Ruled surfaces of finite *II*-type

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Abstract: - In this paper, we consider surfaces in the 3-dimensional Euclidean space E^3 without parabolic points which are of finite *II*-type, that is, they are of finite type, in the sense of B.-Y. Chen, with respect to the second fundamental form. We study an important family of surfaces, namely, ruled surfaces in E^3. We show that ruled surfaces of infinite *II*-type.

Key-Words: - Surfaces in Euclidean space, Surfaces of finite type, Ruled surface, Second fundamental form, Beltrami operator.

1 Introduction

Euclidean immersions of finite type were introduced by B.-Y. Chen about forty years ago and it has been a topic of active research by many differential geometers since then. Many results on this subject have been collected in the book ([5]). Let M^n be an n-dimensional submanifold of an arbitrary dimensional Euclidean space E^m. Denote by Δ^I the Beltrami- Laplace operator on M^n with respect to the first fundamental form I of M^n. Asubmanifold M^n is said to be of finite type with respect to the second fundamental form I, if the position vector $\boldsymbol{x}$ of M^n can be written as a finite sum of nonconstant eigenvectors of the Laplacian Δ^I, that is,

$$\boldsymbol{x}=\boldsymbol{c}+\sum_{i=1}^{k}\boldsymbol{x}_i\,, \qquad (1)$$

where $\Delta^I\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i$, $i = 1,\ldots,k$, $\boldsymbol{c}$ is a constant vector and $\lambda_1, \lambda_2,\ldots, \lambda_k$ are eigenvalues of Δ^I. Moreover, if there are exactly k nonconstant eigenvectors $\boldsymbol{x}_1,\ldots, \boldsymbol{x}_k$ appearing in (1) which all belong to different eigenvalues $\lambda_1,\ldots, \lambda_k$, then M^n is said to be of *I-type* k.However, if $\lambda_i = 0$ for some $i = 1,\ldots,k$, then M^n is said to be of *null I-type k*, otherwise M^n is said to be of infinite type.

The class of finite type submanifolds in an arbitrary dimensional Euclidean space is very large, meanwhile very little is known about surfaces of finite type in the Euclidean 3-space with respect to the first fundamental form. Actually, so far, the only known surfaces of finite type in the Euclidean 3-space are the minimal surfaces, the circular cylinders and the spheres. So in [5] B.-Y. Chen mentions the following problem

Problem1. Classify all surfaces of finite type in E^3.
With the aim of getting an answer to the this problem, important families of surfaces were studied by different authors by proving that finite type ruled surfaces [8], finite type quadrics [9], finite type tubes [7], finite type cyclides of Dupin [10, 11], finite type cones [12], and finite type spiral surfaces [4] are surfaces of the only known examples in E^3. However, for other classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet.

In this area, S. Stamatakis and H. Al-Zoubi restored attention to this theme by introducing the notion of surfaces of finite type with respect to the second or third fundamental forms (see [13]). As an extension of the above problem, we raise the following two questions which seems to be interesting:

Problem 2. Classify all surfaces of finite *II*-type in E^3.

Problem 3. Classify all surfaces of finite *III*-type in E^3.

Therefore, in order to give an answer to the second and third problem, it is worthwhile investigating the classification of the surfaces in Euclidean space E^3 in terms of finite *J*-type, *J*= *II*, *III* by studying the families of surfaces mentioned above.

According to problem 2, in [1] H. Al-Zoubi studied finite type tubes with respect to the second fundamental form and he proved that: All tubes in E^3 are of infinite type. However, for all other classical families of surfaces, the classification of its finite type surfaces is not known yet.

Regarding to problem 3, ruled surfaces and tubes are the only families were studied according to its finite type classification. More precisely, it was shown that: All tubes in E^3 are of infinite type [3]. Meanwhile in [2] H. Al-Zoubi, and others proved that: helicoids are the only ruled surfaces of finite *III*-type in the three-dimensional Euclidean space.

