Equivariant simple singularities and admissible sets of weights

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Abstract: Equivariant maps, i.e., maps that commute with group actions on the source and target, play an important role in the study of manifolds with group actions. It is therefore of interest to classify equivariant maps up to certain equivalence relations. In this paper we study multivariate holomorphic function germs that are equivariant with respect to finite cyclic groups. The natural equivalence relation between such germs is provided by the action of the group of biholomorphic automorphism germs of the source. An orbit of this action is called *equivariant simple* if its sufficiently small neighborhood intersects only a finite number of other orbits. We present a sufficient condition under which there exist no singular equivariant holomorphic function germs; it is also shown that this condition is not necessary. The condition is formulated in terms of *admissible sets of weights*; such sets are defined and classified for all finite cyclic group representations. As an application we describe scalar actions of finite cyclic groups for which there exist no equivariant simple singular function germs.

Key–Words: Equivariant maps, finite group actions, singularity theory, classification of singularities, simple singularities.

1 Introduction

In the study of manifolds with actions of a fixed group it is natural to consider maps that commute with group actions on the source and target.

Definition 1 *Given two actions of a group G on sets M and N, we call a map $f\colon M \to N$ **equivariant** if for all $\sigma \in G$, $m \in M$ the equality $f(\sigma \cdot m) = \sigma \cdot f(m)$ holds.*

In particular, the notion of equivariance can be introduced for germs of holomorphic functions $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and of biholomorphic automorphisms $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ whenever actions of G are defined on $\mathbb{C}^n$ and $\mathbb{C}$.

The group $\mathcal{D}_n^{GG}$ of equivariant biholomorphic germs $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ acts on the space $\mathcal{O}_n^{GG}$ of equivariant holomorphic function germs $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. This infinite-dimensional space is split into orbits of this action, and so are its finite-dimensional subspaces $j_r\mathcal{O}_n^{GG}$ consisting of r-jets at 0 of germs from $\mathcal{O}_n^{GG}$. We introduce the following equivalence relation on $\mathcal{O}_n^{GG}$: two germs will be called equivalent if they belong to the same orbit.

Definition 2 *Two germs $f, g \in \mathcal{O}_n^{GG}$ are called **equivariant right equivalent** if there exists a germ $\Phi \in \mathcal{D}_n^{GG}$ such that $g = f \circ \Phi$.*

It is of interest to study the orbits of the action of $\mathcal{D}_n^{GG}$ on $\mathcal{O}_n^{GG}$, or, in other terms, to classify equivariant function germs with respect to equivariant right equivalence. In the description of the structure of the orbit space, which is often complicated, the following notion is used.

Definition 3 *An orbit $\mathcal{D}_n^{GG}(j_r g) \subset j_r\mathcal{O}_n^{GG}$ is said to be **adjacent** to the orbit $\mathcal{D}_n^{GG}(j_r f)$ if any neighborhood of some point in $\mathcal{D}_n^{GG}(j_r f)$ intersects $\mathcal{D}_n^{GG}(j_r g)$.*

Orbits of equivariant function germs can include both discrete (finite or countable) and continuous families of orbits. Discrete families make up the "simplest" part of the orbit space: its description up to equivariant right equivalence does not require the usage of continuous parameters. This motivates the following definition.

Definition 4 *A germ $f \in \mathcal{O}_n^{GG}$ is called **equivariant simple** if for all $r \in \mathbb{N}$ the orbit $\mathcal{D}_n^{GG}(j_r f) \subset j_r\mathcal{O}_n^{GG}$ has a finite number of adjacent orbits, and this number is bounded from above by a constant independent of r.*

It should be mentioned that an equivariant nonsingular function germ is always equivariant right equivalent to its linear part, and therefore all non-singular equivariant germs are equivariant simple. This is why we are only interested in studying equivariant simple germs with a critical point $0 \in \mathbb{C}^n$.

There exists a general problem to classify equivariant simple singular function germs up to equivariant right equivalence for a given finite abelian group G and a pair of its actions on the source and target. This problem naturally generalizes a similar one for the non-equivariant case solved by V. I. Arnold in 1972 (cf. [1]).

