Generalized Smoothness of the Hermite Type Splines

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Abstract: The smoothness of functions is quite essential in applications. This smoothness can be used in functional calculations, in the construction of the finite element method, in the approximation of those or other numerical data, etc. The interest in smooth approximate spaces is supported by the desire to have a coincidence of smoothness of exact and approximate solutions. A lot of papers have been devoted to this problem. The continuity of the function at a point means equality of the limits on the right and left; the generalization of this situation is the equality of values of two linear functionals (at the prescribed function) with supports located on opposite sides of the mentioned point. Such generalization allows us to introduce the concept of generalized smoothness, which gives the ability to cover various cases of singular behavior functions at some point. The generalized smoothness is called pseudo-smoothness, although, of course, we can talk about the different types of pseudo-smoothness depending on the selected functionals mentioned above. Splines are often used for processing numerical information flows; a lot of scientific papers are devoted to these investigations. Sometimes spline treatment implies to the filtration of the mentioned flows or to their wavelet decomposition. A discrete flow often appears as a result of analog signal sampling, representing the values of a function, and in this case, the splines of the Lagrange type are used. In some cases, there are two interconnected analog signals, one of which represents the values of some function, and the second one represents the values of its derivative. In this case, it is convenient to use the splines of the Hermite type of the first height for processing. In all cases, it is highly desirable that the generalized smoothness of the resulting spline coincides with the generalized smoothness of the original signal. The concepts, which are introduced in this paper, and the theorems, which are proved here, allow us to achieve this result. The paper discusses the existence and uniqueness of spline spaces of the Hermite type of the first height (under condition of fixing the spline grid and the type of generalized smoothness). The purpose of this paper is to discuss generalized smoothness of the Hermite type spline space (not necessarily polynomial). In this paper we use the necessary and sufficient criterion of the generalized smoothness obtained earlier.

Key–Words: splines, smoothness, approximate relations, uniqueness of spline spaces

1 Introduction

It is important to know about the smoothness of discussed functions. For example, in the simplest variant of in finite element method (FEM) a construction of coordinate functions has to be in the energetic space of a suitable self-adjoint operator (see [1]–[8]).

On the other hand, it is often needed to calculate some functionals on the solution (for example, the value of the solution or its derivatives in a point); for that sometimes it needs the additional smoothness of an approximate solution.

We note that the exact solution is often so smooth that it appears to have the desire to have a coincidence of smoothness of exact solution and approximate one (see [9]–[27]).

In paper [9] cell-wise strain smoothing operations are incorporated into conventional finite elements and a smoothed finite element method for 2D elastic problems is proposed. Paper [10] examines the theoretical bases for the smoothed finite element method, which is formulated by incorporating cell-wise strain smoothing operation into standard compatible finite element method. The smoothed finite element method is discussed in [11]. An edge-based smoothed finite element method is implied to improve the accuracy and convergence rate of the standard finite element method for elastic solid mechanics problems and extended to more general cases (see [12]). The cell-based smoothed finite element method [14] is used for the refinement of the accuracy and stability of the standard finite element method.

According to what has been said, a certain investigation of smoothness of approximate solutions is required. There are many research papers devoted to the construction and investigation of spline spaces. Polynomial and non-polynomial splines for equidistant and irregular grids were discussed.

In paper [28] the necessary and sufficient conditions for the smoothness of coordinate functions were obtained.

The smoothness of functions is quite essential in applications. This smoothness can be used in functional calculations, in the construction of the finite element method, in the approximation of those or other numerical data, etc. The interest in smooth approximate spaces is supported by the desire to have a coincidence of smoothness of exact and approximate solutions. A lot of papers have been devoted to this problem. The continuity of the function at a point means equality of the limits on the right and left; the generalization of this situation is the equality of values of two linear functionals (at the prescribed function) with supports, located on opposite sides of the mentioned point. Such generalization allows us to introduce the concept of generalized smoothness, which gives the ability to cover various cases of singular behavior of functions at a fixed point. The generalized smoothness is called pseudo-smoothness, although, of course, we can talk about the different types of pseudo-smoothness depending on the selected functionals mentioned above.

Splines are often used for processing numerical information flows; a lot of scientific works are devoted to these themes. Sometimes the spline treatment implies to the filtration of the mentioned flow or to its wavelet decomposition. Often a discrete flow appears as a result of analog signal sampling, representing the values of a function. In this case, the splines of the Lagrange type are used. In some cases, there are two interconnected analog signals, one of which represents the values of some function, and the second one represents the values of its derivative. In this case, for processing, it is convenient to use splines of the Hermite type of the first height. In all cases, it is highly desirable that the generalized smoothness of the resulting spline coincided with the generalized smoothness of the original signal. The concepts, which are introduced in this paper, and theorems, which are proved here, allow us to achieve this result. The paper discusses the existence and uniqueness of the spline spaces of the Hermite type (under condition of fixing the spline grid and the type of generalized smoothness). The purpose of this paper is to discuss the Hermite type spline space (not necessarily polynomial). In this paper we use the necessary and sufficient criterion of the pseudo-smoothness, obtained earlier.

