Second Order Cauchy Euler Equation and Its Application for Finding Radial Displacement of a Solid Disk using Generalized Trapezoidal Intuitionistic Fuzzy Number

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Abstract: - In this paper, the solution for second order Cauchy Euler equation is derived using generalized trapezoidal intuitionistic fuzzy number. Further as an application we study the problem for finding the radial displacement of a solid disk with given boundary condition in the form of generalized trapezoidal intuitionistic fuzzy number. The obtained solutions are drawn graphically for different radii using (α,β)-cuts.

Key-Words: - Fuzzy Sets; Cauchy-Euler differential equation; α,β–cuts; Trapezoidal Intuitionistic fuzzy number; Generalized Hukuhara Derivative, Solid Disk.

1 Introduction

Various real life dynamical phenomena can be easily modeled mathematically using differential equations. These equations have potential to describe the behaviour of system modeled for real world problems. But practically, in real life situations, the knowledge we use to model in the form of differential equations are often incomplete or vague. That means, while describing real life situations, many times, we encounter some kind of uncertainties. So the differential equation formed using those incomplete or vague information is not able to describe the real life phenomena correctly. It is very difficult to model real life situation precisely using imprecise information. Conventional theory of differential equation fails to handle this kind of vagueness.

The difference between real world phenomena and its modeled differential equations describes the certain kind of uncertainty. Fuzzy set theory introduced by Lofti Zadeh [1] in 1965, becomes a powerful tool for dealing such kind of uncertain environment in a natural way. He modifies the set theory to fuzzy set theory to study the vagueness. He introduced the concept of membership and non membership by assigning a value to each individual in the universal set representing its membership degree. This membership degree describes the similarity with the concept represented by the fuzzy set. The fuzzy set theory is further generalized by Atanassov ([2]-[4]), who introduced the concept of intuitionistic fuzzy set (IFS). An intuitionistic fuzzy set is characterized by degrees of membership and non-membership. These concepts allow defining differential equation under fuzzy context. Thus using these approaches fuzzy differential equations are introduced which becomes the modeling tool for various dynamical systems under the conditions of uncertainty.

Intuitionistic fuzzy set theory is more powerful tool to solve real world problems. In 1989, Atanassov et al. [4] have also developed an interval valued intuitionistic fuzzy sets, a generalization of the IFS. In past few years many authors have applied IFS theory to solve several problems in different application areas [5-9]. The intuitionistic fuzzy sets used in pattern recognition [5], sociometry [6], medicine [7], high school determination [8], academic career [9] and many more.

Ordinary and Partial differential equations have potential to model the real life physical phenomenon with different initial and boundary conditions. A differential equation involves one or more parameters, which plays a significant role to represent real world problems. But due to uncertainties in real life, the mathematical modeling done for physical phenomenon can encounter uncertainties in boundary or in initial conditions.

The initiation of fuzzy derivative was done by Hukuhara [10]. Then after this concept of fuzzy derivative was further studied and extended by many other researchers (see, [11]-[15]). The definition of generalized differentiability and strongly generalized differentiability was well explained in papers [12-15]. In 1987, Kaleva [16] first introduced the concept of fuzzy differential equations and gave the existence and uniqueness theorem for a solution to a fuzzy differential equation. In 1987, Seikkala [17] studied the notion of fuzzy initial value problem by using extension principle and the use of extremal solutions to deterministic initial value problems. The theory on fuzzy differential equations was published by Lakshmikantham and Mohapatra [18] in 2003. Friedman et al. [19] proposed a numerical algorithm for solving fuzzy ordinary differential equation. Diamond [20] in 2000 has derived a Lyapunov stability and periodicity for both time-dependent and autonomous cases.

Thereafter this topic has attracted widespread attention to many researchers and has started work in this direction (see for instance, [21]-[27]). Millani and Chandi ([21]-[22]) discussed the ordinary and partial differential equations under intuitionistic environment. Abbasbandy and Allahviranloo [23] proposed a numerical solution of fuzzy differential equation by Runge-kutta method with intuitionistic fuzzy treatment.

