mathcal{C}_i$ ($i, i' \in \mathcal{K}$) in subdivision of the cover $\mathcal{S}$, if $i \neq i'$ and there exists a point t, belonging to the boundary $\partial\mathcal{C}_i$ of cell $\mathcal{C}_i$, a neighborhood of which belongs to $\mathcal{C}_{i'} \bigcup Cl\ (\mathcal{C}_i)$.*

It's clear to see that if cell $\mathcal{C}_{i'}$ is neighbor to cell $\mathcal{C}_i$ then $\mathcal{C}_i$ is neighbor to cell $\mathcal{C}_{i'}$; the cells $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ are named adjacent cells (in subdivision $\mathcal{C}$ of the family $\mathcal{S}$).

Definitio 5. *Let $\mathcal{S}$ be a q-covered family, let $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ be arbitrary adjacent cells (in subdivision $\mathcal{C}$ of the family $\mathcal{S}$). If the difference $\{j \mid \mathcal{S}_j \supset \mathcal{C}_i\}\backslash\{j' \mid \mathcal{S}_{j'} \supset \mathcal{C}_{i'}\}$ contains p elements (p is a positive integer) then $\mathcal{S}$ is called p-graduating q-covering family for manifold $\mathcal{M}$.*

It is evident that $p \leq q$.

3 Equipment of cover

Consider a family $A = \{\mathbf{a}_j\}_{j\in\mathcal{J}}$ of q-dimensional vectors $\mathbf{a}_j$. The family A is called an equipment of the manifold cover $\mathcal{S}$; thus each set $\mathcal{S}_j$ of the cover $\mathcal{S}$ coincides with vector $\mathbf{a}_j$ of space $\mathbf{R}^q$.

In what follows equipment A of family $\mathcal{S}$ is sometimes denoted $A_{(\mathcal{S})}$, and the vector $\mathbf{a}_j$, coinciding with the set $\mathcal{S}_j$, is denoted $A|_{\mathcal{S}_|}$ (thus in the discussed case $A|_{\mathcal{S}_|} = \mathbf{a}_j$).

Definitio 6. *Let t be a point of manifold $\mathcal{M}$, and let $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathbf{Z}}$ be q-covered family for $\mathcal{M}$. If the vector system*

$$A_{\langle t\rangle} = \{\mathbf{a}_j \mid j \in \mathcal{J}, \mathcal{S}_j \ni t\} \qquad (4)$$

is the basis of space $\mathbf{R}^\mathbf{q}$ almost everywhere for $t \in \mathcal{M}$ then we say that $A_{(\mathcal{S})}$ is the complete equipment of manifold cover.

By (2) – (3), (4) it follows that if A is the complete equipment of family $\mathcal{S}$, $\mathcal{C}$ is equal to $\mathcal{F}(\mathcal{S})$ and i is a fi ed number, $i \in \mathcal{K}$ then relations

$$A_{\langle t'\rangle} = A_{\langle t''\rangle} \qquad \text{for} \quad \forall t', t'' \in \mathcal{C}_i, \qquad (5)$$

are fulfilled

By definition put

$$A_i = A_{\langle t\rangle} \quad \text{for} \quad t \in \mathcal{C}_i. \qquad (6)$$

It is easy to see that if $\mathcal{S}$ is a p-graduated manifold cover and $\mathcal{C}_i$, $\mathcal{C}_{i'}$ are adjacent cells then a number of vectors in sets $A_i\backslash A_{i'}$ is equal to p (for all $i, i' \in \mathcal{K}$).

4 Finite-element spaces (spaces of minimal splines)

We say that function u is define on $\mathcal{M}$, if there is a family of functions $\{u_\zeta(x)\}_{\zeta\in\mathcal{Z},\S\in\mathcal{U}_{\zeta'}}$ such that

$$u_\zeta(\psi_\zeta^{-1}(\xi)) \equiv u_\zeta(\psi_{\zeta'}^{-1}(\xi))$$

$$\forall\xi \in U_\zeta \cap U_{\zeta'}, \quad \zeta, \zeta' \in \mathcal{Z};$$

and $u(\xi) = u_\zeta(\psi_\zeta^{-1}(\xi))$ for $\xi \in U_\zeta$.