2 Auxiliary assertions

Consider a smooth n-dimensional (generally speaking, noncompact) manifold $\mathcal{M}$ (i.e. topological space where each point has a neighborhood which is diffeomorphic to the open n-dimensional ball of Euclidean space $\mathbf{R}^n$).

Let $\{U_\zeta\}_{\zeta\in\mathcal{Z}}$ be a family of opened sets covering $\mathcal{M}$, and such homeomorphisms ψ_ζ, $\psi_\zeta : E_\zeta \mapsto U_\zeta$ opened balls E_ζ of the space $\mathbf{R}^n$ that the maps

$$\psi_\zeta^{-1}\psi_{\zeta'} : \psi_{\zeta'}^{-1}(U_\zeta \cap U_{\zeta'}) \mapsto \psi_\zeta^{-1}(U_\zeta \cap U_{\zeta'})$$

(for all $\zeta, \zeta' \in \mathcal{Z}$, for which the map $U_\zeta \cap U_{\zeta'} \neq \emptyset$) are continuously differential (needed a number of times); here $\mathcal{Z}$ is a set of indices.

We discuss a map $\psi_\zeta : E_\zeta \mapsto U_\zeta$ and a set $\{\psi_\zeta : E_\zeta \mapsto U_\zeta \mid \zeta \in \mathcal{Z}\}$; the last one, called atlas, defines the manifold $\mathcal{M}$.

We say that function u is defined on $\mathcal{M}$, if there is a family of functions $\{u_\zeta(x)\}_{\zeta\in\mathcal{Z},\S\in\mathcal{U}_{\zeta'}}$ such that

$$u_\zeta(\psi_\zeta^{-1}(\xi)) \equiv u_\zeta(\psi_{\zeta'}^{-1}(\xi))$$

$$\forall \xi \in U_\zeta \cap U_{\zeta'}, \quad \zeta, \zeta' \in \mathcal{Z};$$

and $u(\xi) = u_\zeta(\psi_\zeta^{-1}(\xi))$ for $\xi \in U_\zeta$.

Linear spaces of functions prescribed on $\mathcal{M}$ are defined by the atlas with usage of the relevant spaces of functions defined on balls E_ζ.

Let $\mathbf{X}(\mathcal{M})$ be a linear space of (Lebesgue measurable) functions, defined on manifold $\mathcal{M}$, where a symbol $\mathbf{X}$ denotes C^s or L_q^s; thus, the space $\mathbf{X}(\mathcal{M})$ is defined by the equality

$$\mathbf{X}(\mathcal{M}) = \{u \mid u \circ \psi_\zeta \in \mathbf{X}(E_\zeta) \quad \forall \zeta \in \mathcal{Z}\};$$

note that $C^s(E_\zeta)$ and $L_q^s(E_\zeta)$ are the usual spaces of functions defined on E_ζ ($1 \leq q \leq +\infty$, $s = 0, 1, 2, \ldots$).

Let $\mathbf{X}^*$ be the dual space to space $\mathbf{X}$; it consists of functionals f, defined by identity

$$\langle f, u\rangle \equiv \langle f_\zeta, u_\zeta\rangle_\zeta,$$

where $f_\zeta \in (\mathbf{X}(E_\zeta))^* \quad \forall \zeta \in \mathcal{Z}$, and $\{f_\zeta\}_{\zeta\in\mathcal{Z}}$ is a family of functionals representing the functional f.

If the value $\langle f, u\rangle$ of the functional $f \in (\mathbf{X}(\mathcal{M}))^*$ is defined by the values of function u on the set $\Omega \subset \mathcal{M}$ $\forall u \in \mathbf{X}(\mathcal{M})$, then we write $\mathrm{supp} f \subset \Omega$; and if in this case, Ω is a compact set, then we say that functional f has compact support. In what follows, we discuss functionals with compact support.

Let $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathcal{J}}$ be a covering family for manifold $\mathcal{M}$, where subsets $\mathcal{S}_j$ are homeomorphic to opened n-dimensional ball; thus

$$\bigcup_{j\in\mathcal{J}} \mathcal{S}_j = \mathcal{M},$$

where $\mathcal{J}$ is an ordered set of indices. The sets $\mathcal{S}_j$ are called the elements of covering $\mathcal{S}$; the boundary of set $\mathcal{S}_j$ is denoted $\partial\mathcal{S}_j$.

Consider set

$$\mathcal{C}_{(t)} = \bigcap_{j\in\mathcal{J},\, \mathcal{S}_j\ni t} \mathcal{S}_j$$

for each point $t \in \mathcal{M}\setminus\bigcup_{j\in\mathcal{J}}\partial\mathcal{S}_j$. Identifying coincided sets, we see that collection $\{\mathcal{C}_{(t)}\}$ is at most countable; later on, we denote mentioned sets by $\mathcal{C}_i$, $i \in \mathcal{K}$, where $\mathcal{K}$ is an ordered set of indices.

