E-optimality conditions for E-differentiable E-invex multiobjective programming problems

Najeeb Abdulaleem
Department of Mathematics, Hadhramout University, P.O. BOX : (50511-50512), Al-Mahrah, Yemen.
Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland.
nabbas985@gmail.com

Abstract: In this paper, a class of E-differentiable multiobjective programming problems with both inequality and equality constraints is considered. The so-called E-Karush-Kuhn-Tucker necessary optimality conditions are established for such nonsmooth vector optimization problems. Also, the sufficiency of these necessary optimality conditions are proved for E-differentiable multiobjective programming problems under (generalized) E-invexity hypotheses.

Key–Words: E-differentiable vector optimization problem; E-optimality conditions; E-invex set; E-invex function with respect to η.

1 Introduction

Multiobjective programming, has grown remarkably in studying the development of optimality conditions. This is a consequence of the fact that many real world problems can be modeled as optimization problems with several objectives conflicting with one another, that is, by vector optimization problems. Although the concept of convexity plays a vital real in proving the fundamental results in optimization theory, however, not all real life problems can be analyzed as convex multiobjective programming problems. Therefore, various classes of nonconvex vector optimization problems have been defined in optimization literature. One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [12]. In the case of differentiable scalar optimization problems. Namely, Hanson showed that, instead of the usual convexity assumption, if all functions are assumed to be invex (with respect to the same function η), then the sufficient optimality conditions and weak duality can be proved. Over the years, many generalizations of this concept have been introduced in the literature (see, for example, [2], [3], [4], [5], [6], [7], [8], [10], [11], [14], [15], [16], [20], [22], [23], [24], and others).

Another generalization of convexity was derived by Youness [25]. Namely, he introduced the definition of an E-convex set and the definition of an E-convex function and analyzed some properties of these nonconvex sets and functions. Moreover, the results established by Youness [25] were improved by Yang [26]. Further, Megahed et al. [19] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$.

In this paper, a new class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are E-invex. Therefore, the concept of a so-called E-differentiable E-invex function for E-differentiable vector optimization problems is introduced. Further, under the introduced E-Guignard constraint qualification, the so-called E-Karush-Kuhn-Tucker necessary optimality conditions are established for the considered E-differentiable vector optimization problems with both inequality and equality constraints. It is also given an example of such a vector optimization problems with E-differentiable E-invex functions for which the E-Guignard constraint qualification is satisfied but the E-Abadie constraint qualification is not satisfied. It turns out that the E-Karush-Kuhn-Tucker necessary optimality conditions established for such a nonsmooth vector optimization problem are not satisfied in such a case. Moreover, the sufficient optimality conditions are derived for the considered E-differentiable vector optimization problem under E-invexity and/or generalized E-invexity. This result is illustrated by the example of nonconvex E-differentiable vector optimization problem in which the involved functions are E-invex (with respect to the same function η). Thus, in the present paper, tools of differentiable optimization problems are used in proving optimality conditions for (weakly) efficiency of nonsmooth multiobjective programming problems.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;

(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Now, we introduce the definition of an E-invex set as a generalization of an E-convex set given by Youness [25] and the definition of an invex set (with respect to η) given by Mohan and Neogy [22].

Definition 1 *Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set (with respect to $\eta : M \times M \to R^n$) if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation*

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark 2 *If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness [25]).*

Remark 3 *If $E(a) = a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy [22]).*

Now, we present an example of such an E-invex set which is not E-convex.

Example 4 *Let $M = [1, 9] \cup [-9, -1]$ and $E : R \to R$ be an operator defined by*

$$E(x) = \begin{cases} x^2 & \text{if } 0 \leqq x \leqq 3, \\ -x & \text{if } -3 \leqq x \leqq 0, \\ -1 & \text{if } x < -3 \text{ or } x > 3. \end{cases}$$

and $\eta : M \times M \to R$ be defined by

$$\eta(E(x), E(u)) = \begin{cases} x - u & \text{if } x \geqq 0,\ u \geqq 0, \\ x - u & \text{if } x \leqq 0,\ u \leqq 0, \\ -9 - u & \text{if } x > 0,\ u \leqq 0, \\ 1 - u & \text{if } x < 0,\ u \geqq 0. \end{cases}$$

Then, by Definition 1, M is an E-invex set with respect to the function η given above. However, it is not E-convex as can be seen by taking $x = 1$, $u = 4$, and $\lambda = \frac{1}{2}$, we have

$$\lambda E(x) + (1 - \lambda) E(u) = 0 \notin M.$$

Hence, by the definition of an E-convex set (see Remark 2), it follows that M is not E-convex.

Now, we present of an example of such an E-invex set with respect to η which is not invex with respect to η.

Example 5 *Let $M = [1, 4] \cup [-4, -1]$ and $E : R \to R$ be an operator defined by*

$$E(x) = \begin{cases} x^2 & \text{if } -2 \leqq x \leqq 2, \\ -1 & \text{if } x < -2 \text{ or } x > 2. \end{cases}$$

and $\eta : M \times M \to R$ be defined by

$$\eta(x, u) = \begin{cases} x - u & \text{if } x \leqq u, \\ -4 - u & \text{if } x > u. \end{cases}$$

Then, by Definition 1, M is an E-invex set with respect to the function η given above. However, it is not invex with respect to η as can be seen by taking $x = -1$, $u = 1$, and $\lambda = \frac{1}{2}$, we have

$$u + \lambda\eta(x, u) = 0 \notin M.$$

Hence, by the definition of an invex set (see Remark 3), it follows that M is not an invex set with respect to η.

Definition 6 *[19] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x - u)$$

$$+\theta(u, x - u) \|x - u\|, \tag{1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Now, we introduce a new concept of generalized convexity for E-differentiable vector-valued functions.

Definition 7 *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : M \times M \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is an E-invex function with respect to η if, for all $x \in M$,*

$$f_i(E(x)) - f_i(E(u)) \geqq \nabla f_i(E(u))\eta(E(x), E(u)), \; i = 1, ..., k. \tag{2}$$

If inequalities (2) hold for any $u \in M$, then f is E-invex with respect to η on M.

Remark 8 *From Definition 7, there are special cases:*

a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [12] in the scalar case.

b) If $\eta : M \times M \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [19].

c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.

d) If f is differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [25].

Definition 9 *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : M \times M \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is a strictly E-invex function with respect to η if, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities*

$$f_i(E(x)) - f_i(E(u)) > \nabla f_i(E(u))\eta(E(x), E(u)), \; i = 1, ..., k, \tag{3}$$

hold. If inequalities (3) are fulfilled for any $u \in M$ $(E(x) \neq E(u))$, then f is strictly E-invex with respect to η on M.

Now, we present an example of such an E-invex function which is not E-convex.

Example 10 *Let $f : R \to R$ be defined by $f(x) = \cos x$ and $E : R \to R$ be an operator defined by $E(x) = \frac{\pi}{2} - x$ and η be defined by*

$$\eta(E(x), E(u)) = \begin{cases} \frac{\sin u - \sin x}{\cos u} & \text{if } x > u, \\ 0 & \text{if } x = u, \\ \frac{2\sin x - 2\sin u}{\cos u} & \text{if } x < u. \end{cases}$$

Then f is E-invex on R, but it is not E-convex as can be seen by taking $x = 0$, $u = \frac{\pi}{4}$, and $\lambda = \frac{1}{2}$, since the inequality

$$f(\lambda E(x) + (1 - \lambda) E(u)) \geqq \lambda f(E(x)) + (1 - \lambda) f(E(u))$$

holds. Hence, by the definition of an E-convex function [25], it follows that f is not E-convex on R.

Now, we give the necessary condition for E-differentiable E-invexity.

Proposition 11 *Let $E : R^n \to R^n$ and $f : M \to R^k$ be an E-invex (strictly E-invex) function with respect to η on M and $u \in M$. Further, assume that f is E-differentiable at u. Then, the following inequality*

$$\nabla f(E(x)) - \nabla f(E(u))\, \eta(E(x), E(u)) \geqq 0, \quad (>) \tag{4}$$

holds for all x, $u \in M$ $(E(x) \neq E(u))$.

Now, we prove a sufficient condition for an E-differentiable E-invex function with respect to η.