Linear spaces of functions prescribed on $\mathcal{M}$ are define by the atlas with usage of the relevant spaces of functions define on balls E_ζ.

Let $\mathbf{X}(\mathcal{M})$ be a linear space of (Lebesgue measurable) functions define on manifold $\mathcal{M}$, where a symbol $\mathbf{X}$ denotes C^s or L_q^s; thus, the spaces $\mathbf{X}(\mathcal{M})$ defin by qualities

$$\mathbf{X}(\mathcal{M}) = \{u \mid u \circ \psi_\zeta \in \mathbf{X}(E_\zeta) \quad \forall\zeta \in \mathcal{Z}\};$$

note that $C^s(E_\zeta)$ and $L_q^s(E_\zeta)$ are the usual spaces of functions define on E_ζ ($1 \leq q \leq +\infty$, $s = 0, 1, 2, \ldots$).

Let $\mathbf{X}^*$ be dual space to space $\mathbf{X}$; it consists of functionals f, define by identity

$$\langle f, u\rangle \equiv \langle f_\zeta, u_\zeta\rangle_\zeta,$$

where $f_\zeta \in (\mathbf{X}(E_\zeta))^* \quad \forall\zeta \in \mathcal{Z}$, and $\{f_\zeta\}_{\zeta\in\mathcal{Z}}$ is a family of functionals representing the functional f.

If the value $\langle f, u\rangle$ of the functional $f \in (\mathbf{X}(\mathcal{M}))^*$ is define by the values of function u on the set $\Omega \subset \mathcal{M}$ $\forall u \in \mathbf{X}(\mathcal{M})$ then we write $\mathrm{supp} f \subset \Omega$; and if in this case Ω is a compact set then we say that functional f has compact support. In what follows we discuss functionals with compact support.

Introduce space $\mathcal{U}$ as a direct product of spaces $\mathcal{X}(\mathcal{C}_k)$:

$$\mathcal{U} = \bigotimes_{k\in\mathcal{K}} \mathcal{X}(\mathcal{C}_k).$$

By definitio we discuss the trace of function $u \in \mathcal{X}(\mathcal{M})$ on the cell $\mathcal{C}_k$ as an element of the space $\mathcal{X}(\mathcal{C}_k)$; thus we defin natural embedding of the space $\mathcal{X}(\mathcal{M})$ in the space $\mathcal{X}(\mathcal{M})$ $\mathcal{U}$: $\mathcal{X}(\mathcal{M}) \subset \mathcal{U}$.

Consider vector function $\varphi : \quad \mathcal{M} \to \mathbf{R}^{m+1}$ with components $[\varphi]_i(t)$ from space $\mathbf{X}(\mathcal{M})$ (here $m \geq 0$, $i = 0, 1, 2, \ldots, m$, $t \in \mathcal{M}$).

In what follows we discuss q-covering families of sets, where $q = m + 1$.

Theorem 1. *Let $\mathcal{S}$ be $m+1$-covering family (for manifold $\mathcal{M}$), and let $A = \{\mathbf{a}_j\}_{j\in\mathcal{J}}$ be a system of column vectors, forming a complete equipment of the family $\mathcal{S}$. Then a unique vector function (column) $\omega(t) = (\omega_j(t))_{j\in\mathcal{J}}$ exists, which satisfies relations*

$$A\omega(t) = \varphi(t), \qquad \omega_j(t) = 0 \quad \forall t \notin \mathcal{S}_j; \quad (7)$$

here and later on the symbol A is also used for the notation of matrix consisting of column vectors $\mathbf{a}_j$: $A = (\mathbf{a}_j)_{j\in\mathcal{J}}$.