We have $\mathcal{C} = \{\mathcal{C}_i \mid i \in \mathcal{K}\}$, and the next relations are right:

$$\mathcal{C}_{i'} \cap \mathcal{C}_{i''} = \emptyset \quad \text{for} \quad i' \neq i'',\ i', i'' \in \mathcal{K},$$

$$Cl\ (\mathcal{S}_j) = Cl\left(\bigcup_{\mathcal{C}_i\subseteq\mathcal{S}_j} \mathcal{C}_i\right),$$

$$Cl\left(\bigcup_{i\in\mathcal{K}} \mathcal{C}_i\right) = Cl\ (\mathcal{M}); \tag{1}$$

here Cl is the closure in the topology of manifold $\mathcal{M}$.

Thus, the aggregates $\mathcal{M}$ and $\mathcal{S}_j$ are split in sets $\mathcal{C}_i$, so that the covering $\mathcal{S}$ is associated with collection $\mathcal{C}$; the rule of association described above is denoted by $\mathcal{F}$: $\mathcal{C} = \mathcal{F}(\mathcal{S})$. Collection $\mathcal{C}$ is called *the subdivision of the covering* $\mathcal{S}$.

Definition 1. *If all sets $\mathcal{C}_i$ from $\mathcal{F}(\mathcal{S})$ are homeomorphic to an opened ball then $\mathcal{S}$ is called a covering of a simple structure; in this case set $\mathcal{C}_i$ is named a cell.*

Later on we will discuss the covering of a simple structure.

Definition 2. *Let $t \in \mathcal{M}$ be fixed point; a number $\kappa_t(\mathcal{S})$ of elements of the collection $\{j \mid t \in \mathcal{S}_j\}$ is called a multiplicity of the covering of point t by the family $\mathcal{S}$.*

Definition 3. *Suppose natural number q exists, such that an identity*

$$\kappa_t(\mathcal{S}) = q \tag{2}$$

holds almost everywhere for $t \in \mathcal{M}$; then $\mathcal{S}$ is called a q-covering family (for $\mathcal{M}$), and the number q is named a multiplicity of covering of manifold $\mathcal{M}$ by the family $\mathcal{S}$.

Definition 4. *Suppose point t belongs to the intersection $Cl(\mathcal{C}_i) \cap Cl(\mathcal{C}_{i'})$, $i \neq i'$, and a neighborhood $U(t)$ of the point t belongs to the union $Cl(\mathcal{C}_i) \cup Cl(\mathcal{C}_{i'})$; in this case the cells $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ are named adjacent cells.*

Definition 5. *Let $\mathcal{S}$ be a q-covered family, let $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ be arbitrary adjacent cells (in subdivision $\mathcal{C}$ of family $\mathcal{S}$). If difference*

$$\{j \mid \mathcal{S}_j \supset \mathcal{C}_i\}\setminus\{j' \mid \mathcal{S}_{j'} \supset \mathcal{C}_{i'}\}$$

contains p elements (p is a fixed positive integer) then $\mathcal{S}$ is called a p-graduated q-covering family for manifold $\mathcal{M}$.

Consider family $A = \{\mathbf{a}_j\}_{j\in\mathcal{J}}$ of q-dimensional vectors $\mathbf{a}_j$. Family A is called an equipment of the manifold covering $\mathcal{S}$; thus each set $\mathcal{S}_j$ of the covering $\mathcal{S}$ coincides with vector $\mathbf{a}_j$ of space $\mathbf{R}^q$.

In what follows equipment A of family $\mathcal{S}$ is sometimes denoted by $A_{(\mathcal{S})}$, and the vector $\mathbf{a}_j$, coinciding with the set $\mathcal{S}_j$, is denoted by $A|_{\mathcal{S}_j}$ (thus in the discussed case $A|_{\mathcal{S}_j} = \mathbf{a}_j$).

Definition 6. *Let t be a point of manifold $\mathcal{M}$, and let $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathbf{Z}}$ be a q-covered family for $\mathcal{M}$. If the vector system*

$$A_{\langle t\rangle} = \{\mathbf{a}_j \mid j \in \mathcal{J}, \mathcal{S}_j \ni t\} \tag{3}$$

is the basis of space $\mathbf{R}^q$ almost everywhere for $t \in \mathcal{M}$ then we say that $A_{(\mathcal{S})}$ is the complete equipment of manifold covering.

By (1)–(2), (3) it follows that if A is the complete equipment of family $\mathcal{S}$, $\mathcal{C}$ is equal to $\mathcal{F}(\mathcal{S})$ and i is a fixed number, $i \in \mathcal{K}$, then the relations $A_{\langle t'\rangle} = A_{\langle t''\rangle}$ for $\forall t', t'' \in \mathcal{C}_i$,are fulfilled.

By definition, put $A_i = A_{\langle t\rangle}$ for $t \in \mathcal{C}_i$.

It is easy to see that if $\mathcal{S}$ is a p-graduated manifold covering and $\mathcal{C}_i$, $\mathcal{C}_{i'}$ are adjacent cells, then quantities of vectors in sets $A_i\setminus A_{i'}$ are equal to p (for all $i, i' \in \mathcal{K}$).