Theorem 12 *Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at $u \in M$ on M. Further, assume that there exists $\eta : M \times M \to R^n$ such that*

$$f(E(u) + \lambda\eta(E(x), E(u)) \leqq \lambda f(E(x)) + (1 - \lambda) f(E(u)), \tag{5}$$

holds for any $\lambda \in [0, 1]$. Then, f is an E-invex function with respect to η.

Proof: By (5), we have that the inequality

$$f(E(u) + \lambda\eta(E(x), E(u)) - f(E(u)) \leqq$$
$$\lambda[f(E(x)) - f(E(u))] \tag{6}$$

holds for any $\lambda \in [0, 1]$. Thus, the above inequality yields for any $\lambda \in [0, 1]$,

$$\frac{f(E(u) + \lambda\eta(E(x), E(u)) - f(E(u))}{\lambda} \leqq$$
$$f(E(x)) - f(E(u)). \tag{7}$$

By assumption, f is E-differentiable at u. Hence, by Definition 6, it follows that $f \circ E$ is differentiable at u. Therefore, letting $\lambda \to 0$, we obtain the inequality (2). □

Now, we introduce various classes of generalized E-differentiable E-invex functions as a generalization of appropriate generalized E-convex functions and, thus, pseudo-convex and quasi-convex functions.

Definition 13 *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : M \times M \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is a pseudo E-invex function with respect to η if, for all $x \in M$ and $i = 1, ..., k$,*

$$f_i(E(x)) < f_i(E(u)) \Longrightarrow \nabla f_i(E(u))\eta(E(x), E(u)) < 0. \tag{8}$$

If (8) holds for any $u \in M$, then f is pseudo E-invex with respect to η on M.

Definition 14 *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : M \times M \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is a strictly pseudo E-invex function with respect to η if, for all $x \in M$, $x \neq u$, and $i = 1, ..., k$,*

$$f_i(E(x)) \leqq f_i(E(u)) \Longrightarrow \nabla f_i(E(u))\eta(E(x), E(u)) < 0. \tag{9}$$

Note that every strictly pseudo E-invex function is pseudo E-invex and every E-differentiable pseudo E-convex function is pseudo E-invex. Also, every pseudo E-convex function is E-invex and every E-invex function is pseudo E-invex function for the same function η, but the converse is not true.

Now, we present an example of such an E-differentiable pseudo E-invex function which is not E-invex.

Example 15 *Let $f : R \to R$ be defined by $f(x) = e^{\sqrt[3]{x}}$, $\eta : R \times R \to R$ be defined by*

$$\eta(x, u) = \begin{cases} \sqrt[3]{x} - \sqrt[3]{u} & \text{if } x \le u, \\ 10 & \text{if } x > u. \end{cases}$$

and $E : R \to R$ be an operator defined by $E(x) = x^3$. Further, assume that $(f \circ E)(x) < (f \circ E)(u)$. Thus, we have $(f \circ E)(x) = e^x < e^u = (f \circ E)(u)$. This implies that $x < u$ for all $x, u \in R$. Moreover, we have $\nabla (f \circ E)(u)\, \eta (E(x), E(u)) < 0$. Therefore, by Definition 13, f is an E-differentiable pseudo E-invex function on R. However, it is not E-invex (with respect to η) on R. Indeed, if we set $x = \ln 10$, $u = 1$, then we have

$$f(E(x)) - f(E(u)) < \nabla f(E(u))\eta(E(x), E(u)).$$

Hence, by Definition 2, it follows that f is not E-invex.

Definition 16 *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : M \times M \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is a quasi-E-invex function with respect to η if, for all $x \in M$ and $i = 1, ..., k$,*

$$f_i(E(x)) - f_i(E(u)) \leqq 0 \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0. \tag{10}$$

If (10) holds for any $u \in M$, then f is quasi-E-invex with respect to η on M.

Note that E-differentiable quasi E-convex is trivially quasi E-invex and every pseudo E-invex function is quasi E-invex.

Now, we present an example of such a quasi E-invex function but not a quasi E-convex function.

Example 17 *Let $f : R \to R$ be defined by $f(x) = \cos x$, $E : R \to R$ be an operator defined by $E(x) = \frac{\pi}{2} - x$ and η defined by $\eta(E(x), E(u)) = \frac{\sin x - \sin u}{\cos u}$. It can be shown that f is quasi E-invex on R. Assume that $(f \circ E)(x) \leqq (f \circ E)(u)$. We have $(f \circ E)(x) = \sin x \leqq \sin u = (f \circ E)(u)$. This inequality implies that $x \leqq u$. Hence, we have $\nabla (f \circ E)(u)\, \eta (E(x), E(u)) = \sin x - \sin u \leqq 0$. Therefore, by Definition 16, f is quasi E-invex on R. Further, it can be shown that f is not quasi E-convex on R. Assume that $(f \circ E)(x) \leqq (f \circ E)(u)$. We have $(f \circ E)(x) = \sin x \leqq \sin u = (f \circ E)(u)$. This inequality implies that $x \leqq u$. Indeed, if we set $x = \frac{\pi}{6}$, $u = \frac{\pi}{4}$, then we have $\nabla (f \circ E)(u) (E(x) - E(u)) = \cos u(u - x) \geqq 0$. Hence, by the definition of quasi E-convexity, f is not quasi E-convex on R.*

3 E-optimality conditions for E-differentiable multiobjective programming

In some cases, the multiobjective programming problem can be represented as the following unconstrained vector optimization problem:

$$\begin{array}{ll} \text{minimize } f(x) = \left(f_1(x), ..., f_p(x)\right) & \\ x \in R^n, & \end{array} \tag{VP}$$

where f denotes a vector-valued E-differentiable function on R^n.

Now, we give the definitions of a weak Pareto solution and a Pareto solution of the considered vector optimization problem (VP).

Definition 18 *A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution of (VP) if and only if there exists no feasible point x such that*

$$f(x) < f(\overline{x}).$$

Definition 19 *A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution of (VP) if and only if there exists no feasible point x such that*

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be an one-to-one and onto operator. For the considered multiobjective programming problem (VP), we define the vector optimization problem (VP$_E$) as follows

$$\begin{aligned} &\text{minimize } f(E(x)) = \left(f_1(E(x)), ..., f_p(E(x))\right) \\ &\qquad x \in R^n, \end{aligned} \quad (\mathrm{VP}_E)$$

where $f \circ E$ denotes a vector-valued differentiable function on R^n.

Now, we give the definitions of a weak Pareto (weakly efficient) solution and a Pareto (efficient) solution of the vector optimization problem (VP$_E$), which are at the same time a weak E-Pareto solution (weakly E-efficient solution) and an E-Pareto solution (E-efficient solution) of the considered multiobjective programming problem (VP).

Definition 20 *A feasible point $E(\overline{x})$ is said to be a weak E-Pareto solution (weakly E-efficient solution) of (VP) if and only if there exists no feasible point $E(x)$ such that*

$$f(E(x)) < f(E(\overline{x})).$$

Definition 21 *A feasible point $E(\overline{x})$ is said to be an E-Pareto solution (E-efficient solution) of (VP) if and only if there exists no feasible point $E(x)$ such that*

$$f(E(x)) \leq f(E(\overline{x})).$$

As it is known [6], a characteristic property of a scalar invex function with respect to η is the fact that each its stationary point is also its global minimum. It turns out that this property can be generalized to the class of vector E-invex functions with respect to η. For this purpose, we have to define adequately an E-critical point concept for vector-valued functions.

Definition 22 *Let $E : R^n \to R^n$. A point $u \in R^n$ is said to be a vector E-critical point of an E-differentiable vector-valued function $f : R^n \to R^k$, (or, in other words, for the problem (VP)) if there exists a vector $\lambda \in R^k$ with $\lambda \geq 0$ such that $\lambda^T \nabla(f \circ E)(u) = 0$.*

Now, we prove that every weakly efficient point is also an E-vector critical point.