Proof. According to the definitio of set A_i (see also the formulas (5) – (6) by (7) we have

$$\sum_{\mathbf{a}_j\in A_i} \mathbf{a}_j\omega_j(t) = \varphi(t) \qquad \forall t \in \mathcal{C}_i \quad \forall i \in \mathcal{K}. \quad (8)$$

The matrix of system (8) isn't singular because the set of vectors $\{\mathbf{a}_j \mid \mathbf{a}_j \in A_i\}$ is the basis for the space $\mathbf{R}^{m+1}$ according to the definitio of complete equipment; therefore unknown functions $\omega_j(t)$, which are discussed for each fi ed $t \in \mathcal{C}_i$ and for each $i \in \mathcal{K}$, can be determined uniquely. This concludes the proof.

Corollary 1. *The next relations are right:*

$$\omega_j(t) = \frac{\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_i,\ s \neq j\}\ ||'^j\ \varphi(t)\Big)}{\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_i\}\Big)}$$

$$\textit{for}\ \forall t \in \mathcal{C}_i \subset \mathcal{S}_j, \quad \omega_j(t) = 0 \quad \forall t \notin \mathcal{S}_j; \qquad (9)$$

here the columns in the determinants in the numerator and in the denominator have the same order. The symbol $||'^j\ \varphi(t)$ indicates that column vector $\varphi(t)$ is needed in place of column vector $\mathbf{a}_j$.

Let $\mathbf{S}_m = \mathbf{S}_m(\mathcal{S}, A, \varphi)$ be a linear space obtained by closing the linear hull of set $\{\omega_j\}_{j\in\mathcal{J}}$ in the topology of pointwise convergence:

$$\mathbf{S}_m = \mathbf{S}_m(\mathcal{S}, A, \varphi) = Cl_p\{\widetilde{u} \mid \widetilde{u}(t) =$$

$$= \sum_{j\in\mathcal{J}} c_j\omega_j(t) \quad \forall t \in \mathcal{M}\ \ \forall c_j \in \mathbf{R}^1\};$$

(symbol Cl_p denotes closure in mentioned topology). The space $\mathbf{S}_m$ is called *a space of minimal* $(\mathcal{S}, A, \varphi)$-*splines* or *a space of finite elements* (of order m) on manifold $\mathcal{M}$, Triple $(\mathcal{S}, A, \varphi)$ is named *a generator of space* $\mathbf{S}_m$, and functions ω_j are called *coordinate functions* of the space $\mathbf{S}_m$. Correlations 7 are called *approximation relations.*

If the family $\mathcal{S}$ is $r+1$-graduating cover (here r is a positive integer) then we say that $(\mathcal{S}, A, \varphi)$-splines have height r. If $r = 0$ then the splines are named *splines of the Lagrange type*, if $r > 0$ then the splines are called *splines of the Hermite type.*

Theorem 2. *Under the conditions of Theorem 1, the linear independence of the component of vector function $\varphi(t)$ on cell $\mathcal{C}_i$ is equivalent to the linear independence of the function system $\{\omega_j(t) \mid \mathcal{C}_i \subseteq \mathcal{S}_j\}$ on the cell.*

Proof follows from the linear system (8), because the matrix of the mentioned system is nonsingular.

Theorem 3. *Suppose the conditions of Theorem 1 are fulfilled. If the components of vector function $\varphi(t)$ are linear independent on each cell $\mathcal{C}_i$, $i \in \mathcal{K}$, then the system of functions $\{\omega_j(t)\}_{j\in\mathcal{J}}$ is linear independent on the manifold $\mathcal{M}$.*

Proof. Let t be a point belonging to $t \in \mathcal{C}_i$, where i is a fi ed number, $i \in \mathcal{K}$. Considering identity $\sum_{j\in\mathcal{J}} c_j\omega_j(t) \equiv 0$ for $t \in \mathcal{C}_i$, we see that nonzero summands have indices j, which belong to the set $\{j \mid \mathcal{C}_i \subseteq \mathcal{S}_j\}$.