3 Spaces of minimal splines

Consider vector function $\varphi : \mathcal{M} \to \mathbf{R}^{m+1}$ with components from space $\mathbf{X}(\mathcal{M})$ (here $m \geq 0$).

In this section we discuss q-covering families of sets, where $q = m + 1$.

The proofs of the theorems in this section and the applications to splines of the Lagrange type can be found in paper [28].

Theorem 1. *Let $\mathcal{S}$ be $m + 1$-covering family (for manifold $\mathcal{M}$), and let $A = \{\mathbf{a}_j\}_{j\in\mathcal{J}}$ be a system of column vectors, forming a complete equipment of the family $\mathcal{S}$. Then there exists an unique vector function (column) $\omega(t) = (\omega_j(t))_{j\in\mathcal{J}}$, which satisfies the relations below*

$$A\omega(t) = \varphi(t), \qquad \omega_j(t) = 0 \quad \forall t \notin \mathcal{S}_j; \tag{4}$$

here and later on, the symbol A is used also for the notation of a matrix consisting of column vectors $\mathbf{a}_j$: $A = (\mathbf{a}_j)_{j\in\mathcal{J}}$.

Corollary 1. *The next relations are right:*

$$\omega_j(t) = \frac{\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_i,\ s \neq j\}\ ||'^j\ \varphi(t)\Big)}{\det\Big(\{\mathbf{a}_s \mid \mathbf{a}_s \in A_i\}\Big)}$$

$$\text{for } \forall t \in \mathcal{C}_i \subset \mathcal{S}_j, \quad \omega_j(t) = 0 \quad \forall t \notin \mathcal{S}_j;$$

here the columns in the determinants in numerator and in denominator have the same order, and the symbol $||'^j\ \varphi(t)$ *indicates that column vector* $\varphi(t)$ *is needed in place of column vector* $\mathbf{a}_j$.

Let $\mathbf{S}_m(\mathcal{S}, A, \varphi)$ be a linear space obtained by closing the linear span of set $\{\omega_j\}_{j\in\mathcal{J}}$ in the topology of pointwise convergence. The space $\mathbf{S}_m(\mathcal{S}, A, \varphi)$ is called *a space of minimal* $(\mathcal{S}, A, \varphi)$*-splines* (of order m) on manifold $\mathcal{M}$,

$$\mathbf{S}_m(\mathcal{S}, A, \varphi) = Cl_p\{\widetilde{u} \mid \widetilde{u}(t) =$$
$$= \sum_{j\in\mathcal{J}} c_j\omega_j(t) \quad \forall t \in \mathcal{M} \ \ \forall c_j \in \mathbf{R}^1\};$$

(symbol Cl_p denotes closure in the mentioned topology). Triple $(\mathcal{S}, A, \varphi)$ is named *a generator of space* $\mathbf{S}_m(\mathcal{S}, A, \varphi)$, and functions ω_j are called *coordinate functions* of the space $\mathbf{S}_m(X, A, \varphi)$. Correlations (4) are called *approximation relations*.

If the family $\mathcal{S}$ is $r+1$-graduated covering (here r is a positive integer) then we say that $(\mathcal{S}, A, \varphi)$-splines have height r. If $r = 0$ then the splines are named *splines of the Lagrange type,* if $r > 0$ then the splines are called *splines of the Hermite type*. It is easy to see that these definitions correspond to the concepts introduced in the first section.

Theorem 2. *Under the conditions of Theorem 1, linear independence of the components of vector function* $\varphi(t)$ *on cell* $\mathcal{C}_i$ *is equivalent to linear independence of the function system* $\{\omega_j(t) \mid \mathcal{C}_i \subseteq \mathcal{S}_j\}$ *on the cell.*

Theorem 3. *Suppose the conditions of Theorem 1 are fulfilled. If the components of vector function* $\varphi(t)$ *are linear independent on each cell* $\mathcal{C}_i$, $i \in \mathcal{K}$*, then the system of functions* $\{\omega_j(t)\}_{j\in\mathcal{J}}$ *is linear independent on manifold* $\mathcal{M}$.

Let F_k be a linear functional $F_k \in (\mathbf{X}(\mathcal{M}))^*$ with support in cell $\mathcal{C}_k$, $\mathrm{supp}F_k \subset \mathcal{C}_k$. If cells $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ are adjacent then by definition put $A_{k,k'} = \{\mathbf{a}_j \mid \mathbf{a}_j \in A_k \cap A_{k'}\}$. In what follows, we fix an order of column vectors $\mathbf{a}_j$ in the set $A_{k,k'}$. Sometimes we discuss the set $A_{k,k'}$ as a matrix with a mentioned order of columns.

Consider a condition

(A) relation

$$F_k\varphi = F_{k'}\varphi \tag{5}$$

is true.

The next assertions are fulfilled (for the proofs of the theorems in this section see paper [28]).