Theorem 23 *Let $E : R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable vector-valued function, $E(\overline{x})$ be a weakly E-efficient solution of (VP). Then, there exists a vector $\overline{\lambda} \in R^k$ with $\overline{\lambda} \geq 0$ such that $\overline{\lambda}^T \nabla(f \circ E)(\overline{x}) = 0$.*

Proof: Suppose that $\overline{\lambda}^T \nabla(f \circ E)(\overline{x}) \neq 0$. Then, let $d = -\nabla(f \circ E)(\overline{x})$. Hence, we obtain

$$\nabla(f \circ E)(\overline{x})d = -\|\nabla(f \circ E)(\overline{x})\|^2 < 0. \quad (11)$$

By assumption, the objective function f, is E-differentiable at $\overline{x}$. Thus, by Definition 6, we get

$$\begin{aligned} (f \circ E)(x) = (f \circ E)(\overline{x}) + \nabla (f \circ E)(\overline{x})^T (x - \overline{x}) \\ +\theta(\overline{x}, x - \overline{x}) \|x - \overline{x}\|. \end{aligned} \quad (12)$$

Using $\theta(\overline{x}, x - \overline{x}) \to 0$ and $\frac{x-\overline{x}}{\|x-\overline{x}\|} \to d$ as $x \to \overline{x}$ together with (11), we get that the following inequality

$$(f \circ E)(x) < (f \circ E)(\overline{x})$$

holds, which is a contradiction to the assumption that $E(\overline{x})$ is a weakly E-efficient solution of the vector optimization problem (VP). Hence, there exists a vector $\overline{\lambda} \in R^k$ with $\overline{\lambda} \geq 0$ such that $\overline{\lambda}^T \nabla(f \circ E)(\overline{x}) = 0$. The proof of this theorem is completed. □

Now, we prove the converse of the above theorem using the concept of vectorial E-invexity introduced in the paper.

Theorem 24 *Let $E : R^n \to R^n$, $\overline{x}$ be a vector E-critical point of (VP), and let $f \circ E$ be a vector E-invex function at $\overline{x}$ with respect to η. Then $E(\overline{x})$ is a weak E-Pareto solution of (VP).*

Proof: Let $\overline{x}$ be a vector E-critical point. Then, there exists a vector $\overline{\lambda} \in R^k$ with $\overline{\lambda} \geq 0$ such that $\overline{\lambda}^T \nabla(f \circ E)(\overline{x}) = 0$. We proceed by contradiction. Suppose that $E(\overline{x})$ is not a weak E-Pareto solution of (VP). Then, there exists another point $\overline{z} \in R^n$ such that

$$(f \circ E)(\overline{z}) < (f \circ E)(\overline{x}). \quad (13)$$

Thus, by E-invexity of f, we get that

$$f(E(\overline{z})) - f(E(\overline{x})) \geqq \nabla f(E(\overline{x}))\eta(E(\overline{z}), E(\overline{x})). \quad (14)$$

Combining (13) and (14), we get that the inequality

$$\lambda^T \nabla(f \circ E)(\overline{x}) < 0, \text{ for any } \lambda \geq 0$$

holds, which is a contradiction to the assumption that $E(\overline{x})$ is a weak E-Pareto solution for (VP). The proof of this theorem is completed. □

In general, a vector optimization problem is considered with the set of inequality and equality constraints as follows

$$\begin{aligned} &\text{minimize } f(x) = \left(f_1(x), ..., f_p(x)\right) \\ &\text{subject to } g_j(x) \leqq 0, \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0, \ t \in T = \{1, ..., s\}, \end{aligned} \quad (\mathrm{CVP})$$

where the functions $f_i : R^n \to R$, $i \in I = \{1, ..., p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued E-differentiable functions defined on R^n.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. We will write $g := (g_1, ..., g_m) : R^n \to R^m$ and $h := (h_1, ..., h_s) : R^n \to R^s$ for convenience. Let

$$\Omega := \left\{x \in R^n : g_j(x) \leqq 0,\ j \in J, h_t(x) = 0,\ t \in T\right\}$$

be the set of all feasible solutions of (CVP). Further, let us denote by $J(x)$, the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \left\{j \in J : g_j(x) = 0\right\}$

Let $E : R^n \to R^n$ be an one-to-one and onto operator. For the considered constrained multiobjective programming problem (CVP), we define its associated constrained vector optimization problem (CVP$_E$) with both inequality and equality constraints as follows

$$\begin{aligned} &\text{minimize } f(E(x)) = \left(f_1(E(x)), ..., f_p(E(x))\right) \\ &\text{subject to } g_j(E(x)) \leqq 0,\ j \in J = \{1, ..., m\}, \quad (\text{CVP}_E) \\ &\qquad h_t(E(x)) = 0,\ t \in T = \{1, ..., s\}, \end{aligned}$$

where the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T$, are defined in the similar way as for (CVP). We call (CVP$_E$) the E-vector optimization problem (associated to the multiobjective programming problem (CVP)). Let

$$\Omega_E := \{x \in R^n : g_j(E(x)) \leqq 0,\ j \in J,$$
$$h_t(E(x)) = 0,\ t \in T\}$$

be the set of all feasible solutions of (CVP$_E$).

In [1], Antczak and Abdulaleem established the following result

Lemma 25 *[1] Let $E : R^n \to R^n$ be a one-to-one and onto and*

$$\Omega_E = \{z \in R^n : \left(g_j \circ E\right)(z) \leqq 0,\ j \in J,$$
$$(h_t \circ E)(z) = 0,\ t \in T\}.$$

Then $E(\Omega_E) = \Omega$.

In this section, we derive both necessary and sufficient optimality conditions for a new class of nonconvex multicriteria optimization problems. Namely, we consider a class of E-differentiable multiobjective programming problems. Throughout this section, $E : R^n \to R^n$ is assumed to be an one-to-one and onto operator. By Definition 6, the functions constituting the E-vector optimization problem (CVP$_E$) are differentiable at any its feasible solution (in the usual sense). Further, we denote by $J_E(x)$, the set of inequality constraint indices that are active at a feasible solution $E(x)$, that is, $J_E(x) = \left\{j \in J : \left(g_j \circ E\right)(x) = 0\right\}$. Moreover, it can be proved [1] (see Lemma 26 below) that if $\overline{x}$ is a (weak) Pareto solution of the E-vector optimization problem (CVP$_E$), then $E(\overline{x})$ is a (weak) Pareto solution of the original multiobjective programming problem (CVP). We call $E(\overline{x})$ a (weak) E-Pareto solution of the problem (CVP).

Lemma 26 *[1] Let $E : R^n \to R^n$ be a one-to-one and onto and $\overline{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the constrained E-vector optimization problem (CVP$_E$). Then $E(\overline{z})$ is a weak E-Pareto solution (E-Pareto solution) of the considered constrained multiobjective programming problem (CVP).*

Before we establish the Karush-Kuhn-Tucker necessary optimality conditions for problem (CVP), we re-call the Motzkin's theorem of the alternative.

Theorem 27 *[18] (Motzkin's theorem of the alternative). Let A, C, D be given matrices, with A being nonvacuous. Then either the system of inequalities*

$$Ax < 0,\ Cx \leqq 0,\ Dx = 0$$

has a solution x, or the system

$$A^T y_1 + C^T y_2 + D^T y_3 = 0, \quad y_1 \geq 0, y_2 \geqq 0 \qquad (15)$$

has solution y_1, y_2 and y_3, but never both.

Now, we give the definition of the minimal element of a given set (with respect to an order relation).

Definition 28 *[17] Let Y be a given set in R^k ordered by $\leqq$ or by $<$. Specifically, we call the minimal element of Y defined by $\leq$ a minimal vector, and that defined by $<$ a weak minimal vector. Formally speaking, a vector $\overline{y} \in Y$ is called a minimal vector in Y if there exists no vector y in Y such that $y \leq \overline{y}$; it is called a weak minimal vector if there exists no vector y in Y such that $y < \overline{y}$.*

Definition 29 *Let $Y \subseteq R^k$. The Bouligand contingent cone of Y at $\overline{y} \in Y$ is the set $T_Y(\overline{y})$ of all vectors $q \in R^k$ such that there exist a sequence $\{y_n\} \in Y$ and a sequence β_n of strictly positive real number such that*

$$\lim_{n\to\infty} y_n = \overline{y},\ \lim_{n\to\infty} \beta_n = 0,\ \lim_{n\to\infty} \frac{y_n - \overline{y}}{\beta_n} = q.$$

In other words, the Bouligand contingent cone of Y at x is defined by

$$T_Y(x) = \{d \in R^n : \exists_{\{\lambda_n\}\subset R} \lambda_n \to \infty,\ \exists_{\{x_n\}\subset Y} x_n \to x$$

$$s.t.\ \lambda_n (x_n - x) \to d\}$$

A vector $d \in R^n$ belonging to $T_Y(x)$ is called a tangent direction to Y from $x \in cl\, Y$.