In this paper we will pay attention to surfaces of finite *II*-type. Firstly, we will establish a formula for $\Delta^{II}\boldsymbol{x}$ by using tensors calculations. Further, we continue our study by proving finite type ruled surfaces in the Euclidean 3-space.

2 Preliminaries

Let $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ be a regular parametric representation of a surface *S* in the Euclidean space E^3 which does not contain parabolic points. Let $I = g_{ij}du^idu^j$, $II = b_{ij}du^idu^j$ and $III = e_{ij}du^idu^j$ be the first, second and third fundamental forms of *S* respectively. For two sufficient differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ on *S*, the first differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ is defined by [13]

$$\nabla^J(f,g) := a^{ij}f_{/i}g_{/j}, \qquad (2)$$

Where $f_{/i} := \dfrac{\partial f}{\partial u^i}$ and (a^{ij}) denotes the inverse tensor of (g_{ij}), (b_{ij}) and (e_{ij}) for $J=I$, *II* and *III* respectively.

The second differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ of *M* is defined by [13]

$$\Delta^J f := -a^{ij}\nabla^J_i f_{/j}, \qquad (3)$$

where *f* is a sufficiently differentiable function, ∇^J_i is the covariant derivative in the u^i direction with respect to the fundamental form *J* and (α^{ij}) stands, as in definition (2), for the inverse tensor of (g_{ij}), (b_{ij}) and (e_{ij}) for $J = I$, *II* and *III* respectively.

Applying (3) for the position vector $\boldsymbol{x}$ of *S* we have

$$\Delta^{II}\boldsymbol{x} = -b^{ij}\nabla^{II}_j \boldsymbol{x}_{/i}. \qquad (4)$$

Recalling the equations [13, p.128]

$$\nabla^{II}_j \boldsymbol{x}_{/i} = -\frac{1}{2}b^{km}\nabla^I_k b_{ij}\boldsymbol{x}_{/m} + b_{ij}\boldsymbol{n},$$

and inserting these into (4), one finds

$$\Delta^{II}\boldsymbol{x} = \frac{1}{2}b^{ij}b^{km}\nabla^I_k b_{ij}\boldsymbol{x}_{/m} - b^{ij}b_{ij}\boldsymbol{n}, \qquad (5)$$

Where $\boldsymbol{n}$ denotes the Gauss map of *S*. By using the Mainardi-Codazzi equations [13, p.21]

$$\nabla^I_k b_{ij} - \nabla^I_i b_{jk} = 0.$$

Thus relation (5) becomes

$$\Delta^{II}\boldsymbol{x} = \frac{1}{2}b^{ij}b^{mk}\nabla^I_i b_{jk}\boldsymbol{x}_{/m} - 2\boldsymbol{n}. \qquad (6)$$

We consider the Christoffel symbols of the second kind with respect to the first and second fundamental form respectively

$$\Gamma_{ij}{}^k := \frac{1}{2}g^{km}(-g_{ij/m} + g_{im/j} + g_{jm/i}),$$

$$\Pi_{ij}{}^k := \frac{1}{2}b^{kr}(-b_{ij/m} + b_{im/j} + b_{jm/i}),$$

and we put

$$T_{ij}{}^k := \Gamma_{ij}{}^k - \Pi_{ij}{}^k. \qquad (7)$$

It is known that [13, p.125]

$$T_{ij}{}^k = -\frac{1}{2}b^{km}\nabla^I_m b_{ij}, \qquad (8)$$

and

$$\Gamma_{ij}{}^j := \frac{g_{/i}}{2g}, \qquad \Pi_{ij}{}^j := \frac{b_{/i}}{2b}, \qquad (9)$$

where $g := \det(g_{ij})$ and $b := \det(b_{ij})$.