Theorem 4. *Let* $\mathcal{C}_k$ *and* $\mathcal{C}_{k'}$ *be adjacent cells. Suppose the condition* (A) *is fulfilled. Then, the necessary and sufficient conditions for the equalities*

$$F_k\,\omega_j = F_{k'}\,\omega_j \qquad \forall j \in \mathcal{J}, \tag{6}$$

to be valid are the relations below hold:

$$F_k\,\omega_j = 0 \quad \textit{for} \quad \mathbf{a}_j \in A_k \backslash A_{k,k'},$$

$$F_{k'}\,\omega_{j'} = 0 \quad \textit{for} \quad \mathbf{a}_{j'} \in A_{k'} \backslash A_{k,k'}. \tag{7}$$

Under condition (5) we put

$$F_{(k,k')}\varphi = F_k\varphi = F_{k'}\varphi.$$

Theorem 5. *Suppose the conditions of Theorem 4 are fulfilled. Then relation (6) is equivalent to the relation*

$$F_{(k,k')}\varphi \in \mathcal{L}\{\mathbf{a}_s \,|\, \mathbf{a}_s \in A_{k,k'}\}.$$

Corollary 2. *The first relation of formula (7) and the second relation of the mentioned formula are equivalent.*

4 The Hermite type splines

4.1 The Hermite type splines of the first height

Let $\mathcal{M}$ be the interval (α, β) of real axis $\mathbf{R}^1$, let $\mathcal{S}$ be a collection of sets $\mathcal{S}_j$, $j \in \mathbf{Z}$, where $\mathcal{S}_{2s-1} = (x_s, x_{s+2})$, $\mathcal{S}_{2s} = (x_s, x_{s+2})$. We see that the collection of open intervals $\mathcal{C}_i = (x_i, x_{i+1})$, $i \in \mathbf{Z}$, is a cell subdivision. The last one is generated by the grid

$$X: \qquad \ldots x_{-1} < x_0 < x_1 < x_2 < \ldots$$

Thus, $\mathcal{S}$ is a two-step four-times covering of the interval (α, β).

Suppose that vector $\mathbf{a}_j \in \mathbf{R}^4$ corresponds to the set $\mathcal{S}_j$, $j \in \mathbf{Z}$. In the discussed case the system $A_i = \{\mathbf{a}_{2i-3}, \mathbf{a}_{2i-2}, \mathbf{a}_{2i-1}, \mathbf{a}_{2i}\}$ is the equipment of the cell $\mathcal{C}_i = (x_i, x_{i+1})$, if the system is linear independent; the last one is equivalent to the condition

(H_1)

$$\det(\mathbf{a}_{2i-3}, \mathbf{a}_{2i-2}, \mathbf{a}_{2i-1}, \mathbf{a}_{2i}) \neq 0 \qquad \forall i \in \mathbf{Z}.$$

It is clear that the cell $\mathcal{C}_{k'}$ $k' = k+1$: $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$ is adjacent to the cell $\mathcal{C}_k$; we have $A_{k+1} = \{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2}\}$, and $A_{k,k'} = A_{k,k+1} = \{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}\}$, so that

$$A_k \backslash A_{k,k'} = \{\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}\},$$

$$A_{k+1} \backslash A_{k,k'} = \{\mathbf{a}_{2k+1}, \mathbf{a}_{2k+2}\}.$$

The Hermite type splines of the first height are defined by the relations

$$\sum_{j\in\mathbf{Z}} \mathbf{a}_{2j-1}\omega_{2j-1}(t) + \mathbf{a}_{2j}\omega_{2j}(t) = \varphi(t), \tag{8}$$

$$\operatorname{supp}\omega_{2j-1}, \operatorname{supp}\omega_{2j} \subset [x_j, x_{j+2}], \qquad (9)$$

where $\varphi : \ (\alpha,\beta) \longmapsto \mathbf{R}^4$, $\varphi \in \mathbf{X}(\alpha,\beta)$.

For $t \in (x_k, x_{k+1})$ we define j such that

$$[x_j, x_{j+2}] \cap (x_k, x_{k+1}) \neq \emptyset,$$

and we find $j = k-1, k$. Now by (8) – (9) we have

$$\mathbf{a}_{2k-3}\omega_{2k-3}(t) + \mathbf{a}_{2k-2}\omega_{2k-2}(t) +$$

$$+\mathbf{a}_{2k-1}\omega_{2k-1}(t) + \mathbf{a}_{2k}\omega_{2k}(t) = \varphi(t) \qquad (10)$$

$$\forall t \in (x_k, x_{k+1}). \qquad (11)$$

Suppose $t \in (x_k, x_{k+1})$. According to condition (H_1) from relations (10) – (11) we obtain

$$\omega_{2k-3}(t) = \frac{\det(\varphi(t), \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}, \qquad (12)$$

$$\omega_{2k-2}(t) = \frac{\det(\mathbf{a}_{2k-3}, \varphi(t), \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}, \qquad (13)$$

$$\omega_{2k-1}(t) = \frac{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \varphi(t), \mathbf{a}_{2k})}{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}, \qquad (14)$$

$$\omega_{2k}(t) = \frac{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \varphi(t))}{\det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k})}. \qquad (15)$$

The functions ω_j are called *coordinate splines of the Hermite type of the first height.* The closure of their linear span in pointwise topology is named *the Hermite type spline space of the first height.*

The last space is denoted with $H^1(X, A, \varphi)$.