Many authors (see, [24]-[25]) have used the concept of fuzzy Laplace transform for solving various differential equations and their applications. Thus, the intuitionistic fuzzy differential equations model real life problems more precisely. In 2015, Mondal and Roy [26] described the generalized intuitionistic fuzzy Laplace transform method for solving first order generalized intuitionistic fuzzy differential equations. Mondal and Roy [27] solved second order intuitionistic fuzzy differential equation using generalized trapezoidal intuitionistic fuzzy number.

Fuzzy differential equations plays very important role in the modeling of real life problems. We can easily find many articles on the applications of fuzzy differential equations. For instance predator-prey model using fuzzy differential Duffing's equation is discussed by Ahmad et al. [28]. An application of first order linear fuzzy differential equations using variation of constants is discussed by Vasavi et al. [30]. Similarly, we can easily find a lot of application oriented research work on fuzzy ordinary differential equations using both analytical and numerical methods (see for instance, [29-33]).

As discussed above, a lot of research work is done on the fuzzy differential equations – ordinary as well as partial. Cauchy-Euler differential equation is a special form of a linear ordinary differential equation with variable coefficients. The second order Cauchy–Euler equations are used in various fields of science and engineering such as in time-harmonic vibrations of a thin elastic rod, problems on annual and solid disc, wave mechanics, etc. This paper deals with the Cauchy-Euler homogeneous second order linear differential equation with intuitionistic fuzzy boundary conditions. In addition, we apply this approach to solve the problem of solid disk whose differential equation is the second order linear ordinary differential equation with variable coefficients.

2 Basic Definitions

Definition 1 Fuzzy Set [1]: Let X be a fixed set. A fuzzy set A in X is a set of ordered pairs defined as $A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0, 1]\}$.

Definition 2 Height [30]: Let X be a fixed set. The height for a fuzzy set is defined as the largest value of the membership grade of any of the elements in the fuzzy set A , i.e. $h(A) = sup(\mu_A(x))$.

Definition 3 Intuitionistic Fuzzy Set [2]: Let X be any fixed set. An intuitionistic fuzzy set is defined as the set of the form:

$$A = \left\{ \begin{matrix} (x, \mu_A(x), \nu_A(x)) : x \in X, \mu_A(x) \in [0, 1], \\ \nu_A(x) \in [0, 1] \;\; 0 \le \mu_A(x) + \nu_A(x) \le 1 \end{matrix} \right\}$$

Here, $\mu_A(x), \nu_A(x) : X \to [0, 1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set A.

Definition 4 α–cut on Fuzzy set [24]: Let A be a fuzzy interval in R and $\alpha \in (0,1]$. The α -cut of A is the crisp set $[A]^\alpha$ that contains all elements with membership degree in A greater than or equal to α, i.e.

$$[A]^\alpha = \{x \in R | A(x) \ge \alpha\}.$$

Definition 5 Convex Fuzzy set [35]: A fuzzy set A on a real Euclidean space X is said to be convex fuzzy set iff the $\alpha -$ level set of A, denoted by $[A]^\alpha$ is a convex subset of X. If $X = R^n$, with R being the set of real numbers, then the fuzzy set A is convex iff the following condition holds: Given any two different points x and y in $[A]^\alpha$, then for any $\in [0,1]$, $ax + (1 - a)y \in [A]^\alpha$.

Definition 6 Normal Fuzzy set [24]: A fuzzy set A defined on fixed set X is said to be normal if and only if $\underset{x \in X}{sup} \mu_A(x) = 1$.

Definition 7 Fuzzy Number [24]: A fuzzy number is a fuzzy set on the real line that satisfies the conditions of normality and convexity.

Definition 8 Generalized Fuzzy Number [36]: Generalized fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; \omega)$, where $0 < \omega \le 1$ and

a_1, a_2, a_3, a_4 $(a_1 < a_2 < a_3 < a_4)$ are real numbers.