Taking into account the nonvanishing of the determinant of system (8) and the linear independence of the component of vector function $\varphi(t)$ on cell $\mathcal{C}_i$, we see that all coefficien c_j with mentioned indices are equal to zero. Because we can fin index $i = i(j)$ for each $j \in \mathcal{J}$ so that $\mathcal{C}_i \subseteq \mathcal{S}_j$, therefore all coeffi cients c_j are equal to zero.

5 Pseudo-continuity of splines (or finit elements)

Let F_k be a linear functional $F_k \in (\mathbf{X}(\mathcal{M}))^*$ with support in the cell $\mathcal{C}_k$, $\mathrm{supp} F_k \in \mathcal{C}_k$.

If cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent then by definitio put $A_{k,k'} = \{\mathbf{a}_j \,|\, \mathbf{a}_j \in A_k \cap A_{k'}\}$. In what follows we fi an order of column vectors $\mathbf{a}_j$ in the set $A_{k,k'}$. Sometimes we discuss the set $A_{k,k'}$ as a matrix with a mentioned order of columns.

Consider a condition

(A) Relation

$$F_k\varphi = F_{k'}\varphi \qquad (10)$$

is right.

Lemma 1. *Suppose condition (A) is right and indices $k, k' \in \mathcal{K}$ are fixed. Let $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ be adjacent cells, and let F_k, $F_{k'}$ be corresponding functionals.*

Then for the relation

$$F_k\,\omega_j = 0 \quad \textit{for} \quad \mathbf{a}_j \in A_k \backslash A_{k,k'}, \qquad F_{k'}\,\omega_{j'} = 0$$

$$\textit{for} \quad \mathbf{a}_{j'} \in A_{k'} \backslash A_{k,k'}, \qquad (11)$$

to be right it is necessary, and if the system of vectors $(A_k \cup A_{k'})\backslash A_{k,k'}$ is linear independent, then it is sufficient, to have relations

$$F_k\omega_j = F_{k'}\,\omega_j \qquad \forall j \in A_{k,k'}. \qquad (12)$$

Proof. We have

$$\sum_{\mathbf{a}_j\in A_k} \mathbf{a}_j\omega_j(t) = \varphi(t) \qquad \forall t \in \mathcal{C}_k, \qquad (13)$$

$$\sum_{\mathbf{a}_{j'}\in A_{k'}} \mathbf{a}_{j'}\omega_j(t) = \varphi(t) \qquad \forall t \in \mathcal{C}_{k'}. \qquad (14)$$

applying functionals F_k, $F_{k'}$ to relations (13) and (14) accordingly, we get

$$\sum_{\mathbf{a}_j \in A_k} \mathbf{a}_j F_k \omega_j = F_k \varphi, \qquad (15)$$

$$\sum_{\mathbf{a}_{j'} \in A_{k'}} \mathbf{a}_{j'} F_{k'} \omega_{j'} = F_{k'} \varphi. \qquad (16)$$

Comparing (15) and (16) and applying suppositions (10) – (11), we have

$$\sum_{\mathbf{a}_j \in A_{k,k'}} \mathbf{a}_j F_k \omega_j = \sum_{\mathbf{a}_j \in A_{k,k'}} \mathbf{a}_j F_{k'} \omega_j$$

Using the linear independence of system $A_{k,k'}$, we obtain formula (12). The necessity has been proved.

The proof of sufficien y is trivial: if the vector system $(A_k \cup A_{k'}) \backslash A_{k,k'}$ is linear independent then by (10) and (12) it follows relations (11). This completes the proof.

Theorem 4. *Let $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ be adjacent cells. Suppose condition (A) is fulfilled. Then for the equalities*

$$F_k \omega_j = F_{k'} \omega_j \qquad \forall j \in \mathcal{J}, \qquad (17)$$

to be right it is necessary and sufficient for the relations (11) to be fulfilled.