For the Gauss curvature *K* of *S* we know that

$$K = \frac{b}{g}.$$

Using (7), (8) and (9), it follows that

$$\frac{K_{/k}}{K} = \frac{b_{/k}}{b} - \frac{g_{/k}}{g} = 2(\Pi_{kj}^{\ j} - \Gamma_{kj}^{\ j})$$
$$= -2T_{kj}^{\ j} = b^{ij}\nabla_i^I b_{kj},$$

therefore

$$\frac{1}{2K}\nabla^{III}(K,\boldsymbol{n}) = \frac{1}{2K}e^{ks}K_{/k}\boldsymbol{n}_{/s}$$

$$= \frac{1}{2}e^{ks}b^{ij}\nabla_i^I b_{kj}\boldsymbol{n}_{/s}.$$

On the other hand, from the Weingarten equations [13, p.128]

$$\boldsymbol{n}_{/j} = -e_{jk}b^{km}\boldsymbol{x}_{/m} = -b_{jk}g^{km}\boldsymbol{x}_{/m},$$

we get

$$\frac{1}{2K}\nabla^{III}(K,\boldsymbol{n}) = -\frac{1}{2}e^{ks}b^{ij}e_{sr}b^{rm}\nabla_i^I b_{kj}\boldsymbol{x}_{/m}$$

$$= -\frac{1}{2}b^{ij}b^{km}\nabla_i^I b_{kj}\boldsymbol{x}_{/m}.$$

Hence, by simple substitution in (6) we obtain the following relation

$$\Delta^{II}\boldsymbol{x} = -\frac{1}{2K}\,\mathrm{grad}^{III}K - 2\boldsymbol{n}. \qquad (10)$$

From (10) we obtain the following results which were proved in [14]:

Theorem 1 A surface S in E^3 is of II-type 1 if and only if S is part of a sphere.

Theorem 2 The Gauss map of a surface S in E^3 is of II-type 1 if and only if S is part of a sphere.

Up to now, the only known surfaces of finite II-type in E^3 are parts of spheres. In the next section we focus our attention on the class of ruled surfaces. Our main result is the following

Theorem 3. All ruled surfaces in the three-dimensional Euclidean space are of infinite II-type.

3 Proof of the main theorem

In the three-dimensional Euclidean space E^3 let S be a ruled C^r-surface, $r \geq 3$, of nonvanishing Gaussian curvature defined by an injective C^r-immersion $\boldsymbol{x} = \boldsymbol{x}(s, t)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$ [33]. The surface Scan be expressed in terms of a directrix curve Γ : $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ and a unit vector field $\boldsymbol{\beta}(s)$ pointing along the rulings as follows

$$S: \boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s), \quad s \in J, \quad -\infty < t < \infty$$

Moreover, we can take the parameter s to be the arc length along the spherical curve $\boldsymbol{\beta}(s)$.Thus for the curves $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ we have

$$<\boldsymbol{\alpha}', \boldsymbol{\beta}> = 0, \quad <\boldsymbol{\beta}, \boldsymbol{\beta}> = 1, \quad <\boldsymbol{\beta}', \boldsymbol{\beta}'> = 1,$$

where the differentiation with respect to s is denoted by a prime and $<\ ,\ >$ denotes the standard scalar product in E^3. It can be easily verified that the first and the second fundamental forms of S are given b

$$I = q ds^2 + dt^2,$$
$$II = \frac{p}{\sqrt{q}}ds^2 + \frac{2A}{\sqrt{q}}dtds,$$

where

$$q = <\boldsymbol{\alpha}', \boldsymbol{\alpha}'> + 2<\boldsymbol{\alpha}', \boldsymbol{\beta}'>t + t^2,$$

$$p = (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\alpha}'') + [(\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}'') + (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\alpha}'')]t$$
$$+ (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}'')t^2,$$

$$A = (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}')$$

If, for simplicity, we put

$$\kappa := <\boldsymbol{\alpha}', \boldsymbol{\alpha}'>, \quad \lambda := <\boldsymbol{\alpha}', \boldsymbol{\beta}'>,$$

$$\mu := (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}''), \quad \nu := (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}'') + (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\alpha}''),$$