Several results are known in the equivariant setting for finite cyclic groups. In [2] simple singularities of functions on manifolds with boundary are classified, and the complex analogue of this result is the classification of simple singularities that are even in the first coordinate (i.e., equivariant with respect to the action of $\mathbb{Z}_2$ on $\mathbb{C}^n$ in the first coordinate and the trivial action on $\mathbb{C}$). A somewhat similar problem arises in [3] in connection with the classification of simple functions on space curves. In [4] the classification of odd (i.e., equivariant with respect to non-trivial scalar actions of $\mathbb{Z}_2$ on $\mathbb{C}^n$ and on $\mathbb{C}$) simple germs is given (it is proved, in particular, that no such germs exist for $n \geq 3$). In [5] and [6] the problem is solved for bivariate functions that are equivariant simple with respect to certain actions of $\mathbb{Z}_3$. Cyclic-equivariant singularities with finite monodromy groups are studied in [10].

Simple singularities of square and symmetric matrix families are studied in [7]–[9]. Some recent results on the classification of equivariant maps, vector fields and differential equations can be found in [11]–[15]. Certain calculation techniques for the classification of singularities with special attention to the equivariant case are presented in [16]–[18].

In this paper we study conditions on finite cyclic group actions under which there exist no equivariant simple singularities. A sufficient condition for nonexistence of equivariant simple singularities is given, which is also shown not to be necessary. As an application we describe scalar actions of finite cyclic groups for which there exist no equivariant simple singular functions.

In Section 2 we describe equivariance conditions for holomorphic function and automorphism germs. In Section 3 we introduce the notion of an admissible set of weights and describe all such sets for any given pair of finite cyclic group actions on $\mathbb{C}^n$ and on $\mathbb{C}$. In Section 4 a sufficient condition for nonexistence of equivariant simple singularities in terms of dimensions of certain vector spaces defined by group actions is given, as well as an example showing that this condition is not necessary. In the same section we also give a sufficient condition for existence of equivariant singular holomorphic function germs that are not equivariant simple. In Section 5 we study equivariant simple singularities for the scalar action of $\mathbb{Z}_m$ on $\mathbb{C}^n$. In Section 6 we sum up the results of the paper and list some open questions.

2 Equivariant function and automorphism germs

It is known that an action of a finite group on a vector space can be linearized in a suitable system of coordinates due to a particular case of Bochner's linearization theorem (cf. [19]). Throughout this paper we assume that the generator $\sigma \in G = \mathbb{Z}_m$ acts on $\mathbb{C}^n$ and on $\mathbb{C}$ in the following way:

$$\sigma \cdot (z_1, \dots, z_n;\, w) = (\tau^{p_1} z_1, \dots, \tau^{p_n} z_n;\, \tau^q w)\,, \quad (1)$$

where $\tau = \exp\left(\frac{2\pi i}{m}\right)$, $(z_1, \dots, z_n) \in \mathbb{C}^n$, $w \in \mathbb{C}$ and the integers $p_1, \dots, p_n, q$ are considered modulo m. In fact we will always choose $0 < p_1, \dots, p_n, q \leq m$.

Remark 5 Without loss of generality we can assume that $\gcd(p_1, \dots, p_n, q) = 1$. If $\gcd(p_1, \dots, p_n, q) = d > 1$ and $d \nmid m$, then one can divide all p_s and q by d and obtain a pair of actions that is equivalent to the original one (these two cases coincide up to the choice of generator in $\mathbb{Z}_m$). If $\gcd(p_1, \dots, p_n, q) = d > 1$ and $d \mid m$, then the given actions of the group $\mathbb{Z}_m$ can be considered as actions of its subgroup $\mathbb{Z}_{m/d}$. Moreover, we can assume that $\gcd(p_1, \dots, p_n) = 1$. If $\gcd(p_1, \dots, p_n) = d > 1$, but $\gcd(p_1, \dots, p_n, q) = 1$, then $d \nmid q$, which implies that no holomorphic function germs are equivariant with respect to actions (1).