Proof. Sufficien y. If relation (11) is true then (according to Lemma 1) relation (12) is right so that

$$F_k \omega_j = F_{k'} \omega_j \qquad \text{for} \quad \mathbf{a}_j \in A_k \cap A_{k'}.$$

If $\mathbf{a}_j \notin A_k \cap A_{k'}$ then

$$\mathrm{supp} F_k \cap \mathrm{supp}\omega_j = \emptyset, \qquad \mathrm{supp} F_{k'} \cap \mathrm{supp}\omega_j = \emptyset,$$

and therefore

$$F_k \omega_j = F_{k'} \omega_j \qquad \mathbf{a}_j \notin A_k \cup A_{k'}.$$

Thus, relation (17) is true. Sufficien y has been proved.

Necessity. Now we suppose that relation (17) is right. In particular the equalities

$$F_k \omega_j = F_{k'} \omega_j \qquad \mathbf{a}_j \in A_k \backslash A_{k,k'} \qquad (18)$$

and

$$F_k \omega_j = F_{k'} \omega_j \qquad \mathbf{a}_j \in A_{k'} \backslash A_{k,k'} \qquad (19)$$

are fulfilled Because for $\mathbf{a}_j \in A_k \backslash A_{k,k'}$ we have $\mathrm{supp} F_k \cap \mathrm{supp}\omega_j = \emptyset$, then the relation (18) can be written in the form

$$F_k \omega_j = 0 \qquad \mathbf{a}_j \in A_k \backslash A_{k,k'}.$$

Thus the firs relation of (11) has been received. Analogously by (19) we get the second relation of (11). The necessity has been established. This concludes the proof.

Under condition (10) we put

$$F_{(k,k')} \varphi = F_k \varphi = F_{k'} \varphi.$$

Theorem 5. *Suppose the conditions of Theorem 4 are fulfilled. Then relations (17) is equivalent to relation*

$$F_{(k,k')} \varphi \in \mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_{k,k'}\}. \qquad (20)$$

Proof. Taking into account Theorem 4 we see that it is sufficien to prove that relation (20) is equivalent to formulas (11). Substituting the right part of formula (9) in the firs formula of relations (11) we obtain

$$\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_k,\, s \neq j\} \,||'^j\, F_{(k,k')}\varphi\Big) = 0$$

$$\mathbf{a}_j \in A_k \backslash A_{k,k'}. \qquad (21)$$

Relations (21) show that the vector $F_{(k,k')}\varphi$ is situated in the linear spans $\mathcal{L}_j = \mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_k, s \neq j\}$, where j satisfie condition $\mathbf{a}_j \in A_k \backslash A_{k,k'}$. Hence the vector $F_{(k,k')}\varphi$ is contained in the intersection of the mentioned spans. The last one is equivalent to formula (20).

Considering the second formula of relations (11), analogously we obtain

$$\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_{k'},\, s \neq j'\} \,||'^{j'}\, F_{k'}\varphi\Big) = 0$$

$$\mathbf{a}_{j'} \in A_{k'} \backslash A_{k,k'},$$

and again we get formula (20). Using the equivalence of discussed formulas and taking into account Theorem 4, we see that necessity and sufficien y have been proved.

Corollary 2. *The first relation of formula (11) and the second relation of the mentioned formula are equivalent.*

6 Maximal pseudo-smoothness. Embedding of spaces

Let $\mathcal{F}_k$ be a set of linear functionals F belonging to $\mathcal{U}^*$ and $\mathrm{supp} F \in \mathcal{F}_k$. Let $\mathcal{U}_0$ be linear subspace of the space $\mathcal{U}$ and $\mathcal{X}(\mathcal{M}) \subset \mathcal{U}_0$.

Suppose that for each pair of adjacent cells ($\mathcal{C}_k$ and $\mathcal{C}_{k'}$) there are functionals F_k $F_{k'}$ such that relations

$$F_k u = F_{k'} u \qquad \forall u \in \mathcal{U}_0. \qquad (22)$$

are right.