4.2 The Hermite type splines of the second height

As before we put $\mathcal{M} = (\alpha,\beta) \subset \mathbf{R}^1$. Let $\mathcal{S}$ be a collection of sets $\mathcal{S}_{3s-2} = \mathcal{S}_{3s-1} = \mathcal{S}_{3s} = (x_s, x_{s+2})$ $\forall s \in \mathbf{Z}$. We see that the collection of open intervals $\mathcal{C}_i = (x_i, x_{i+1})$, $i \in \mathbf{Z}$, is a cell subdivision.

Thus, $\mathcal{S}$ is a 3th-step 6th-times covering of the interval (α,β).

Suppose that vector $\mathbf{a}_j \in \mathbf{R}^6$ corresponds to the set $\mathcal{S}_j$, $j \in \mathbf{Z}$. In the discussed case the system $A_k = \{\mathbf{a}_{3k-5}, \mathbf{a}_{3k-4}, \ldots, \mathbf{a}_{3k}\}$ is the equipment of the cell $\mathcal{C}_k = (x_k, x_{k+1})$ if the system is linear independent; the last one is equivalent to the condition

(H_2)

$$\det(\mathbf{a}_{3i-5}, \mathbf{a}_{3i-4}, \ldots, \mathbf{a}_{3i}) \neq 0 \qquad \forall i \in \mathbf{Z}.$$

For the cell $\mathcal{C}_{k'}$ $k' = k+1$: $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$ we have $A_{k+1} = \{\mathbf{a}_{3k-2}, \mathbf{a}_{3k-1}, \ldots, \mathbf{a}_{3k+3}\}$, and $A_{k,k'} = A_{k,k+1} = \{\mathbf{a}_{3k-2}, \mathbf{a}_{3k-1}, \mathbf{a}_{3k}\}$, so that

$$A_k \backslash A_{k,k'} = \{\mathbf{a}_{3k-5}, \mathbf{a}_{3k-4}, \mathbf{a}_{3k-3}\},$$

$$A_{k+1} \backslash A_{k,k'} = \{\mathbf{a}_{3k+1}, \mathbf{a}_{3k+2}, \mathbf{a}_{3k+3}\}.$$

The Hermite type splines of the second order are defined by the relations

$$\sum_{j\in\mathbf{Z}} \mathbf{a}_{3j-2}\omega_{3j-2}(t) + \mathbf{a}_{3j-1}\omega_{3j-1}(t) +$$

$$+\mathbf{a}_{3j}\omega_{3j}(t) = \varphi(t), \qquad (16)$$

$$\operatorname{supp}\omega_{3j-2}, \operatorname{supp}\omega_{3j-1},$$

$$\operatorname{supp}\omega_{3j} \subset [x_j, x_{j+2}], \qquad (17)$$

where $\varphi : \ (\alpha,\beta) \longmapsto \mathbf{R}^6$, $\varphi \in \mathbf{X}(\alpha,\beta)$.

Now by (16) – (17) we have

$$\sum_{j=3k-5}^{3k} \mathbf{a}_j\omega_j(t) = \varphi(t), \qquad \forall t \in (x_k, x_{k+1}). \quad (18)$$

Thus according to condition (H_2) and relations (18) we obtain

$$\omega_j(t) =$$

$$= \frac{\det\Big(\mathbf{a}_{3k-5}, \mathbf{a}_{3k-4}, \ldots, \mathbf{a}_{3k} \ ||'^j \ \varphi(t)\Big)}{\det\Big(\mathbf{a}_{3k-5}, \mathbf{a}_{3k-4}, \ldots, \mathbf{a}_{3k}\Big)}, \qquad (19)$$

where $t \in (x_k, x_{k+1}) \subset [x_j, x_{j+2}]$, $\omega_j(t') = 0$ $\forall t' \notin [x_j, x_{j+2}]$.

The closure (in pointwise topology) of linear span of system $\{\omega_j\}_{j\in\mathbf{Z}}$ is named *the Hermite type spline space of the second height*; it is denoted with $H^2(X, A, \varphi)$.

4.3 The Hermite type splines of the third height

The Hermite type splines of the third height are constructed analogously. In that case we introduce a next notion: $\mathcal{M} = (\alpha,\beta) \subset \mathbf{R}^1$, $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\mathbf{Z}}$, where $\mathcal{S}_{4s-3} = \mathcal{S}_{4s-2} = \mathcal{S}_{4s-1} = \mathcal{S}_{4s} = (x_s, x_{s+2})$ $\forall s \in \mathbf{Z}$. As before we obtain the collection of open intervals $\mathcal{C}_i = (x_i, x_{i+1})$, $i \in \mathbf{Z}$, which give subdivision of $\mathcal{M}$. Now we have a covering of the interval (α,β) by he collection $\mathcal{S}$, where $q = 8$ and $p = 4$.