Remark 30 *Note that Lin [17] named any Bouligand contingent vector, that is, any vector $q \in T_Y(\overline{y})$, a convergence vector for the set Y at $\overline{y}$.*

Now, we extend the result established by Lin [17] for the E-vector optimization problem (CVP$_E$).

Theorem 31 *If $\overline{x} \in \Omega_E$ is locally (weak) minimal for $f \circ E$ on Ω_E then no Bouligand contingent vector for $f(E(\Omega_E))$ at $\overline{y} = f(E(\overline{x}))$ is strictly negative.*

Definition 32 *The tangent cone (also called contingent cone or Bouligand cone) of Ω_E at $\overline{x} \in cl\,\Omega_E$ is defined by*

$$T_{\Omega_E}(\overline{x}) = \{d \in R^n : \exists_{\{d_n\} \subset R^n} d_n \to d,$$

$$\exists_{\{t_n\} \subset R} t_n \downarrow 0 \ s.t.\ \overline{x} + t_n d_n \in \Omega_E\}.$$

Before we prove the Karush-Kuhn-Tucker necessary optimality conditions for the differentiable constrained E-vector optimization problem with inequality and equality constraint (CVP$_E$), we introduce the so-called E-Guignard constraint qualification. In order to do this, for the constrained E-vector optimization problem (CVP$_E$), we introduce the E-linearized cone $L_E(\overline{x})$.

Definition 33 *For the constrained E-vector optimization problem (CVP$_E$), the E-linearized cone at $\overline{x} \in \Omega_E$, denoted by $L_E(\overline{x})$, is defined by*

$$L_E(\overline{x}) = \{d \in R^n : \nabla g_j(E(\overline{x}))\, d \leqq 0,\ j \in J_E(\overline{x}),$$

$$\nabla h_t(E(\overline{x}))\, d = 0,\ t \in T\}.$$

It is easy to see that $L_E(\overline{x})$ is a nonempty closed convex cone.

The following lemma shows the relationship between the Bouligand contingent cone $T_{\Omega_E}(\overline{x})$ and the E-linearizing $L_E(\overline{x})$ cone.

Lemma 34 *If $\overline{x} \in \Omega_E$ is a Pareto solution of the constrained E-vector optimization problem (CVP$_E$), then*

$$cl\ conv\ T_{\Omega_E}(\overline{x}) \subseteq L_E(\overline{x}).$$

Proof: Let $\overline{x} \in \Omega_E$ be given and $d \in T_{\Omega_E}(\overline{x})$. Then, by Definition 32, there exists a sequence $\{x_n\} \in \Omega_E$ such that $\lim_{n\to\infty} x_n = \overline{x}$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{x_n - \overline{x}}{\beta_n} = d$. By assumption, all constraint functions g_j, $j \in J$ and h_t, $t \in T$, are E-differentiable at $\overline{x}$. Hence, by Definition 6, it follows that

$$g_j(E(x_n)) = g_j(E(\overline{x})) + \nabla g_j(E(\overline{x}))^T (x_n - \overline{x})$$

$$+\theta_{g_j}(\overline{x}, x_n - \overline{x}) \|x_n - \overline{x}\|,\ j \in J, \tag{16}$$

$$h_t(E(x_n)) = h_t(E(\overline{x})) + \nabla h_t(E(\overline{x}))^T (x_n - \overline{x})$$

$$+\theta_{h_t}(\overline{x}, x_n - \overline{x}) \|x_n - \overline{x}\|,\ t \in T, \tag{17}$$

where $\theta_{g_j}(\overline{x}, x_n - \overline{x}) \to 0$, $j \in J$, $\theta_{h_t}(\overline{x}, x_n - \overline{x}) \to 0$, $t \in T$, as $x_n \to \overline{x}$. Assume that $d_n := \frac{x_n - \overline{x}}{\beta_n}$. Then $x_n = \overline{x} + \beta_n d_n$ and we obtain that

$$g_j(E(x_n)) = g_j(E(\overline{x} + \beta_n d_n)) \leqq 0 = g_j(E(\overline{x})),\quad j \in J_E(\overline{x}), \tag{18}$$

$$h_t(E(x_n)) = h_t(E(\overline{x} + \beta_n d_n)) = 0 = h_t(E(\overline{x})),\quad t \in T. \tag{19}$$

From (18), (19), and Definition 32, it follows that

$$\nabla g_j(E(\overline{x}))\, d \leqq 0,\ j \in J_E(\overline{x}), \tag{20}$$

$$\nabla h_t(E(\overline{x}))\, d = 0,\ t \in T. \tag{21}$$

By (20), (21) and Definition 33, we have

$$T_{\Omega_E}(\overline{x}) \subseteq L_E(\overline{x}).$$

Since $L_E(\overline{x})$ is closed and convex, we get

$$cl\ conv\ T_{\Omega_E}(\overline{x}) \subseteq L_E(\overline{x}).$$

□

Now, we give the so-called E-Abadie constraint qualification for the E-differentiable vector optimization problem (CVP) with both inequality and equality constraints which was introduced in [1].

Definition 35 *It is said that the so-called E-Abadie constraint qualification (ACQ$_E$) holds at $\overline{x} \in \Omega_E$ for the differentiable E-vector optimization problem (CVP$_E$) with both inequality and equality constraints if*

$$T_{\Omega_E}(\overline{x}) = L_E(\overline{x}). \tag{22}$$

Now, we introduce the so-called E-Guignard constraint qualification for the E-differentiable constrained vector optimization problem (CVP) with both inequality and equality constraints.

Definition 36 *It is said that the so-called E-Guignard constraint qualification (GCQ$_E$) holds at $\overline{x} \in \Omega_E$ for the differentiable constrained E-vector optimization problem (CVP$_E$) with both inequality and equality constraints if*

$$cl\ conv\ T_{\Omega_E}(\overline{x}) = L_E(\overline{x}). \tag{23}$$

Now, we present an example of such a nondifferentiable vector optimization problem for which the E-Guignard constraint qualification is satisfied but E-Abadie constraint qualification introduced in [1] does not hold.

Example 37 *Consider the following nonconvex nondifferentiable vector optimization problem*

$$\begin{aligned} &\text{minimize } f(x) = \left(2x_2 - 2\sqrt[3]{x_1}\,,\ -\sqrt[3]{x_1}x_2\right) \\ &\quad s.t.\ g(x) = x_2 - \sqrt[3]{x_1} \leqq 0. \end{aligned} \qquad (CVP1)$$

Note that $\Omega = \left\{(x_1, x_2) \in R^2 : x_2 - \sqrt[3]{x_1} \leqq 0\right\}$. *Let* $E : R^2 \to R^2$ *be an one-to-one and onto mapping defined as follows* $E(x_1, x_2) = \left(x_1^3, x_2\right)$. *For the considered vector optimization problem (CVP1), we define its associated E-vector optimization problem (CVP$_E$1) as follows*

$$\begin{aligned} &\text{minimize } f(E(x)) = (2x_2 - 2x_1\,,\ -x_1x_2) \\ &\quad g(E(x)) = x_2 - x_1 \leqq 0. \end{aligned} \qquad (CVP_E1)$$

Note that $\Omega_E = \left\{(x_1, x_2) \in R^2 : x_2 - x_1 \leqq 0\right\}$ *and* $\overline{x} = (0, 0)$ *is a feasible solution. Then, by the definition of the E-linearized cone, we have that* $L_E(\overline{x}) = \left\{(d_1, d_2) \in R^2 : d_2 \leqq d_1\right\}$. *Further, by the definition of the Bouligand contingent cone, we have that* $T_{\Omega_E}(\overline{x}) = \left\{(d_1, d_2) \in R^2 : -d_1 d_2 \leqq 0 \wedge d_1 \geqq d_2\right\}$. *Therefore,* $L_E(\overline{x}) = cl\,conv\, T_{\Omega_E}(\overline{x})$, *but* $L_E(\overline{x}) \nsubseteq T_{\Omega_E}(\overline{x})$. *Hence, the E-Guignard constraint qualification is satisfied at* $\overline{x}$ *but E-Abadie constraint qualification is not satisfied.*

Now, we prove the Karush-Kuhn-Tucker necessary optimality conditions for the differentiable constrained E-vector optimization problem (CVP$_E$) and, thus, the so-called E-Karush-Kuhn-Tucker necessary optimality conditions for not necessarily differentiable constrained multiobjective programming problem (CVP) in which the involved functions are E-differentiable and for which the E-Guignard constraint qualification holds.