The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}(x) : R \to [0,1]$, (ii) $\mu_{\tilde{A}}(x) = 0$ for $x \le a_1$, (iii) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \le x \le a_2$, (iv) $\mu_{\tilde{A}}(x) = \omega$ for $a_2 \le x \le a_3$, (v) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \le x \le a_4$, (vi) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \le x$.

Definition 9 Generalized Triangular Fuzzy Number [36]: If $a_2 = a_3$, then $\tilde{A}$ is called a generalized triangular fuzzy number as $\tilde{A} = (a_1, a_2, a_4; \omega)$ or $(a_1, a_3, a_4; \omega)$ with membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\left(\dfrac{x - a_1}{a_2 - a_1}\right) & if \quad a_1 \le x \le a_2 \\ \omega\left(\dfrac{a_4 - x}{a_4 - a_2}\right) & if \quad a_2 \le x \le a_4 \\ 0 & otherwise \end{cases}$$

Definition 10 Triangular Fuzzy Number[36]: If $a_2 = a_3$ and $\omega = 1$ then $\tilde{A}$ is called a triangular fuzzy number as $\tilde{A} = (a_1, a_2, a_3)$ or $\tilde{A} = (a_1, a_3, a_4)$

Definition 11 Intuitionistic Fuzzy Number [30]: An intuitionistic fuzzy number $\tilde{A}^i$ is defined as follows (i) an intuitionistic fuzzy subset of real line, (ii) normal, (iii) a convex set for the membership function $\mu_{\tilde{A}^i}(x)$, i.e.

$$\mu_{\tilde{A}^i}(\lambda x_1 + (1 - \lambda)x_2) \ge min\left(\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)\right), \quad \forall\, x_1, x_2 \in R, \lambda \in [0,1]$$

(iv) a concave set for the non-membership function $\nu_{\tilde{A}^i}(x)$, i.e.

$$\mu_{\tilde{A}^i}(\lambda x_1 + (1 - \lambda)x_2) \ge max\left(\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)\right), \quad \forall\, x_1, x_2 \in R, \lambda \in [0,1]$$

Definition 12 Triangular Intuitionistic fuzzy number [30]: A triangular intuitionistic fuzzy number $\tilde{A}^i$ is a subset of intuitionistic fuzzy number in R with following membership and non-membership function as follows:

$$\mu_{\tilde{A}^i}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases},$$

$$\nu_{\tilde{A}^i}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'}, & a_1' \le x \le a_2 \\ \frac{x-a_2}{a_3'-a_2}, & a_2 \le x \le a_3' \\ 1, & otherwise \end{cases}$$

where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and triangular intuitionistic fuzzy number is denoted by $\tilde{A}^i{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$.

Definition 13 Trapezoidal Fuzzy Number [35]: A Trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$ is a fuzzy set, where a_1, a_2, a_3 and a_4 are real numbers and $a_1 \le a_2 \le a_3 \le a_4$ with membership function defined as

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & for\ a_1 \le x < a_2 \\ 1, & for\ a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & for\ a_3 < x \le a_4 \\ 0, & otherwise \end{cases}$$

Definition 14 Trapezoidal Intuitionistic Fuzzy Number [35]: A Trapezoidal intuitionistic fuzzy number is denoted by $A = ((a_1, a_2, a_3, a_4), (a_1', a_2, a_3, a_4'))$ where $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$ with membership and non-membership function defined as

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 < x \le a_4 \\ 0, & otherwise \end{cases},$$

$$\nu_A(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'}, & a_1' \le x < a_2 \\ 0, & a_2 \le x \le a_3 \\ \frac{x-a_3}{a_4'-a_3}, & a_3 < x \le a_4' \\ 1, & otherwise \end{cases}$$

with $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$.