Suppose that the actions of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$ are given by formulae (1). Any holomorphic function germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ in a neighborhood of 0 can be represented by a power series

$$f(\mathbf{z}) = \sum_{J \in \mathbb{Z}^n_{\geq 0}} a_J \mathbf{z}^J. \quad (2)$$

Here $J = (j_1, \dots, j_n), \mathbf{z} = (z_1, \dots, z_n)$, $a_J \in \mathbb{C}$, $\mathbf{z}^J = z_1^{j_1} \dots z_n^{j_n}$. It is obvious that $f \in \mathcal{O}_n^{GG}$ if and only if $a_J = 0$ whenever $\sum_{s=1}^n p_s j_s \not\equiv q \pmod{m}$.

Any germ of a biholomorphic automorphism $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ in a neighborhood of 0 can be represented by n power series of the form

$$z_k = \sum_{J \in \mathbb{Z}^n_{\geq 0}} a_{k,J} \tilde{\mathbf{z}}^J, \quad (3)$$

where $a_{k,J} \in \mathbb{C}$, $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are new variables, $\tilde{\mathbf{z}}^J = \tilde{z}_1^{j_1} \dots \tilde{z}_n^{j_n}$ and the matrix of the linear part of

Φ is non-degenerate. It is obvious that $\Phi \in \mathcal{D}_n^{GG}$ if and only if $a_{k,J} = 0$ whenever $\sum_{s=1}^n p_s j_s \not\equiv p_k \pmod{m}$.

The equivariance conditions for function and automorphism germs given by power series admit a geometric interpretation. To each monomial $\mathbf{z}^J = z_1^{j_1} \dots z_n^{j_n}$ we associate the point $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$. All points in $\mathbb{Z}^n_{\geq 0}$ associated to equivariant monomials lie in hyperplanes with the normal vector $(p_1, \dots, p_n)$. For monomials of a germ f defined by (2) these hyperplanes are defined by equations of the form

$$p_1 j_1 + \dots + p_n j_n = km + q \quad (k \in \mathbb{Z}_{\geq 0}), \qquad (4)$$

while for monomials of maps $z_l = z_l(\tilde{\mathbf{z}})$ defined by (3) they are defined by equations of the form

$$p_1 j_1 + \dots + p_n j_n = km + p_l \quad (k \in \mathbb{Z}_{\geq 0}). \qquad (5)$$

Note that under the choice of $p_1, \dots, p_n, q$ made above these hyperplanes intersect all coordinate axes at points with positive rational (but not necessarily integral) coordinates, and thus the intersection of such a hyperplane with the positive octant $\mathbb{Z}^n_{\geq 0}$ contains only a finite number of integral points.

For the description and classification of equivariant holomorphic function germs, the following two notions are convenient.

Definition 6 *Given an n-tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of natural numbers (**weights**), we define the **quasi-degree with weights** $\underline{\alpha}$ of a monomial $\mathbf{z}^J = z_1^{j_1} \dots z_n^{j_n}$ to be equal to*

$$\deg_{\underline{\alpha}} \left(\mathbf{z}^J\right) = \langle \underline{\alpha}, J \rangle = \alpha_1 j_1 + \dots + \alpha_n j_n.$$

The quasi-degree of a polynomial is defined to be the highest quasi-degree of its monomials.

Definition 7 *The r-**quasi-jet with weights** $\underline{\alpha}$ of a germ f given by power series (2) is the sum of all its monomials that have quasi-degrees with weights $\underline{\alpha}$ not exceeding r:*

$$j_r^{\underline{\alpha}} f = \sum_{\substack{J \in \mathbb{Z}^n_{\geq 0}: \\ \langle \underline{\alpha}, J \rangle \leq r}} a_J \mathbf{z}^J.$$

All r-quasi-jets of holomorphic function germs with given weights $\underline{\alpha}$ form a finite-dimensional vector space, which is exactly the space of polynomials of quasi-degree r with weights $\underline{\alpha}$. We denote this space by $j_r^{\underline{\alpha}} \mathcal{O}_n$.

3 Admissible sets of weights

For the space $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ of function germs equivariant with respect to actions (1) there exists a natural choice of weights $\underline{\alpha}$. Namely, one can take $\alpha_s = p_s$ for $s = 1, \dots, n$. Under this choice of weights, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\alpha}$ equals $km + q$ for some $k \in \mathbb{Z}_{\geq 0}$. This implies that the corresponding r-quasi-jet spaces can only increase when r increases by m:

$$\emptyset = j_1^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \dots = j_{q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset j_q^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \\ = \dots = j_{m+q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset j_{m+q}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} = \dots = \\ = j_{2m+q-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \subset \dots .$$

However, this is not the only set of weights with such a property, as one can see from the following example.