Theorem 38 *(E-Karush-Kuhn-Tucker necessary optimality conditions). Let* $\overline{x} \in \Omega_E$ *be a weak Pareto solution of the constrained E-vector optimization problem (CVP$_E$) (and, thus,* $E(\overline{x})$ *be a weak E-Pareto solution of the considered constrained multiobjective programming problem (CVP)). Further, let f, g, h be E-differentiable at* $\overline{x}$ *and the E-Guignard constraint qualification be satisfied at* $\overline{x}$. *Then there exist Lagrange multipliers* $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^s$ *such that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla \left(g_j \circ E\right)(\overline{x})$$

$$+ \sum_{t=1}^{s} \overline{\xi}_t \nabla (h_t \circ E)(\overline{x}) = 0, \qquad (24)$$

$$\overline{\mu}_j \left(g_j \circ E\right)(\overline{x}) = 0, \ \ j \in J(E(\overline{x})), \qquad (25)$$

$$\overline{\lambda} \geq 0, \overline{\mu} \geqq 0. \qquad (26)$$

Proof: By assumption, $\overline{x} \in \Omega_E$ is a weak Pareto solution in the E-vector optimization problem (CVP$_E$) (and, thus, $E(\overline{x})$ is a weak E-Pareto solution of the considered multiobjective programming problem (CVP)). Let $d \in T_{\Omega_E}(\overline{x})$ and x_n be the corresponding sequence of feasible solutions in E-vector optimization problem (CVP$_E$) converging to $\overline{x}$ and $\{\beta_n\}$ be the corresponding sequence of scalars such that $\beta_n > 0$ for each integer n converging to 0 (see Definition 32). We denote by $f(E(\Omega_E)) \subset R^k$ and $\overline{y} = f(E(\overline{x}))$. Since $\overline{x} \in \Omega_E$ is a weak Pareto point in the E-vector optimization problems (CVP$_E$), $\overline{y} = f(E(\overline{x}))$ is a weak minimal vector in $f(E(\Omega_E))$ (see Theorem 31). Further, we consider the sequence of vectors $\{y_n\} \in f(E(\Omega_E))$, where $y_n = f(E(x_n))$. By assumption, the objective functions f_i, $i \in I$, are E-differentiable at $\overline{x}$, Thus, by Definition 6, we have

$$(f_i \circ E)(x_n) - (f_i \circ E)(\overline{x}) = \nabla (f_i \circ E)(\overline{x})^T (x_n - \overline{x})$$

$$+\theta_i(\overline{x}, x_n - \overline{x}) \|x_n - \overline{x}\|, \qquad (27)$$

where $\theta_i(\overline{x}, x_n - \overline{x}) \to 0$, $i \in I$, as $x_n \to \overline{x}$. By the above equality, we obtain, for any $i \in I$,

$$\begin{aligned} \frac{y_n - \overline{y}}{\beta_n} &= \frac{1}{\beta_n}((f_i \circ E)(x_n) - (f_i \circ E)(\overline{x})) \\ &= \nabla (f_i \circ E)(\overline{x})^T \frac{(x_n - \overline{x})}{\beta_n} + \theta_i(\overline{x}, x_n - \overline{x}) \frac{\|x_n - \overline{x}\|}{\beta_n}. \end{aligned} \qquad (28)$$

By assumption, $\{x_n\}$ is a sequence of feasible solutions in the constrained E-vector optimization problems (CVP$_E$) converging to $\overline{x}$. In view of E-differentiability of the functions f_i, $i \in I$, at $\overline{x}$, it follows that $(f \circ E)$ is differentiable at $\overline{x}$ and, hence, it is also a continuous function at $\overline{x}$. Therefore, the sequence $\{y_n\}$ converges to $\overline{y} = f(E(\overline{x}))$. Hence, by (28), it follows that

$$q = \lim_{n\to\infty} \frac{y_n - \overline{y}}{\beta_n} = \nabla (f_i \circ E)(\overline{x})^T d. \qquad (29)$$

Then, by Definition 32, q is a Bouligand contingent vector for $f(E(\Omega_E))$ at $\overline{y}$. From the E-Guignard constraint qualification, it follows that d is a Bouligand contingent vector (convergence vector) for Ω_E at $\overline{x}$ if and only if d is a solution to the system

$$\nabla \left(g_j \circ E\right)(\overline{x})^T d \leqq 0, \ \ j \in J, \qquad (30)$$

$$\nabla (h_t \circ E)(\overline{x})^T d = 0, t \in T. \tag{31}$$

Since $\overline{y}$ is a weak Pareto of $f(E(\Omega_E))$, there is no Bouligand contingent (convergence vector) for $f(E(\Omega_E))$ at $\overline{y}$ strictly negative (see Theorem 31). Therefore, the system

$$\nabla (f_i \circ E)(\overline{x})^T d < 0, \quad i \in I, \tag{32}$$

$$\nabla \left(g_j \circ E\right)(\overline{x})^T d \leqq 0, \quad j \in J, \tag{33}$$

$$\nabla (h_t \circ E)(\overline{x})^T d = 0, t \in T. \tag{34}$$

is inconsistent. From Motzkin's theorem of the alternative (see Theorem 27), it follows that the system

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x}) + \sum_{j \in J(E(\overline{x}))} \overline{\zeta}_j \nabla \left(g_j \circ E\right)(\overline{x})$$

$$+ \sum_{t=1}^{s} \overline{\xi}_t \nabla (h_t \circ E)(\overline{x}) = 0,$$

$$\overline{\lambda} \in R^p, \ \overline{\lambda} \geq 0, \ \overline{\zeta} \in R^{J(E(\overline{x}))}, \ \overline{\zeta} \geqq 0, \ \overline{\xi} \in R^s$$

is consistent. Let $(\overline{\lambda}, \overline{\zeta}, \overline{\xi})$ be a solution to the above system. Then, we define $\overline{\mu} \in R^q_+$ as follows

$$\overline{\mu}_j = \overline{\zeta}_j, \quad j \in J(E(\overline{x})),$$
$$\overline{\mu}_j = 0, \quad j \notin J(E(\overline{x})).$$

Thus, we conclude that $(\overline{\lambda}, \overline{\mu}, \overline{\xi})$ satisfies the E-Karush-Kuhn-Tucker necessary optimality conditions (24)-(26). Hence, the proof of this theorem is completed. □

In order to show that the E-Karush-Kuhn-Tucker necessary optimality conditions cannot be fulfilled without the E-Guignard constraint qualification, we present the example of such an E-differentiable vector optimization problem.