Definition 15 Generalized Trapezoidal Intuitionistic Fuzzy Number [35]: A generalized Trapezoidal intuitionistic fuzzy number is denoted by $A = \left((a_1, a_2, a_3, a_4; w_A), (a_1', a_2, a_3, a_4'; u_A)\right)$ with membership and non-membership function defined as

$$\mu_A(x) = \begin{cases} w_A\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \le x \le a_2 \\ w_A, & a_2 \le x \le a_3 \\ w_A\left(\frac{a_4-x}{a_4-a_3}\right), & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases},$$

$$\nu_A(x) = \begin{cases} u_A\left(\frac{a_2-x}{a_2-a_1'}\right), & a_1' \le x < a_2 \\ u_A, & a_2 \le x \le a_3 \\ u_A\left(\frac{x-a_3}{a_4'-a_3}\right), & a_3 < x \le a_4' \\ 0, & otherwise \end{cases}$$

with $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$.

Definition 16 (α,β) - cuts on Intuitionistic fuzzy number [35]: Let $[A_1(\alpha), A_2(\alpha), A_1{}'(\beta), A_2{}'(\beta)]$ be the (α, β)-cuts of a trapezoidal intuitionistic fuzzy number A and ω, σ be the gradation of membership and non-membership function respectively then the intuitionistic fuzzy number is given by

$$A =$$

$$((A_1(\alpha = 0), A_1(\alpha = \omega), A_2(\alpha = \omega), A_2(\alpha = 0); \omega)$$

$$(A_1{}'(\beta = \sigma), A_1{}'(\beta = 0), A_2{}'(\beta = 0), A_2{}'(\beta = \sigma); \sigma)$$

Definition 17 Generalized Hukuhara derivative for first order [34]: The generalized Hukuhara derivative of a fuzzy valued function $f: (a, b) \to R_F$ at t_0 is defined as $f'(t_0) = lim_{h\to 0} \frac{f(t_0+h)\Theta_{gH} f(t_0)}{h}$.
If $f'(t_0) \in R_F$ exists, we say that f is generalized Hukuhara differentiable at t_0.
Also we say that $f(t)$ is (i) – gH differentiable at t_0 if
$\left|f'(t_0)\right|^\alpha = \left|f_1'(t_0, \alpha), f_2'(t_0, \alpha)\right|$
and $f(t)$ is (ii) - gH differentiable at t_0 if
$\left|f'(t_0)\right|^\alpha = \left|f_2'(t_0, \alpha), f_1'(t_0, \alpha)\right|$.

Definition 18 Generalized Hukuhara derivative for second order [24]: The second order generalized Hukuhara derivative of a fuzzy valued function $f:(a,b) \to R_F$ at t_0 is defined as

$f''(t_0) = lim_{h\to 0} \frac{f'(t_0+h)\Theta_{\mathrm{gH}} f'(t_0)}{h}$.

If $f''(t_0) \in R_F$, we say that $f'(t_0)$ is generalized Hukuhara at t_0.

Also we say that $f'(t_0)$ is (i) $-gH$ differentiable at t_0 if

$$f''(t_0,\alpha) = \begin{cases} \left(f_1'(t_0,\alpha), f_2'(t_0,\alpha)\right) \; if\ f\ is\ (i) \\ -gH\ differentiable\ on\ (a,b) \\ \left(f_2'(t_0,\alpha), f_1'(t_0,\alpha)\right) \; if\ f\ is\ (ii) \\ -gH\ differentiable\ on\ (a,b) \end{cases}$$

for all $\alpha \in [0,1]$, and that $f'(t_0)$ is (ii) – gH differentiable at t_0 if

$$f''(t_0,\alpha) = \begin{cases} \left(f_2'(t_0,\alpha), f_1'(t_0,\alpha)\right) \quad if\ f\ is\ (i) \\ -gH\ differentiable\ on\ (a,b) \\ \left(f_1'(t_0,\alpha), f_2'(t_0,\alpha)\right) \quad if\ f\ is\ (ii) \\ -gH\ differentiable\ on\ (a,b) \end{cases}$$

for all $\alpha \in [0,1]$.