Let $\mathcal{F}_k^0$ be a set of functionals with property (22) ($\forall k \in \mathcal{K}$); thus $\mathcal{F}_k^0$ is the set of functionals, for each of which there is a cell $\mathcal{C}_{k'}$ (ajecent to $\mathcal{C}_k$) and functional $F_{k'}$ (such that $\mathrm{supp} F_{k'} \subset \mathcal{C}_{k'}$) with property (22). By definitio we put $\mathcal{F} = \bigcup_{k\in\mathcal{K}} \mathcal{F}_k^0$. If property (22) is fulfille then the function u is called *$\mathcal{F}$-smooth function*. The set of $\mathcal{F}$-smooth function is denoted by $\mathcal{U}_\mathcal{F}$. It is clear to see that $\mathcal{U}_0 \subset \mathcal{U}_\mathcal{F}$.

Discuss the next condition

(B) The vector function $\varphi(t)$ is $\mathcal{F}$-smooth (that is its component are $\mathcal{F}$-smooth functions) so that the condition (A) is fulfille for all pairs $(\mathcal{C}_k, \mathcal{C}_{k'})$ of adjacent cells and relevant functionals.

By previous results it follows the next assertion.

Theorem 6. *Suppose the condition (B) is correct. Then for the coordinate functions ω_j to be $\mathcal{F}$-smoothness it is necessary and sufficient for relevant vectors $F_{(k,k')}\varphi$ defined by relation $F_{(k,k')}\varphi = F_k\varphi = F_{k'}\varphi$ to be in linear hull $\mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_{k,k'}\}$.*

Proof. The assertion formulated above follows immediately by Theorem 5.

By $\mathcal{L}\{\mathcal{F}_k^0\}$ denote the linear hull of set of functionsals $\mathcal{F}_k^0$. If relations

$$\dim \mathcal{L}\{\mathcal{F}_k^0\} = m+1 \qquad \forall k \in \mathcal{K},$$

are right then $\mathcal{F}$-smoothness is called *maximal smoothness*.

Theorem 7. *If $\varphi \in \mathcal{U}_\mathcal{F}$ and $\mathcal{F}$-smoothness is maximal then under condition (B) the functions $\omega_j(t)$ are defined by trace of vector function $\varphi(t)$ on the set $\mathrm{supp}\,\omega_j$ (more precisely by values of components of the vector function $\varphi(t)$ on the all cells, which belong to the set $\mathcal{S}_j$, and by values of functionals $F \in \mathcal{F}_k^0$ on the traces of mentioned components).*

Proof. Under condition (B) all vectors belonging to the set A_i can be represented as linear combinations of vectors $F\varphi$, where functionals F belong to the set of functionals $\mathcal{F}_i^0$; the last ones have support in the cell $\mathcal{C}_k$. By formula (9) we see that the function ω_j on the cell $\mathcal{C}_i$ is define with values of vector function $\varphi(t)$ on mentioned cell. Taking into account this circumstance for each cell of the set $\mathcal{S}_j$, we see that this completes the proof.

In what follows we suppose that the assumption of Theorem 7 are fulfilled

Let one more cover $\widehat{\mathcal{S}}$ be introduced on the manifold $\mathcal{M}$, different from the previous one by only a finit number of covering sets, and so that the multiplicity of the cover remains the same, and the corresponding subdivision $\widehat{\mathcal{C}} = \mathcal{F}(\widehat{\mathcal{S}})$ is an enlargement of the previous one. The equipment of the former covering sets we save, and the new covering sets we equip with vectors (one vector from $\mathcal{R}^{m+1}$ for each set) so that the resulting equipment is complete. As before, we construct the family of functions $\widehat{\omega}_j$ from approximation relations of the form

$$\sum_{\widehat{\mathbf{a}}_j \in \widehat{A}_i} \widehat{\mathbf{a}}_j \widehat{\omega}_j(t) = \varphi(t) \qquad \forall t \in \widehat{\mathcal{C}}_i \quad \forall i \in \widehat{\mathcal{K}}, \tag{23}$$

and also the notation appearing here refers to the enlargement and acquire a clear meaning if we return to the formula (8). From (23) the functions $\widehat{\omega}_j(t)$ are uniquely determined. Using (23) and (8), we arrive at the identity

$$\sum_{j\in\widehat{\mathcal{J}}} \widehat{\mathbf{a}}_j \widehat{\omega}_j(t) = \sum_{j\in\mathcal{J}} \mathbf{a}_j \omega_j(t) \qquad \forall t \in \mathcal{M} \quad \forall \varphi \in \mathcal{U}_\mathcal{F}. \tag{24}$$