Example 39 *Consider the following nondifferentiable vector optimization problem*

$$\begin{aligned} & f(x) = (f_1(x), f_2(x)) = \\ & \left(\sin \sqrt[3]{x_1}, \quad \cos x_2 + \sqrt[3]{x_1^2}\right) \to V\text{-}\min \\ & g(x) = \sin \sqrt[3]{x_1} - \cos x_2 \leqq 0, \\ & h(x) = \sqrt[3]{x_1}(\tfrac{\pi}{2} - x_2) = 0. \end{aligned} \qquad (CVP2)$$

Note that the set of all feasible solutions of the considered constrained vector optimization problem (CVP2) is $\Omega = \{(x_1, x_2) \in R^2 : \ \sin \sqrt[3]{x_1} - \cos x_2 \leqq 0, \ \ \sqrt[3]{x_1}(\frac{\pi}{2} - x_2) = 0\}$. *Further, note that the functions constituting problem (CVP2) are nondifferentiable at* $(0,0)$. *It can be shown by Definition 19 that the feasible solution* $\overline{x} = (0,0)$ *is an E-Pareto solution of the considered nondifferentiable constrained multiobjective programming problem (CVP2).*
Let $E : R^2 \to R^2$ *be defined as follows:* $E(x_1, x_2) = \left(x_1^3, \frac{\pi}{2} - x_2\right)$. *For the considered constrained vector optimization problem (CVP2), we define its associated constrained E-vector optimization problem ($CVP_E 2$) as follows*

$$\begin{aligned} & f(E(x)) = (f_1(E(x)), f_2(E(x))) = \\ & \left(\sin x_1, \quad \sin x_2 + x_1^2\right) \to V\text{-}\min \\ & g(E(x)) = \sin x_1 - \sin x_2 \leqq 0, \\ & h(E(x)) = x_1 x_2 = 0. \end{aligned} \qquad (CVP_E 2)$$

Note that the set of all feasible solutions of the constructed constrained E-vector optimization problem ($CVP_E 2$) is $\Omega_E = \{(x_1, x_2) \in R^2 : \sin x_1 - \sin x_2 \leqq 0 \wedge x_1 x_2 = 0\}$ *and* $\eta(E(x), E(u)) = (2\sin x_1, \ 2\sin x_2)$. *Then, by the definition of the E-linearized cone, we have that* $L_E(\overline{x}) = \left\{(d_1, d_2) \in R^2 : d_2 \geqq d_1\right\}$. *Further, by the definition of the Bouligand contingent cone, we have that* $T_{\Omega_E}(\overline{x}) = \left\{(d_1, d_2) \in R^2 : d_2 \geqq 0 \wedge d_1 \leqq 0 \wedge d_1 d_2 = 0\right\}$. *Therefore,* $L_E(\overline{x}) \neq cl\ conv\, T_{\Omega_E}(\overline{x})$. *Hence, the E-Guignard constraint qualification is not fulfilled at* $\overline{x}$. *Now, we show that E-Karush-Kuhn-Tucker necessary optimality conditions are not satisfied at* $\overline{x}$. *Indeed, we have* $\nabla (f_1 \circ E)(\overline{x}) = [1, 0]^T$, $\nabla (f_2 \circ E)(\overline{x}) = [0, 1]^T$, $\nabla (g \circ E)(\overline{x}) = [1, -1]^T$, $\nabla (h \circ E)(\overline{x}) = [0, 0]^T$. *However, note that the E-Karush-Kuhn-Tucker necessary optimality conditions are not satisfied at* $\overline{x} = (0,0)$. *Namely, by (24), it follows that* $\overline{\lambda}_1 + \overline{\lambda}_2 = 0$, *are fulfilled only in the case when* $\overline{\lambda}_1 = 0$ *and* $\overline{\lambda}_2 = 0$, *what is impossible.*

Definition 40 $\left(E(\overline{x}), \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p \times R^m \times R^s$ *is said to be an E-Karush-Kuhn-Tucker point for the considered constrained vector optimization problem (CVP) if the E-Karush-Kuhn-Tucker necessary optimality conditions (24)-(26) are satisfied at* $E(\overline{x})$ *with Lagrange multiplier* $\overline{\lambda}$, $\overline{\mu}$, $\overline{\xi}$.

Definition 41 $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ *is said to be a Karush-Kuhn-Tucker point for the considered constrained E-vector optimization problem (CVP_E) if the Karush-Kuhn-Tucker necessary optimality conditions (24)-(26) are satisfied at* $\overline{x}$ *with Lagrange multiplier* $\overline{\lambda}$, $\overline{\mu}$, $\overline{\xi}$.

Now, we prove the sufficiency of the E-Karush-Kuhn-Tucker necessary optimality conditions for constrained vector optimization problem (CVP) under E-invexity hypotheses.

Theorem 42 *Let* $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}) \in \Omega_E \times R^p \times R^m \times R^s$ *be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (CVP$_E$). Let* $T_E^+(E(\overline{x})) = \{t \in T : \overline{\xi}_t > 0\}$ *and* $T_E^-(E(\overline{x})) = \{t \in T : \overline{\xi}_t < 0\}$. *Furthermore, assume the following hypotheses are fulfilled:*

a) each objective function f is an E-invex with respect to η at $\overline{x}$ on Ω_E,

b) each inequality constraint g_j, $j \in J(E(\overline{x}))$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E,

c) each equality constraint h_t, $t \in T^+(E(\overline{x}))$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E,

d) each function $-h_t$, $t \in T^-(E(\overline{x}))$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E.

Then $\overline{x}$ is a weak Pareto solution of the problem (CVP$_E$) and, thus, $E(\overline{x})$ is a weak E-Pareto solution of the problem (CVP).

Proof: By assumption, $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}) \in \Omega_E \times R^p \times R^m \times R^s$ is a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (CVP$_E$). Then, by Definition 41, the Karush-Kuhn-Tucker necessary optimality conditions (24)-(26) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^s$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution of the problem (CVP$_E$). Hence, by Definition 18, there exists another $\widetilde{x} \in \Omega_E$ such that

$$f(E(\widetilde{x})) < f(E(\overline{x})). \tag{35}$$

Using hypotheses a)-d), by Definition 7 and Theorem 12, the following inequalities

$$f_i(E(\widetilde{x})) - f_i(E(\overline{x})) \geqq \nabla f_i(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ i \in I, \tag{36}$$

$$g_j(E(\widetilde{x})) - g_j(E(\overline{x})) \geqq$$
$$\nabla g_j(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{37}$$

$$h_t(E(\widetilde{x})) - h_t(E(\overline{x})) \geqq$$
$$\nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{38}$$

$$-h_t(E(\widetilde{x})) + h_t(E(\overline{x})) \geqq$$
$$-\nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^-(E(\overline{x})) \tag{39}$$

hold, respectively. Combining (35)-(36) and then multiplying the resulting inequalities by the corresponding Lagrange multipliers and adding both their sides, we get

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla(f_i \circ E)(\overline{x})\right] \eta((E(\widetilde{x}), E(\overline{x}))) < 0. \tag{40}$$

Multiplying inequalities (37)-(39) by the corresponding Lagrange multipliers, respectively, we obtain

$$\overline{\mu}_j g_j(E(\widetilde{x})) - \overline{\mu}_j g_j(E(\overline{x})) \geqq$$
$$\overline{\mu}_j \nabla g_j(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J(E(\overline{x})), \tag{41}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq$$
$$\overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{42}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq$$
$$\overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^-(E(\overline{x})). \tag{43}$$

Using the E-Karush-Kuhn-Tucker necessary optimality condition (25) together with $\widetilde{x} \in \Omega_E$ and $\overline{x} \in \Omega_E$, we get, respectively,

$$\overline{\mu}_j \nabla g_j(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})) \leqq 0,\ j \in J(E(\overline{x})), \tag{44}$$

$$\overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})) \leqq 0,\ t \in T^+(E(\overline{x})), \tag{45}$$

$$\overline{\xi}_t \nabla h_t(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})) \leqq 0,\ t \in T^-(E(\overline{x})). \tag{46}$$

Adding both sides of the above inequalities, by (40), we obtain that the following inequality

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i \nabla(f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x}))$$
$$+ \sum_{t=1}^{s} \overline{\mu}_t \nabla h_t(E(\overline{x}))\Big]\eta(E(\widetilde{x}), E(\overline{x})) < 0$$

holds, which is a contradiction to the the E-Karush-Kuhn-Tucker necessary optimality condition (24). By assumption, $E : R^n \to R^n$ is an one-to-one and onto operator. Since $\overline{x}$ is a weak Pareto solution of the problem (CVP$_E$), by Lemma 26, $E(\overline{x})$ is a weak E-Pareto solution of the problem (CVP). Thus, the proof of this theorem is completed. □

Remark 43 *As it follows from the proof of Theorem 42, the sufficient conditions are also satisfied if all or some of the functions g_j, $j \in J(E(\overline{x}))$, h_t, $t \in T^+(E(\overline{x}))$, $-h_t$, $t \in T^-(E(\overline{x}))$ are E-differentiable quasi E-invex function at $\overline{x}$ on Ω with respect to η.*

Theorem 44 *Let* $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}) \in \Omega_E \times R^p \times R^m \times R^s$ *be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (CVP$_E$). Furthermore, assume that the following hypotheses are fulfilled:*

a) each objective function f is strictly E-invex with respect to η at $\overline{x}$ on Ω_E,

b) each inequality constraint g_j, $j \in J(\overline{x})$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E,

c) *each equality constraint h_t, $t \in T^+(E(\overline{x}))$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E,*

d) *each function $-h_t$, $t \in T^-(E(\overline{x}))$, is an E-invex function with respect to η at $\overline{x}$ on Ω_E.*

Then $\overline{x}$ is a Pareto solution of the problem (CVP$_E$) and, thus, $E(\overline{x})$ is an E-Pareto solution of the problem (CVP).