3 General Second Order Cauchy Euler Equation

Let us consider a homogeneous general second order Cauchy-Euler equation with boundary conditions in the form TIFN. In mathematical terms, it is written as

$$b_0 x^2 \frac{d^2y}{dx^2} + b_1 x \frac{dy}{dx} + b_2 y = 0 \qquad (1)$$

with the given boundary conditions $y(0) = \tilde{a}$ and $y(L) = \tilde{b}$. Here b_0, b_1, b_2 are arbitrary constants and $\tilde{a}$, $\tilde{b}$ represents generalized trapezoidal intuitionistic fuzzy number.

Here

$\tilde{a} = \left((a_1, a_2, a_3, a_4;\ w_1\ ), (a_1', a_2, a_3, a_4';\ \sigma_1)\right)$

and

$\tilde{b} = \left((b_1, b_2, b_3, b_4;\ w_2\ ), (b_1', b_2, b_3, b_4';\ \sigma_2)\right)$.

Using the generalized Hukuhara derivative, the four cases will arise, which will be discussed in the next section for a solid disk without rotation.

The Cauchy-Euler equation is solved by the method of substitution where the independent variable is chosen as $= e^z$. Then by the use of chain rule

$x\frac{dy}{dx} = D_1 y$ and $x^2\frac{d^2y}{dx^2} = D_1(D_1 - 1)y$, where $D_1 y = \frac{dy}{dz}$. (2)

Thus, using (2) in (1), we get

$$(b_0 D_1(D_1 - 1) + b_1 D_1 + b_2)y = 0\ . \qquad (3)$$

The equation (3) obtained is a differential equation with constant coefficients whose solution can be obtained by the method of complementary function.

4 Radial Displacement of a Solid Disk under Intuitionistic Fuzzy Environment

The problem of a solid disk is considered with boundary conditions in the form of TIFN. The differential equation modeling the problem of a solid disk is second order Cauchy-Euler ordinary differential equation, where the dependent variable is radial displacement (u) and the independent variable is radii (r) . The boundary condition is defined at $r = 0$ and $r = L$.

The differential equation model for a solid disk is given as

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0 \qquad (4)$$

where u is the radial displacement in a solid disc at a distance r from the axis with boundary conditions defined as: $u = 0$, when $r = 0$ and $r = L$.

Let $u(0) = \tilde{a}$ and $u(L) = \tilde{b}$, where $\tilde{a}, \tilde{b}$ are generalized trapezoidal intuitionistic fuzzy number.

Here

$\tilde{a} = \left((a_1, a_2, a_3, a_4;\ w_1\ ), (a_1', a_2, a_3, a_4';\ \sigma_1)\right)$

and $\tilde{b} = \left((b_1, b_2, b_3, b_4;\ w_2\ ), (b_1', b_2, b_3, b_4';\ \sigma_2)\right)$.

The four cases arises depending on (i) and (ii) – gH differentiability

Case 1: When $u(r)$ is (i) –gH differentiable and $\frac{du(r)}{dr}$ is (i) –gH differentiable.

For membership function, we have

$$r^2 \frac{d^2u_1(r,\alpha)}{dr^2} + r\frac{du_1(r,\alpha)}{dr} - u_1(r,\alpha) = 0$$

$$r^2 \frac{d^2u_2(r,\alpha)}{dr^2} + r\frac{du_2(r,\alpha)}{dr} - u_2(r,\alpha) = 0$$