Example 8 Assume that the generator $\sigma \in G = \mathbb{Z}_3$ acts on $\mathbb{C}^2$ and on $\mathbb{C}$ as follows:

$$\sigma \cdot (z_1, z_2;\, w) = (\tau z_1,\, \tau^2 z_2;\, \tau z).$$

For weights $\underline{\alpha} = (1, 2)$ suggested above, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\alpha}$ equals $3k + 1$ for some $k \in \mathbb{Z}_{\geq 0}$. For weights $\underline{\beta} = (2, 1)$, a monomial is equivariant if and only if its quasi-degree with weights $\underline{\beta}$ equals $3k + 2$ for some $k \in \mathbb{Z}_{\geq 0}$). In both cases the corresponding r–quasi-jet spaces increase when r increases by 3.

This example motivates the following definition.

Definition 9 *A set of weights $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called **admissible** with respect to actions (1) of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$ if it satisfies the following conditions:*
1) $\gcd(\alpha_1, \dots, \alpha_n) = 1$;
2) for all $s \in [1, n]$ the inequalities $1 \leq \alpha_s \leq m$ hold;
3) a monomial is equivariant with respect to actions (1) if and only if its quasi-degree (with weights $\underline{\alpha}$) has a certain excess $\bmod m$.

Remark 10 In the case of actions (1) the choice of weights $\alpha_s = p_s$ $(s = 1, \dots, n)$ suggested above provides an admissible set of weights. The fact that this set of weights satisfies conditions 1 and 2 of the definition above follows from the assumption $\gcd(p_1, \dots, p_n) = 1$ explained in Remark 5 and the choice of $p_1, \dots, p_n, q$ made prior to that remark.

Remark 11 It is obvious that the notion of an admissible set of weights only depends on the group action on the source and not on its action on the target. Therefore, one can replace Condition 3 of Definition

9 by an equivalent one:
3') a monomial is equivariant with respect to actions (1) *with* $q = 0$ *if and only if its quasi-degree (with weights* $\underline{\alpha}$*) is divisible by* m.

Admissible sets of weights play an important role for the formulation and application of results obtained in the following sections. Therefore, a natural problem is to describe all admissible sets of weights for a given pair of group actions of the form (1). Such a description is given by the following theorem.

Theorem 12 *Suppose that the actions of the group* $\mathbb{Z}_m$ *on* $\mathbb{C}^n$ *and on* $\mathbb{C}$ *are given by* (1)*. A set of weights* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *with the property* $1 \leq \alpha_s \leq m$ *for all* $s \in [1, n]$ *is admissible with respect to these actions if and only if there exists a primitive element* $\gamma \in \mathbb{Z}_m$ *such that for all* $s \in [1, n]$ *the condition*

$$\alpha_s \equiv \gamma p_s \pmod m \tag{6}$$

is satisfied.