$$\rho := (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\alpha}'').$$

Then we obtain the following relations

$$q = t^2 + 2\lambda t + \kappa, \qquad p = \mu t^2 + \nu t + \rho.$$

Furthermore, the Gaussian curvature K of S is given by

$$K = -\frac{A^2}{q^2}$$

The Beltrami operator with respect to the second fundamental form, after a long computation, can be expressed as follows

$$\Delta^{II} = \frac{\sqrt{q}}{A}\left(-2\frac{\partial^2}{\partial s \partial t} + \frac{p}{A}\frac{\partial^2}{\partial t^2} + \frac{p_t}{A}\frac{\partial}{\partial t}\right), \quad (11)$$

where $p_t := \frac{\partial p}{\partial t}$.

Applying (11) for the position vector $\boldsymbol{x}$ we find

$$\Delta^{II}\boldsymbol{x} = \frac{1}{\sqrt{q}}\left(-\frac{2q}{A}\,\boldsymbol{\omega} + \frac{qp_t}{A^2}\,\boldsymbol{\omega}\right) = \frac{1}{\sqrt{q}}\boldsymbol{P}_1(t),$$

where $\boldsymbol{P}_1(t)$ is a vector whose components are polynomials in t with functions in s as coefficients of degree less than or equal 3. More precisely we have

$$\boldsymbol{P}_1(t) = \frac{1}{A^2}\,[2\mu\boldsymbol{\beta}t^3 + ((4\lambda\mu + \nu)\boldsymbol{\beta} + 2A\boldsymbol{\beta}')t^2$$

$$+ ((2\kappa\mu + 2\lambda\nu)\boldsymbol{\beta} + 4\lambda A\boldsymbol{\beta}')t + (\kappa\nu\boldsymbol{\beta} + 2\kappa A\boldsymbol{\beta}')].$$

Before we start the proof of our theorem we give the following Lemma which can be proved by a straightforward computation.

Lemma 1. Let g be a polynomial in t with functions in s as coeficients and $\deg(g) = d$. Then

$$\Delta^{II}\left(\frac{g}{q^m}\right) = \frac{\hat{g}}{q^{m+\frac{3}{2}}},$$

where $\hat{g}$ is a polynomial in t with functions in s as coeficients and $\deg(\hat{g}) \leq d + 4$.

From now on we suppose that S is of finite II-type k. Hence there exist real numbers$c_1, c_2, \ldots, c_k$ such that

$$(\Delta^{II})^{k+1}\boldsymbol{x} + c_1(\Delta^{II})^k\boldsymbol{x} + \ldots + c_k(\Delta^{II})\boldsymbol{x} = \boldsymbol{0}, \quad (12)$$

see [6]. By lemma 1, we conclude that there is an E^3-vector-valued function $\boldsymbol{P}_k(t)$ in the variable t with some functions in s as coeficients, such that

$$(\Delta^{II})^k\boldsymbol{x} = \frac{1}{q^m}\boldsymbol{P}_k(t),$$

where $\deg(\boldsymbol{P}_k) \leq 4k - 1$ and $m = \frac{3}{2}k - 1$, $k \in \mathbb{N}$. Now, if k goes up by one, the degree of each component of $\boldsymbol{P}_k$ goes up at most by 4 while the degree of the denominator goes up by $\frac{3}{2}k - 1$. Therefore, the sum (12) can never be zero, unless of course $\Delta^{II}\boldsymbol{x} = \boldsymbol{P}_1 = \boldsymbol{0}$. But then

$$-2\boldsymbol{\beta}' + \frac{p_t}{A}\boldsymbol{\beta} = \boldsymbol{0}. \quad (13)$$

By taking the derivative of $<\boldsymbol{\beta}, \boldsymbol{\beta}> = 1$, we observe that the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are linearly independent. Thus (13) cannot be achieved unless $\boldsymbol{\beta}$ is constant, which implies that $K \equiv 0$.This is clearly impossible for the surfaces under consideration. The proof of the theorem is completed.

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