For non-membership, it is written as

$$r^2 \frac{d^2u_1'(r,\beta)}{dr^2} + r\frac{du_1'(r,\beta)}{dr} - u_1'(r,\beta) = 0$$

$$r^2 \frac{d^2u_2'(r,\beta)}{dr^2} + r\frac{du_2'(r,\beta)}{dr} - u_2'(r,\beta) = 0$$

with boundary conditions, $u_1(0,\alpha) = a_1 + \frac{\alpha}{w} l_{\tilde{a}}$, $u_2(0,\alpha) = a_4 - \frac{\alpha}{w} r_{\tilde{a}}$, $u_1(L,\alpha) = b_1 + \frac{\alpha}{w} l_{\tilde{b}}$, $u_2(L,\alpha) = b_4 - \frac{\alpha}{w} r_{\tilde{b}}$, $u_1'(0,\beta) = a_2 - \frac{\beta}{\sigma} l_{\tilde{a}}$, $u_2'(0,\beta) = a_3 + \frac{\beta}{\sigma} r'_{\tilde{a}}$, $u_1'(L,\beta) = b_2 - \frac{\beta}{\sigma} l_{\tilde{b}}$, $u_2'(L,\beta) = b_3 + \frac{\beta}{\sigma} r'_{\tilde{b}}$, where $l_{\tilde{a}} = a_2 - a_1$, $r_{\tilde{a}} = a_4 - a_3$, $l_{\tilde{b}} = b_2 - b_1$, $r_{\tilde{b}} = b_4 - b_3$, $l'_{\tilde{a}} = a_2 - a_1'$, , $r'_{\tilde{a}} = a_4' - a_3$, $l'_{\tilde{b}} = b_2 - b_1'$, $r'_{\tilde{b}} = b_4' - b_3$

and $w = \min\{w_1, w_2\}$, $\sigma = \min\{\sigma_1, \sigma_2\}$.

Let $x = e^z$, then (4) reduces to

$$(D_1(D_1 - 1) + D_1 - 1)u = 0. \qquad (5)$$

The general solution of differential equation (5) is

$$u(r) = C_1 r + \frac{C_2}{r}.$$

Using the initial condition $u_1(0,\alpha) = a_1 + \frac{\alpha}{w} l_{\tilde{a}}$, we get $C_2 = 0$.

Using the condition at the boundary $u_1(L,\alpha) = b_1 + \frac{\alpha}{w} l_{\tilde{b}}$, we get $C_1 = \frac{b_1 + \frac{\alpha}{w}(b_2 - b_1)}{L}$.

Thus, $u_1(r,\alpha) = \frac{\left[b_1 + \frac{\alpha}{w}(b_2 - b_1)\right]}{L}.r$

Similarly, using $u_2(0,\alpha) = a_4 - \frac{\alpha}{w}(a_4 - a_3)$, $u_2(L,\alpha) = b_4 - \frac{\alpha}{w}(b_4 - b_3)$. we get, $u_2(r,\alpha) = \frac{1}{L}\left[b_4 - \frac{\alpha}{w}(b_4 - b_3)\right].r$

Using, other conditions such as $u_1'(0,\beta) = a_2 - \frac{\beta}{\sigma} l_{\tilde{a}}$ and $u_1'(L,\beta) = b_2 - \frac{\beta}{\sigma} l_{\tilde{b}}$, we get

$u_1'(r,\beta) = \frac{1}{L}\left[b_2 - \frac{\beta}{\sigma}(b_2 - b_1')\right].r$ and using

conditions $u_2'(0,\beta) = a_3 + \frac{\beta}{\sigma} r'_{\tilde{a}}$ and

$u_2'(L,\beta) = b_3 + \frac{\beta}{\sigma} r'_{\tilde{b}}$, we get

$$u_2'(r,\beta) = \frac{1}{L}\left[b_3 + \frac{\beta}{\sigma}(b_4' - b_3)\right].r$$

Case 2: When $u(r)$ is (ii) –gH differentiable and $\frac{du(r)}{dr}$ is (i) – gH differentiable.

For membership function, we have

$$r^2 \frac{d^2u_2(r,\alpha)}{dr^2} + r\frac{du_2(r,\alpha)}{dr} = u_1(r,\alpha)$$

$$r^2 \frac{d^2u_1(r,\alpha)}{dr^2} + r\frac{du_1(r,\alpha)}{dr} = u_2(r,\alpha)$$

For non-membership, it is written as

$$r^2 \frac{d^2u_2'(r,\beta)}{dr^2} + r\frac{du_2'(r,\beta)}{dr} = u_1'(r,\beta)$$

$$r^2 \frac{d^2u_1'(r,\beta)}{dr^2} + r\frac{du_1'(r,\beta)}{dr} = u_2'(r,\beta)$$

On solving above differential equation, we get

$$u_1(r,\alpha) = C_1 r + \frac{C_2}{r} + rC_3\cos(\log r)$$

$+rC_4\sin(\log r)$.