Proof: It is easy to see that any set of weights described in the theorem is admissible with respect to actions (1). Condition 1 of Definition 9 is satisfied due to the fact that $\gamma \in \mathbb{Z}_m$ is a primitive element and the assumption $\gcd(p_1, \ldots, p_n) = 1$ explained in Remark 5. Condition 2 of Definition 9 is satisfied due to the assumptions of the theorem. Condition 3 of Definition 9 is satisfied because a monomial $\mathbf{z}^J = z_1^{j_1} \ldots z_n^{j_n}$ is equivariant with respect to actions (1) if and only if $p_1 j_1 + \ldots + p_n j_n \equiv q \pmod m$, which is equivalent to saying that $\deg_{\underline{\alpha}}\left(\mathbf{z}^J\right) = \alpha_1 j_1 + \ldots + \alpha_n j_n \equiv \gamma(p_1 j_1 + \ldots + p_n j_n) \equiv \gamma q \pmod m$ (the two conditions are equivalent because $\gamma \in \mathbb{Z}_m$ is a primitive element). It remains to check that any set of weights $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ that is admissible with respect to actions (1) satisfies conditions (6) for some primitive element $\gamma \in \mathbb{Z}_m$. First of all, recall that, as explained in Remark 11, equalities $\alpha_s = p_s$, $s \in [1, n]$ define an admissible set of weights, and this set satisfies condition (6) for $\gamma = 1$. Without loss of generality one can assume that $p_1 = \alpha_1 = 1$ (this can always be achieved by reordering the coordinates on $\mathbb{C}^n$ and choosing another generator in $\mathbb{Z}_m$). Next, assume that $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ is another set of weights admissible with respect to actions (1). Then β_1 represents a primitive element in $\mathbb{Z}_m$ (otherwise some non-invariant monomials would have quasi-degrees with weights β divisible by m, which contradicts Condition 3' from Remark 11). Consider an invariant monomial z_1^m. One has $\deg_{\underline{\alpha}}(z_1^m) = 1 \cdot m \equiv 0 \pmod m$, $\deg_{\underline{\beta}}(z_1^m) = \beta_1 \cdot m \equiv 0 \pmod m$. Now, consider another invariant monomial $z_1^{m-\alpha_2} z_2$. One has $\deg_{\underline{\alpha}}(z_1^{m-\alpha_2} z_2) = 1 \cdot (m - \alpha_2) + 1 \cdot \alpha_2 = m \equiv 0 \pmod m$, $\deg_{\underline{\beta}}(z_1^{m-\alpha_2}) = \beta_1 \cdot (m - \alpha_2) + 1 \cdot \beta_2 \equiv -\beta_1 \alpha_2 + \underline{\beta_2} \pmod m$. Because of the requirement $\deg_{\underline{\beta}}(z_1^{m-\alpha_2}) \equiv 0 \pmod m$ that is necessary for $\underline{\beta}$ to be an admissible set of weights, one gets $-\beta_1\alpha_2 + \beta_2 \equiv 0 \pmod m$, and therefore, $\beta_2 \equiv \beta_1 \alpha_2 \pmod m$. Similarly, one can prove that $\beta_s \equiv \beta_1 \alpha_s \pmod m$ for $s = 3, \ldots, n$. Put $\gamma = \beta_1$. Then $\beta_s \equiv \gamma \alpha_s \equiv \gamma p_s \pmod m$ for $s = 1, \ldots, n$, and thus for the chosen γ conditions (6) are satisfied. This finishes the proof of Theorem 12. □

4 Sufficient nonexistence condition

Remark 13 It is worth mentioning that equivariant simple singularities can be defined in terms of quasi-jet spaces $j_r^{\underline{\alpha}} \mathcal{O}_n^{GG}$ in a way similar to the original Definition 4. It is straightforward to check that equivariant simplicity in terms of quasi-jets (with any admissible set of weights) is equivalent to equivariant simplicity in terms of ordinary jets.

In the following sections we will mostly check equivariant simplicity in terms of quasi-jets, which will be especially convenient for admissible sets of weights. Therefore we are interested in studying the way in which a biholomorphic equivariant automorphism germ of form (3) acts on quasi-jet spaces $j_r^{\underline{\alpha}} \mathcal{O}_n^{GG}$.

The classification of function germs is usually performed step by step starting from non-trivial jets of the lowest degree. In the equivariant case the same is done for quasi-jets. The following lemma describes the action of automorphism germs from $\mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ on the non-trivial quasi-jet space of lowest quasi-degree.

Lemma 14 *Let* $\Phi \in \mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ *be a biholomorphic automorphism germ equivariant with respect to action* (1) *of the group* $G = \mathbb{Z}_m$ *on* $\mathbb{C}^n$ *defined by* n *power series of form* (3)*. Let* $f \in \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ *be a holomorphic function germ equivariant with respect to actions* (1) *of* G *on* $\mathbb{C}^n$ *and on* $\mathbb{C}$*. Suppose that* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *is a set of weights admissible with respect to actions* (1)*, and* $j_{r-1}^{\underline{\alpha}} f = 0 \neq j_r^{\underline{\alpha}} f$*. Then the quasi-jet* $j_r^{\underline{\alpha}}(f \circ \Phi)$ *depends only on those terms of series* (3) *whose exponents satisfy one of the following conditions:*

$$\sum\nolimits_{s=1}^{n} \alpha_s j_s = \alpha_k. \tag{7}$$

Lemma 14 follows directly from the multiplication rule for power series.