Discuss vectors $\mathbf{a}_j \in \mathbf{R}^8$ with property

(H_3)

$$\det(\mathbf{a}_{4i-7}, \mathbf{a}_{4i-6}, \ldots, \mathbf{a}_{4i}) \neq 0 \qquad \forall i \in \mathbf{Z}.$$

In the discussed case the system $A_k = \{\mathbf{a}_{4k-7}, \mathbf{a}_{4k-6}, \ldots, \mathbf{a}_{4k}\}$ is the equipment of the cell $\mathcal{C}_k = (x_k, x_{k+1})$.

For the adjacent cell $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$ we have

$$A_{k+1} = \{\mathbf{a}_{4k-3}, \mathbf{a}_{4k-2}, \ldots, \mathbf{a}_{4k+4}\},$$

and $A_{k,k+1} = \{\mathbf{a}_{4k-3}, \mathbf{a}_{4k-2}, \mathbf{a}_{4k-1}, \mathbf{a}_{4k}\}$, so that

$$A_k \backslash A_{k,k'} = \{\mathbf{a}_{4k-7}, \dots, \mathbf{a}_{4k-4}\}, \qquad (20)$$

$$A_{k+1} \backslash A_{k,k'} = \{\mathbf{a}_{4k+1}, \dots, \mathbf{a}_{4k+4}\}.$$

The Hermite type splines of the third height are defined by the approximate relations

$$\sum_{j\in\mathbf{Z}} \mathbf{a}_{4j-3}\omega_{4j-3}(t) + \mathbf{a}_{4j-2}\omega_{4j-2}(t) + \mathbf{a}_{4j-1}\omega_{4j-1}(t) +$$

$$+\mathbf{a}_{4j}\omega_{4j}(t) = \varphi(t) \qquad \forall t \in (\alpha, \beta), \qquad (21)$$

$$\operatorname{supp}\omega_{4j-3}, \operatorname{supp}\omega_{4j-2},$$

$$\operatorname{supp}\omega_{4j-1}, \operatorname{supp}\omega_{4j} \subset [x_j, x_{j+2}], \qquad (22)$$

where $\varphi : (\alpha, \beta) \longmapsto \mathbf{R}^8$, $\varphi \in \mathbf{X}(\alpha, \beta)$.

Now by (21) – (22) we have

$$\sum_{j=4k-7}^{4k} \mathbf{a}_j\omega_j(t) = \varphi(t) \qquad \forall t \in (x_k, x_{k+1}). \qquad (23)$$

Thus according to condition (H_3) and relations (23) we obtain

$$\omega_j(t) =$$

$$\frac{\det\Big(\mathbf{a}_{4k-7}, \mathbf{a}_{4k-6}, \dots, \mathbf{a}_{4k} \;||'^j\; \varphi(t)\Big)}{\det\Big(\mathbf{a}_{4k-7}, \mathbf{a}_{4k-6}, \dots, \mathbf{a}_{4k}\Big)}, \qquad (24)$$

where $t \in (x_k, x_{k+1}) \subset [x_j, x_{j+2}]$, $\omega_j(t') = 0$ $\forall t' \notin [x_j, x_{j+2}]$.

The functions ω_j are called *coordinate splines of the Hermite type of the third height*, and the closure of their linear span in pointwise topology $H^3(X, A, \varphi)$ is named *the Hermite type spline space of the third height*.

5 Pseudo-smoothness of the Hermite type splines

5.1 Pseudo-smoothness of the Hermite type splines of the first height

If $t \in (x_{k+1}, x_{k+2})$ then analogously to formulas (12)–(15) we have (see subsection 4.1)

$$\omega_{2k-1}(t) = \frac{\det(\varphi(t), \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})}{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})},$$

$$\omega_{2k}(t) = \frac{\det(\mathbf{a}_{2k-1}, \varphi(t), \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})}{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})},$$

$$\omega_{2k+1}(t) = \frac{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \varphi(t), \mathbf{a}_{2k+2})}{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})}, \qquad (25)$$

$$\omega_{2k+2}(t) = \frac{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \varphi(t))}{\det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \mathbf{a}_{2k+2})}. \qquad (26)$$

Consider the linear spaces $\mathbf{X}(x_j, x_{j+1})$, which consist of functions $u(t)$, $t \in (x_j, x_{j+1})$. For example, we can assume that $\mathbf{X}(x_j, x_{j+1}) = C^s(x_j, x_{j+1})$, or $\mathbf{X}(x_j, x_{j+1}) = W_p^s(x_j, x_{j+1})$, where $1 \leq p$, p is real number, s is nonnegative integer.

Let $\mathcal{X}$ be a direct production of the spaces:

$$\mathcal{X} = \ldots \times \mathbf{X}(x_{-1}, x_0) \times \mathbf{X}(x_0, x_1) \times \mathbf{X}(x_1, x_2) \times \ldots$$

Let F_k and F_{k+1} be linear functionals in $\mathcal{X}$ with supports in adjacent cells $\mathcal{C}_k = (x_k, x_{k+1})$ and $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$.