Similarly other equations can be derived.

$u_2(r,\alpha) = C_1 r + \frac{C_2}{r} - rC_3\cos(\log r) - rC_4\sin(\log r)$,

$u_1'(r,\beta) = d_1 r + \frac{d_2}{r} + d_3\cos(\log r) + d_4\sin(\log r)$,

$u_2'(r,\beta) = d_1 r + \frac{d_2}{r} - d_3 \cos(\log r) - d_4 \sin(\log r)$.

Using the initial conditions $u_1(0,\alpha) = a_1 + \frac{\alpha}{w} l_{\tilde{a}}$, $u_2(0,\alpha) = a_4 - \frac{\alpha}{w} r_{\tilde{a}}$, we get $C_2 = 0$. From the conditions $u_1(L,\alpha) = b_1 + \frac{\alpha}{w} l_{\tilde{b}}$ and

$u_2(L,\alpha) = b_4 - \frac{\alpha}{w} r_{\tilde{b}}$, we get,

$\mathrm{L}. u_1(L,\alpha) = C_1 . L^2 + L. C_3 \cos(\log L) + L. C_4 \sin(\log L)$

$\mathrm{L}. u_2(L,\alpha) = C_1 . L^2 -$

$L. [C_3 \cos(\log L) + C_4 \sin(\log L)]$.

Adding above two equations, we get $C_1 = \frac{u_1(L,\alpha)+u_2(L,\alpha)}{2L}$.

As it is not possible to get the values of other two constants, we will get the general solution, but not particular solution for case 2.

Case 3: When $u(r)$ is (i) –gH differentiable and $\frac{du(r)}{dr}$ is (ii) –gH differentiable. For membership function, we have

$$r^2 \frac{d^2 u_2(r,\alpha)}{dr^2} + r \frac{du_2(r,\alpha)}{dr} - u_1(r,\alpha) = 0$$

$$r^2 \frac{d^2 u_1(r,\alpha)}{dr^2} + r \frac{du_1(r,\alpha)}{dr} - u_2(r,\alpha) = 0$$

For non-membership, it is written as

$$r^2 \frac{d^2 u_1'(r,\beta)}{dr^2} + r \frac{du_1'(r,\beta)}{dr} - u_1'(r,\beta) = 0$$

$$r^2 \frac{d^2 u_2'(r,\beta)}{dr^2} + r \frac{du_2'(r,\beta)}{dr} - u_2'(r,\beta) = 0$$

The boundary conditions will remain same as per case 1 and 2. The case 3 is same as that of case 2.

Case 4: When $u(r)$ is (ii) –gH differentiable and $\frac{du(r)}{dr}$ is (ii) –gH differentiable.

For membership function, we have

$$r^2 \frac{d^2 u_1(r,\alpha)}{dr^2} + r \frac{du_1(r,\alpha)}{dr} - u_1(r,\alpha) = 0$$

$$r^2 \frac{d^2 u_2(r,\alpha)}{dr^2} + r \frac{du_2(r,\alpha)}{dr} - u_2(r,\alpha) = 0$$

For non-membership, it is written as

$$r^2 \frac{d^2 u_1'(r,\beta)}{dr^2} + r \frac{du_1'(r,\beta)}{dr} - u_1'(r,\beta) = 0$$

$$r^2 \frac{d^2 u_2'(r,\beta)}{dr^2} + r \frac{du_2'(r,\beta)}{dr} - u_2'(r,\beta) = 0 .$$

Case 4 is same as that of case 1 and having the same boundary conditions. The solutions for cases 1 and 4 are same.