Now, under the concepts of generalized E-invexity, we prove the sufficient optimality conditions for a feasible solution to be a weak E-Pareto solution of problem (CVP).

Theorem 45 *Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (CVP$_E$). Furthermore, assume that the following hypotheses are fulfilled:*

a) *each objective function f is an pseudo E-invex function with respect to η at $\overline{x}$ on Ω_E,*

b) *each inequality constraint g_j, $j \in J(E(\overline{x}))$, is an quasi E-invex function with respect to η at $\overline{x}$ on Ω_E,*

c) *each equality constraint h_t, $t \in T^+(E(\overline{x}))$, is an quasi E-invex function with respect to η at $\overline{x}$ on Ω_E,*

d) *each function $-h_t$, $t \in T^-(E(\overline{x}))$, is an quasi E-invex function with respect to η at $\overline{x}$ on Ω_E.*

Then $\overline{x}$ is a weak Pareto solution of the problem (CVP$_E$) and, thus, $E(\overline{x})$ is a weak E-Pareto solution of the problem (CVP).

Proof: By assumption, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Omega_E \times R^p \times R^m \times R^s$ is a Karush-Kuhn-Tucker point in the considered constrained E-vector optimization problem (CVP$_E$). Then, by Definition 41, the Karush-Kuhn-Tucker necessary optimality conditions (24)-(26) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^s$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution in problem (CVP$_E$). Hence, by Definition 19, there exists another $\widetilde{x} \in \Omega_E$ such that

$$f_i(E(\widetilde{x})) \leq f_i(E(\overline{x})), \; i \in I. \tag{47}$$

By hypothesis (a), the objective function f is E-differentiable pseudo E-invex at $\overline{x}$ on Ω_E. Then, (47) gives

$$\nabla (f_i \circ E)(\overline{x})\, \eta(E(\widetilde{x}), E(\overline{x})) < 0, \; i \in I, \tag{48}$$

By the E-Karush-Kuhn-Tucker necessary optimality condition (26), inequality (48) yields

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x})\right] \eta(E(\widetilde{x}), E(\overline{x})) < 0. \tag{49}$$

Since $E(\widetilde{x}) \in \Omega$, $E(\overline{x}) \in \Omega$, therefore, the E-Karush-Kuhn-Tucker necessary optimality conditions (25) and (26) imply

$$g_j(E(\widetilde{x})) - g_j(E(\overline{x})) \leqq 0, \;\; j \in J(E(\overline{x})).$$

From the assumption, each g_j, $j \in J$, is an E-differentiable quasi E-invex function at $\overline{x}$ on Ω_E. Then, by Definition 16, we get

$$\nabla g_j(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0, \;\; j \in J(E(\overline{x})). \tag{50}$$

Thus, by the E-Karush-Kuhn-Tucker necessary optimality condition (26), and by Definition 16, (50) gives

$$\sum_{j \in J(E(\overline{x}))} \overline{\mu}_j \nabla g_j(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0.$$

Hence, taking into account $\overline{\mu}_j = 0$, $j \notin J(E(\overline{x}))$, we have

$$\sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0. \tag{51}$$

From $\widetilde{x} \in \Omega$, $\overline{x} \in \Omega$, $\widetilde{x} = E(\widetilde{x})$ and $\overline{x} = E(\overline{x})$, we get

$$h_t(E(\widetilde{x})) - h_j(E(\overline{x})) = 0, \, t \in T^+(E(\overline{x})), \tag{52}$$

$$-h_t(E(\widetilde{x})) - \left(-h_j(E(\overline{x}))\right) = 0, \, t \in T^-(E(\overline{x})). \tag{53}$$

Since each equality constraint h_t, $t \in T^+(E(\overline{x}))$, and each function $-h_t$, $t \in T^-(E(\overline{x}))$, is an E-differentiable quasi E-invex function at $\overline{x}$ on Ω_E, then by Definition 16, (52) and (53) give, respectively,

$$\nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0, \, t \in T^+(E(\overline{x})), \tag{54}$$

$$-\nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0, \, t \in T^-(E(\overline{x})). \tag{55}$$

Thus, (54) and (55) yield

$$\Bigg[\sum_{t \in T^+(E(\overline{x}))} \overline{\xi}_t \nabla h_t(E(\overline{x})) + \sum_{t \in T^-(E(\overline{x}))} \overline{\xi}_t \nabla h_t(E(\overline{x}))\Bigg] \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0.$$

Hence, taking into account $\overline{\xi}_t = 0$, $t \notin T^+(E(\overline{x})) \cup T^-(E(\overline{x}))$, we have

$$\sum_{t=1}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})) \leqq 0. \tag{56}$$

Combining (49), (51) and (56), we get that the following inequality

$$\Big[\sum_{i=1}^{p}\overline{\lambda}_i\nabla(f_i\circ E)(\overline{x})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(E(\overline{x}))$$

$$+\sum_{t=1}^{s}\overline{\xi}_t\nabla h_t(E(\overline{x}))\Big]\eta(E(\widetilde{x}),E(\overline{x}))<0,$$

which is a contradiction to the E-Karush-Kuhn-Tucker necessary optimality condition (24). The result that $E(\overline{x})$ is a weak E-Pareto solution follows directly from Lemma 26. Thus, the proof of this theorem is completed. □

In order to illustrate the sufficient optimality conditions established in the paper, we now present an example of an E-differentiable vector optimization problem in which the involved functions are (generalized) E-invex.

Example 46 *Consider the following nondifferentiable vector optimization problem*

$$\begin{gathered} f(x)=(f_1(x),f_2(x))= \\ \left(\sqrt[3]{x_1}-\cos x_2+4,\quad \sqrt[3]{x_1}-\cos x_2+2\right)\to V\text{-}\min \\ g_1(x)=\sin\sqrt[3]{x_1}-4\cos x_2+\tfrac{24}{7}\leqq 0, \\ g_2(x)=2\sin\sqrt[3]{x_1}+7\cos x_2+\sqrt[3]{x_1}-6\leqq 0, \\ g_3(x)=4\sqrt[3]{x_1^2}+4(\tfrac{\pi}{2}-x_2)^2-9\leqq 0, \qquad (CVP3) \\ g_4(x)=2\sqrt[3]{x_1}+2(\tfrac{\pi}{2}-x_2)-3\leqq 0, \\ g_5(x)=-\sin\sqrt[3]{x_1}\leqq 0, \\ g_6(x)=-\cos x_2\leqq 0. \end{gathered}$$