After the reduction of the same components in the right and left parts of identities (24) we arrive at an analogous identity, in which the number of terms in the sums of the left and right sides are finite

$$\sum_{j\in\widehat{\mathcal{J}}_0} \widehat{\mathbf{a}}_j \widehat{\omega}_j(t) = \sum_{j\in\mathcal{J}_0} \mathbf{a}_j \omega_j(t) \qquad \forall t \in \mathcal{M} \quad \forall \varphi \in \mathcal{U}_\mathcal{F}. \tag{25}$$

Consider the resulting identity (25) as a system of linear equations for the unknown $\widehat{\omega}_i(t)$ with a full rank matrix. Taking into account that $\{\widehat{\mathbf{a}}_i\}$ is the complete equipment, we can fin a principle minor to express $\widehat{\omega}_i(t)$ as linear combination of the functions $\omega_j(t)$.

Using the linear independence of the system $\{\omega_j\}_{j\in\mathcal{J}}$, we fin the calibration relations

$$\widehat{\omega}_i(t) = \sum_{j\in\mathcal{J}} G_j\widehat{\omega}_i \cdot \omega_j(t), \tag{26}$$

where $\{G_j\}_{j\in\mathcal{J}}$ is a system of functionals, which are biorthogonal to the system of functions $\{\omega_j\}_{j\in\mathcal{J}}$.

Denoting by $\widehat{\mathcal{S}}_m$ the linear hull of functions $\{\widehat{\omega}_j\}_{j\in\widehat{\mathcal{J}}}$ and taking into account the relations (26), we obtain the relation $\widehat{\mathcal{S}}_m \subset \mathcal{S}_m$.

7 Conclusion

The smoothness of functions belonging to approximate spaces in FEM is define by the smoothness of the coordinate functions used for the construction of the spaces.

The smoothness of coordinate functions inside cells are define by the smoothness of the generating vector function in approximate relations, but the smoothness of coordinate functions on the boundary of adjacent cells requires additional discussion.

Sometimes a number of pairs of adjacent cells in support of coordinate function are very large; therefore the investigation of the smoothness of all mentioned pairs is laborious. The result of this paper permits to restrict oneself to such investigation only on the boundary of support of the coordinate function.

This paper discusses general smoothness as a coincidence of values of two linear functionals on the appropriate functions where mentioned functionals have their supports in adjacent cells. It gives the opportunity to discuss different sorts of smoothness.

For example, for adjacent cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ with smooth boundary σ between them we put

$$F_k u = \lim_{\tau\to+0}\int_\sigma u(\xi+\tau n(\xi))d\xi,$$

$$F_{k'} u = \lim_{\tau\to+0}\int_\sigma u(\xi-\tau n(\xi))d\xi,$$

where $n(\xi)$ is a normal vector to the boundary σ in the point ξ; in that case the equality $F_k u = F_{k'} u$ is "mean smoothness".

Consider another example:

$$F_k u = \lim_{\tau\to+0}\psi(\tau)\frac{\partial u}{\partial n}(\xi+\tau n(\xi)),$$

$$F_{k'} u = \lim_{\tau\to+0}\psi(\tau)\frac{\partial u}{\partial n}(\xi-\tau n(\xi)), \xi\in\sigma,$$

where $\frac{\partial u}{\partial n}$ is the derivative with respect to vector n, and $\psi(\tau)$ is a weight function; now the equality $F_k u = F_{k'} u$ illustrates "weight smoothness" (see also [28] – [29]).

We would like to add that the embedding of FEM spaces is very important in different investigations of the approximate properties of FEM spaces, in multigrid methods and in the developing of wavelet decomposition.

In future we suppose to demonstrate the application of the obtained results to spline-wavelet treatment of numerical fl ws.

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