To illustrate the above example, we take particular values of the above trapezoidal ntuitionistic fuzzy numbers for case 1.

Let

$u(0) = \tilde{a} = \begin{pmatrix} (10, 15, 20, 25;\ 0.6), \\ (8, 15, 20, 28;\ 0.3) \end{pmatrix}$ and

$u(5) = \tilde{b} = \begin{pmatrix} (30, 35, 40, 45;\ 0.7), \\ (29, 35, 40, 48;\ 0.3) \end{pmatrix}$.

Here $L = 5$, $w = min(0.6,0.7) = 0.6$ and

$\sigma = min(0.3,0.3) = 0.3$.

And $\alpha \in [0, w]$ and $\beta \in [\sigma, 1]$. Thus $\alpha \in [0,0.6]$ and $\beta \in [0.3,1]$.

TABLE 1
Solutions for $r = 1$

α	$u_1(r,\alpha)$	$u_2(r,\alpha)$	β	$u_1'(r,\beta)$	$u_2'(r,\beta)$
0	6	9	0.3	5.8	9.6
0.1	6.1667	8.8333	0.4	5.4	10.1333
0.2	6.3333	8.6667	0.5	5	10.6667
0.3	6.5	8.5	0.6	4.6	11.2
0.4	6.6667	8.3333	0.7	4.2	11.7333
0.5	6.8333	8.1667	0.8	3.8	12.2667
0.6	7	8	0.9	3.4	12.8
			1.0	3	13.3333

TABLE 2
Solutions for $r = 4$

α	$u_1(r,\alpha)$	$u_2(r,\alpha)$	β	$u_1'(r,\beta)$	$u_2'(r,\beta)$
0	24.0000	36	.3	23.2	38.4
0.1	24.6667	35.3333	.4	21.6	40.5333
0.2	25.3333	34.6667	.5	20	42.6668
0.3	26.0000	34.0000	.6	18.4	44.8
0.4	26.6667	33.3333	.7	16.8	46.9333
0.5	27.3333	32.6667	.8	15.2	49.0668
0.6	28	32	.9	13.6	51.2
			1	12	53.3333

The tables are drawn for radii at $r = 1$ and $r = 4$. It is observed that with the increase in value of α, $u_1(r,\alpha)$ increases and $u_2(r,\alpha)$ decreases, whereas with the increase in β, $u_1'(r,\beta)$ decreases and $u_2'(r,\beta)$ increases.

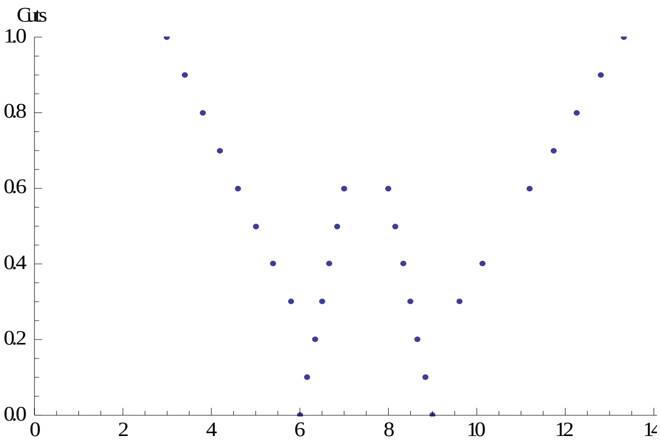

Fig. 1 The radial displacement at $r = 1$.

It is observed form Fig. 1, that the radial displacement is varying in the form of trapezoidal fuzzy number and hence solution matches with the boundary conditions considered.

4 Conclusion

In this paper, we have considered the boundary value problem of a solid disk under intuitionistic fuzzy environment. The problem is modeled in the form of fuzzy linear Cauchy Euler second order ordinary differential equation. The radial displacement at different radii is calculated using trapezoidal intuitionistic fuzzy numbers and the results are shown in the form of tables and figure for different radii.

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