Note that the set of all feasible solutions of the considered vector optimization problem (CVP3) is

$$\Omega=\{(x_1,x_2)\in R^2:\sin\sqrt[3]{x_1}-4\cos x_2+\frac{24}{7}\leqq 0,$$

$$2\sin\sqrt[3]{x_1}+7\cos x_2+\sqrt[3]{x_1}-6\leqq 0,$$

$$4\sqrt[3]{x_1^2}+4(\frac{\pi}{2}-x_2)^2-9\leqq 0,\ \ 2\sqrt[3]{x_1}+2(\frac{\pi}{2}-x_2)-3\leqq 0,$$

$$\sin\sqrt[3]{x_1}\geqq 0,\quad \cos x_2\geqq 0\}.$$

Further, note that the functions constituting problem (CVP3) are nondifferentiable at $\left(0,\cos^{-1}\frac{6}{7}\right)$. *Let* $E:R^2\to R^2$ *be an one-to-one and onto mapping defined as follows* $E(x_1,x_2)=\left(x_1^3,\frac{\pi}{2}-x_2\right)$ *and* $\eta(E(x),E(u))=(\frac{\sin x_1-\sin u_1}{\cos u_1},\frac{\sin x_2-\sin u_2}{\cos u_2})$. *Now, for the considered E-invex nondifferentiable constrained multiobjective programming problem (CVP3), we define its associated constrained E-vector optimization problem (CVP$_E$3) as follows*

$$\begin{gathered} f(E(x))=(f_1(E(x)),f_2(E(x)))= \\ (x_1-\sin x_2+4,\quad x_1-\sin x_2+2)\to V\text{-}\min \\ g_1(E(x))=\sin x_1-4\sin x_2+\tfrac{24}{7}\leqq 0, \\ g_2(E(x))=2\sin x_1+7\sin x_2+x_1-6\leqq 0, \\ g_3(E(x))=4x_1^2+4x_2^2-9\leqq 0, \qquad (CVP_E3) \\ g_4(E(x))=2x_1+2x_2-3\leqq 0, \\ g_5(E(x))=-\sin x_1\leqq 0, \\ g_6(E(x))=-\sin x_2\leqq 0. \end{gathered}$$

Note that the set of all feasible solutions of the considered E-vector optimization problem (CVP$_E$3) is

$$\Omega_E=\{(x_1,x_2)\in R^2:\sin x_1-4\sin x_2+\frac{24}{7}\leqq 0,$$

$$2\sin x_1+7\sin x_2+x_1-6\leqq 0,\ \ 4x_1^2+4x_2^2-9\leqq 0,$$

$$2x_1+2x_2-3\leqq 0,\ \ \sin x_1\geqq 0,\quad \sin x_2\geqq 0\}.$$

and $\left(0,\sin^{-1}\frac{6}{7}\right)$ *is a feasible solution of the problem (CVP$_E$3). Further, note that all functions constituting the considered vector optimization problem (CVP3) are E-differentiable E-invex at* $\left(0,\sin^{-1}\frac{6}{7}\right)$. *Then, it can also be shown that the E-Karush-Kuhn-Tucker necessary optimality conditions (24)-(26) are fulfilled at* $\left(0,\sin^{-1}\frac{6}{7}\right)$ *with Lagrange multipliers* $\overline{\lambda}_1+\overline{\lambda}_2=1$, $\overline{\mu}_2=\frac{1}{7}$, *and* $\overline{\mu}_5=\frac{10}{7}$. *Further, it can be proved that* f, g_3, *and* g_4 *are an E-invex function at* $\overline{x}$ *on* Ω_E, *the constraint function* g_1, g_2 *are quasi E-invex at at* $\overline{x}$ *on* Ω_E, *the function* g_5, g_6 *are (strictly) pseudo E-invex at at* $\overline{x}$ *on* Ω_E. *Hence,* $\overline{x}=\left(0,\sin^{-1}\frac{6}{7}\right)$ *is a Pareto solution of the E-vector optimization problem (CVP$_E$3) and, thus,* $E(\overline{x})$ *is an E-Pareto solution of the considered multiobjective programming problem (CVP3).*

4 Concluding remarks

In this paper, a new class of nonconvex nondifferentiable vector optimization problems has been defined. Namely, an E-differentiable multiobjective programming problem with both inequality and equality constraint has been considered. Further, the so-called E-Karush-Kuhn-Tucker necessary optimality conditions with both inequality and equality constraints under the introduced E-Guignard constraint qualification have been established for the considered E-differentiable vector optimization problem. Moreover, the sufficient optimality conditions have been derived for such nonconvex nonsmooth vector optimization problems under the introduced concepts of E-invexity and generalized E-invexity. In order to illustrate the optimality results established in the paper, the examples of E-differentiable multiobjective programming problems have been given.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

References:

[1] T. Antczak, N. Abdulaleem: E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with inequality and equality constraints, accepted for publication in Journal of Nonlinear Sciences and Applications, 2018.

[2] T. Antczak, (p, r)-invex sets and functions, *Journal of Mathematical Analysis and Applications*, **263**(2), (2001) 355-379.

[3] T. Antczak, A class of $B-(p, r)$-invex functions and mathematical programming, *Journal of Mathematical Analysis and Applications*, **286**(1), (2003) 187-206.

[4] T. Antczak, $B-(p, r)$-pre-invex functions, *Folia Mathematica Acta Universitatis Lodziensis*, **11**, (2004) 3-15.

[5] T. Antczak, r-preinvexity and r-invexity in mathematical programming, *Computers & Mathematics with Applications*, **50**(3-4), (2005) 551-566.

[6] A. Ben-Israel, B. Mond, What is invexity?, *Journal of the Australian Mathematical Society*, **28**(1), (1986) 1-9.

[7] D. Bhatia, A. Sharma, New-invexity type conditions with applications to constrained dynamic games, *European Journal of Operational Research*, **148**(1), (2003) 48-55.

[8] B. D. Craven, Invex functions and constrained local minima, *Bulletin of the Australian Mathematical Society* **24**(3), (1981) 357-366.

[9] B. D. Craven and B.M. Glover: Invex functions and duality, *Journal of the Australian Mathematical Society* **39**(1), (1985) 1-20.

[10] M. A. Hanson, B. Mond, Further generalizations of convexity in mathematical programming, *Journal of Information and Optimization Sciences*, **3**(1), (1982) 25-32.

[11] M. A. Hanson, B. Mond, Necessary and sufficient conditions in constrained optimization, *Mathematical programming*, **37**(1), (1987) 51-58.

[12] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications*, **80**(2), (1981) 545-550.

[13] V. Jeyakumar, B. Mond, On generalised convex mathematical programming, *The Anziam Journal*, **34**(1), (1992) 43-53.

[14] V. Jeyakumar, Strong and weak invexity in mathematical programming, *Mathematical Methods of Operations Research*, **55**, (1985) 109-125.

[15] R. N. Kaul, S. K. Suneja, and Srivastava, M. K., Optimality criteria and duality in multiple-objective optimization involving generalized invexity, *Journal of Optimization Theory and Applications*, **80**(3), (1994) 465-482.

[16] Luo, Zhiming, Some properties of semi-E-preinvex maps in Banach spaces, *Nonlinear Analysis: Real World Applications*, **12**(2), (2011) 1243-1249.

[17] J. G. Lin, Maximal vectors and multi-objective optimization, *Journal of Optimization Theory and Applications*, **18**(1), (1976) 41-64.

[18] O. L. Mangasarian, Nonlinear programming, *Society for Industrial and Applied Mathematics*, 1994.

[19] A. A. Megahed, H. G. Gomma, E. A. Youness, A. Z. El-Banna, Optimality conditions of E-convex programming for an E-differentiable function, *Journal of Inequalities and Applications*, **2013**(1), (2013) 246.

[20] S. Mititelu, I. M. Stancu-Minasian, Invexity at a point: generalisations and classifications, *Bulletin of the Australian Mathematical Society*, **48**(1), (1993) 117-126.

[21] B. Mond, T. Weir: Generalized concavity and duality, in: Schaible, W.T. Ziemba (eds.), Generalized Concavity in Optimization and Economics, Academic press, New York, (1981), 263-275.

[22] S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications*, **189**(3), (1995) 901-908.

[23] S. K. Suneja, S. Khurana, Generalized nonsmooth invexity over cones in vector optimization, *European Journal of Operational Research*, **186**(1), (2008) 28-40.

[24] T. Weir, B. Mond, Preinvex functions in multiple objective optimization, *Journal of Mathematical Analysis and Applications*, **136**(1), (1988) 29-38.

[25] E. A. Youness, E-convex sets, E-convex functions and E-convex programming, *Journal of Optimization Theory and Applications*, **102**(2), (1999) 439-450.

[26] X. M. Yang, On E-convex sets, E-convex functions and E-convex programming, *Journal of Optimization Theory and Applications*, **109**(3), (2001) 699-704.