Remark 15 Geometrically, equation (7) for each $k = 1, \dots, n$ defines a hyperplane in $\mathbb{Z}^n$ with normal vector $(\alpha_1, \dots, \alpha_n)$ passing through the point $(0, \dots, 1_k, \dots, 0)$.

Therefore, under the conditions of Lemma 14 a group of transformations depending on parameters acts on the quasi-jet space $j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$. The number of these parameters (i.e., the dimension of the group) equals the number of solutions in $\mathbb{Z}^n_{\geq 0}$ to system of equations (7) in the variables $\left\{ j_s^{(k)} \right\}$ ($k = 1, \dots, n$), which is also equal to the number of integer points with non-negative coordinates in hyperplanes (7). We denote this number by $D_r^{\underline{\alpha}}$. We also put $d_r^{\underline{\alpha}} = \dim \left(j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} / j_{r-1}^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m} \right)$, which is the dimension of the space of quasi-degree r (with weights $\underline{\alpha}$) equivariant polynomials. It follows from Lemma 14 that if $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \ldots = d_{r-1}^{\underline{\alpha}} \neq d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$, then the orbits of the action of $\mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ on $j_r^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ form at least $(d_r^{\underline{\alpha}} - D_r^{\underline{\alpha}})$-parameter families. This implies the following statement.

Theorem 16 *Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an admissible set of weights with respect to actions* (1) *of the group $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$. If $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \ldots = d_{r-1}^{\underline{\alpha}} \neq d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$ (in the notation chosen above), then there exist no holomorphic function germs in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ that are equivariant simple with respect to actions* (1).

Theorem 16 gives a sufficient condition for nonexistence of equivariant simple singular germs (or, equivalently, a necessary condition for existence of equivariant simple singular germs). However, this sufficient condition is not necessary, which can be demonstrated by the following example.

Example 17 Assume that the generator $\sigma \in G = \mathbb{Z}_3$ acts on $\mathbb{C}^3$ and on $\mathbb{C}$ as follows:

$$\sigma \cdot (z_1, z_2, z_3;\, w) = (z_1,\, \tau z_2, \tau z_3;\, \tau w). \tag{8}$$

In this case any equivariant germ is of the form $f(z_1, z_2, z_3) = Q_1(z_2, z_3) + z_1 \cdot Q_2(z_2, z_3) + H(z_1, z_2, z_3)$, where Q_1 and Q_2 are quadratic forms, and $H(z_1, z_2, z_3)$ consists of terms of higher order. Both quadratic forms can be made non-degenerate by an arbitrarily small perturbation of their coefficients. The eigenvalues of the pair of quadratic forms are invariant under equivariant changes of coordinates (z_2, z_3), because these forms are only influenced by the linear parts of the coordinate changes. Therefore, each orbit of an equivariant germ has an infinite family of adjacent equivariant germ orbits such that different orbits are characterized by different eigenvalues of the pair (Q_1, Q_2). This implies that in this case there exist no equivariant simple germs.

However, in this example the assumptions of Theorem 16 are not satisfied. Due to Theorem 12, the only admissible sets of weights for actions (8) are $\underline{\alpha} = (3, 1, 1)$ and $\underline{\beta} = (3, 2, 2)$. One obviously has $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} < d_2^{\underline{\alpha}} = 3$, because $\{0\} = j_0^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_3 \mathbb{Z}_3} = j_1^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_3 \mathbb{Z}_3} \subset j_2^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_3 \mathbb{Z}_3} = \langle z_2^2, z_2 z_3, z_3^2 \rangle$, but $D_2^{\underline{\alpha}} = 4$, because all nontrivial linear transformations of coordinates (z_2, z_3) act nontrivially on $j_2^{\underline{\alpha}} \mathcal{O}_n^{\mathbb{Z}_3 \mathbb{Z}_3}$. For similar reasons, $0 = d_0^{\underline{\beta}} = d_1^{\underline{\beta}} = d_2^{\underline{\beta}} = d_3^{\underline{\beta}} < d_4^{\underline{\beta}} = 3$, but $D_4^{\underline{\beta}} = 4$. This implies that the sufficient nonexistence condition for equivariant simple function germs given by Theorem 16 is not necessary.

Similarly to Theorem 16, one can obtain a sufficient condition for existence of non-simple equivariant germs.

Theorem 18 *Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an admissible set of weights with respect to actions* (1) *of the group $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and on $\mathbb{C}$. If there exists such a number $r \in \mathbb{N}$ that $d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$ (in the notation chosen above), then there exist holomorphic function germs in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ that are not equivariant simple with respect to actions* (1).

Proof: If $d_r^{\underline{\alpha}} > D_r^{\underline{\alpha}}$ for some $r \in \mathbb{N}$, then the classification of quasi-homogeneous polynomials of quasi-degree r (with weights $\underline{\alpha}$) contains continuous parameters, and therefore any germ $f \in \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ with the property $j_{r-1}^{\underline{\alpha}} f = 0$ is not equivariant simple. □

5 Scalar actions of $G = \mathbb{Z}_m$, $m \geq 3$

In this section we study equivariant simple singularities in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ in the case when the action of $G = \mathbb{Z}_m$, $m \geq 3$ on $\mathbb{C}^n$ is scalar. We will only consider the case $n \geq 2$ (the case $n = 1$ is trivial). Without loss of generality we can assume that the actions of the group on the source and target are given by the formulae

$$\sigma \cdot (z_1, \dots, z_n;\, w) = (\tau z_1, \dots, \tau z_n;\, \tau^q w)\,, \tag{9}$$

where $\sigma \in \mathbb{Z}_m$ is a generator, $\tau = \left(\frac{2\pi i}{m}\right)$. The result essentially depends on the excess $q \bmod m$. For the rest of this section we choose $q \in [1, m]$.

Theorem 19 *(cf. [5, Theorem 1]) Suppose that the actions of the group $\mathbb{Z}_m$ on $\mathbb{C}^n$ and $\mathbb{C}$ are given by formulae* (9) *with $q = 1$. For $m \geq 3$, $n \geq 2$ there exist no equivariant simple singular function germs in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$.*

Proof: Take $\underline{\alpha} = (1, \ldots, 1)$. Then $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \ldots = d_m^{\underline{\alpha}}$, while $d_{m+1}^{\underline{\alpha}} = \binom{n+m}{n-1}$. At the same time, $D_{m+1}^{\underline{\alpha}} = n^2$, because due to Lemma 14, the $(m+1)$-jets of singular equivariant germs depend only on the linear parts of automorphism germs from $\mathcal{D}_n^{\mathbb{Z}_m \mathbb{Z}_m}$. Finally, it is straightforward to check (e.g. by induction on m) that whenever $m \geq 3$, $n \geq 2$, the inequality $\binom{n+m}{n-1} > n^2$ holds. Therefore, the statement of the theorem follows from Theorem 16. □

Remark 20 *For $m = 2$ equivariant simple singular germs with respect to actions* (9) *with $q = 1$ are classified in [4].*

Theorem 21 *Suppose that the actions of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and $\mathbb{C}$ are given by formulae* (9) *with $q = 2$. A singular equivariant germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivariant simple with respect to the given actions if and only if it is equivalent to one of the following germs:*

$$(z_1, \ldots, z_n) \mapsto z_1^{mk+2} + z_2^2 + \ldots + z_n^2 \; (k \in \mathbb{Z}_{\geq 0}). \tag{10}$$

Proof: The proof is based on the following two lemmas.

Lemma 22 *In a neighborhood of the origin there exists an equivariant change of coordinates $x = x(\tilde{x}, \tilde{y})$, $y = y(\tilde{x}, \tilde{y})$ that gives the germ f the form $f(\tilde{x}, \tilde{y}) = \varphi(\tilde{x}) + Q(\tilde{y})$, where Q is a non-degenerate quadratic form,* $\dim\{\tilde{y}\} = rk(d^2 f|_0) = \rho$, $\dim\{\tilde{x}\} = n - \rho$.

Proof of Lemma 22: The lemma is proved similarly to [1, Lemma 4.1]. The only required modification for the equivariant case is the Morse lemma with parameter: we need to prove that a family of equivariant functions that depends analytically on the parameter and has a critical point analytically depending on the parameter with critical value 0 is equivariant right equivalent to a sum of squares. The corresponding coordinate change can be obtained in the same way as in the proof of the ordinary Morse lemma (cf. [20, Lemma 2.2]); the equivariance of this coordinate change follows from its explicit form. □

Lemma 23 *In the notation of Lemma 22 the inequality $\rho \geq n - 1$ holds.*

Proof of Lemma 23: If $\rho < n-1$, then φ is an equivariant function germ in two or more variables with a trivial $(m+1)$-jet. Therefore its lowest degree nontrivial jet is of degree greater than or equal to $m + 2 \geq 5$. But the classification of forms of degree 5 and higher in two or more variables contains moduli (i.e., continuous parameters), and therefore, in this case the germ f will not be equivariant simple. □

Now we can finish the proof of Theorem 21. If $\rho = n$, then f is a non-degenerate quadratic form in n variables that is linearly equivalent to the sum of squares, i.e., has the form (10) with $k = 0$. If $\rho = n - 1$, consider the equivariant function φ in one variable. If all of its derivatives vanish at the origin, then f is not equivariant simple (all orbits $\tilde{x}^{3k+2} + Q(\tilde{y})$ are adjacent to the orbit of f.) If $\varphi^{(i)}(0) = 0$ for $0 \leq i \leq 3k + 1$, but $\varphi^{(3k+2)}(0) \neq 0 \, (k \in \mathbb{N})$, then the germ f is equivalent to germ (10) with the same k. Each of germs (10) is equivariant simple: adjacent orbits in $j_r \mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$ for $r \geq 3k + 2$ are $z_1^{3l+2} + z_2^2 + \ldots + z_n^2$ with $0 \leq l \leq k$. Any two germs of form (10) with different values of k are not equivalent because they have different multiplicities of zero at the origin. □

Theorem 24 *Suppose that the actions of $G = \mathbb{Z}_m$ on $\mathbb{C}^n$ and $\mathbb{C}$ are given by formulae* (9) *with $q \geq 3$. If $q = 3$, $n = 2, 3$ or $q \geq 4$, $n \geq 2$, then there exist no equivariant simple function germs in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$.*

Proof: The proof is similar to the proof of Theorem 19. Take $\underline{\alpha} = (1, \ldots, 1)$. Then (in the notation of Theorem 16) $0 = d_0^{\underline{\alpha}} = d_1^{\underline{\alpha}} = \ldots = d_{q-1}^{\underline{\alpha}}$, while $d_q^{\underline{\alpha}} = \binom{n+q-1}{q-1}$, and $D_q^{\underline{\alpha}} = n^2$. From Theorem 16 it follows that if the inequality $\binom{n+q-1}{q-1} > n^2$ is satisfied, then there exist no equivariant simple function germs in $\mathcal{O}_n^{\mathbb{Z}_m \mathbb{Z}_m}$. For $q = 3$ this inequality holds only for $n = 2$ and $n = 3$. For $q \geq 4$ and $n \geq 2$ this inequality is always true, which can be proved by induction on q. □

Remark 25 The assumptions of Theorem 24 depend on the excess $q \bmod m$ and not on m itself. In particular, the statement of the theorem holds for $q = m$.

6 Conclusion

We obtained a sufficient condition for nonexistence of equivariant simple singular holomorphic function germs for finite cyclic group actions (and applied it in the case of scalar group actions on the source), as well as a sufficient condition for existence of equivariant singular holomorphic function germs that are not equivariant simple. These results can be used as the first step in classifying equivariant simple singularities for all possible actions of a given finite cyclic group on the source and target. However, we have also shown that the first of these conditions is not necessary, while the necessity of the second one is an open question. Moreover, the application of these results can meet some technical difficulties, because for a non-scalar action of the group on the source the calculation of dimensions that are used in the conditions amounts to finding the number of non-negative integer solutions to certain systems of diophantine equations. In particular, the result of Theorem 20 is incomplete: the sufficient nonexistence condition does not allow to study the case $q = 3$, $n \geq 4$ straightforwardly. Further development of calculation techniques and possibly computer algorithms for such calculations that might help to solve this problem is the aim of our future research.

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