We introduce a condition

(D_1)

$$F_k\varphi = F_{k+1}\varphi.$$

Under condition (D_1) we define

$$\Phi_k = F_k\varphi = F_{k+1}\varphi. \qquad (27)$$

According to Theorem 4, conditions (6) and (7) are equivalent. In the discussed case, the last one can be written in the form

$$F_k\,\omega_j = 0 \quad \text{for } \mathbf{a}_j \in \{\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}\}, \qquad (28)$$

$$F_{k'}\,\omega_{j'} = 0 \quad \text{for } \mathbf{a}_{j'} \in \{\mathbf{a}_{2k+1}, \mathbf{a}_{2k+2}\}. \qquad (29)$$

By (12) – (13) and (27) – (28) we have

$$F_k\omega_{2k-3} = 0 \iff \det(\mathbf{a}_{2k-2}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \Phi_k) = 0,$$

$$F_k\omega_{2k-2} = 0 \iff \det(\mathbf{a}_{2k-3}, \mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \Phi_k) = 0;$$

now we deduce

$$\Phi_k \in \mathcal{L}\{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}\}. \qquad (30)$$

By (25) – (26) and (27) – (29) analogously we obtain

$$F_{k+1}\omega_{2k+1} = 0 \iff$$

$$\iff \det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \Phi_k) = 0,$$

$$F_{k+1}\omega_{2k+2} = 0 \iff$$

$$\iff \det(\mathbf{a}_{2k-1}, \mathbf{a}_{2k}, \mathbf{a}_{2k+1}, \Phi_k) = 0.$$

It is clear to see that we obtain the implication (30) again.

In addition, we assume that functionals F'_k and F'_{k+1} have supports in adjacent cells $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ accordingly.

Suppose that the next condition is true

(D'_1)

$$F'_k\varphi = F'_{k+1}\varphi.$$

Later we take into account the notation

$$\Phi'_k = F'_k\varphi = F'_{k+1}\varphi. \qquad (31)$$

Using the previous discussion, we obtain

$$\Phi'_k \in \mathcal{L}\{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}\}.$$

Suppose the next condition is fulfilled

(E_1) the conditions (D_1) and (D'_1) are true, and vectors defined by relations (27) and (31) are linear independent (for each fixed $k \in \mathbf{Z}$).

Under condition (E_1) the next relation is true

$$\mathcal{L}\{\Phi_k, \Phi'_k\} = \mathcal{L}\{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}\} \qquad \forall k \in \mathbf{Z}. \qquad (32)$$

Let $\Phi_{(1)}$ be a sequence of pairs (Φ_k, Φ'_k) so that $\Phi_{(1)} = \{(\Phi_k, \Phi'_k)\}_{k\in\mathbf{Z}}$.

If the conditions (H_1) and (E_1) are true, then the space $H^1(X, A, \varphi)$ is named *a space of the Hermite type splines with psuedo-smoothness* $\Phi_{(1)}$.

The previous discussion proves the next assertion.

Theorem 6. *If grid X and vector function $\varphi(t)$ are fixed, then the space of the Hermite type splines with psuedo-smoothness $\Phi_{(1)}$ is unique.*

Proof. Taking into account condition (32), we have

$$\mathbf{a}_{2k-3} = \alpha_{k-1}\Phi_{k-1} + \alpha'_{k-1}\Phi'_{k-1}, \qquad (33)$$

$$\mathbf{a}_{2k-2} = \beta_{k-1}\Phi_{k-1} + \beta'_{k-1}\Phi'_{k-1}, \qquad (34)$$

$$\mathbf{a}_{2k-1} = \alpha_k\Phi_k + \alpha'_k\Phi'_k, \qquad (35)$$

$$\mathbf{a}_{2k-2} = \beta_k\Phi_k + \beta'_k\Phi'_k. \qquad (36)$$

Using formulas (33) – (36) in (10) we get the system of equations

$$\Phi_{k-1}\widetilde{\omega}_{2k-3} + \Phi'_{k-1}\widetilde{\omega}_{2k-2} +$$

$$+\Phi_k\widetilde{\omega}_{2k-1} + \Phi'_k\widetilde{\omega}_{2k} = \varphi(t), \qquad (37)$$

where

$$\widetilde{\omega}_{2k-3} = \alpha_{k-1}\omega_{2k-3} + \beta_{k-1}\omega_{2k-2}, \qquad (38)$$

$$\widetilde{\omega}_{2k-2} = \alpha'_{k-1}\omega_{2k-3} + \beta'_{k-1}\omega_{2k-2}, \qquad (39)$$

$$\widetilde{\omega}_{2k-1} = \alpha_k\omega_{2k-1} + \beta_k\omega_{2k}, \qquad (40)$$

$$\widetilde{\omega}_{2k} = \alpha'_k\omega_{2k-1} + \beta'_k\omega_{2k}. \qquad (41)$$

According to supposition (E_1), vectors Φ_k and Φ'_k are linear independent. By conditions (32) it follows that the vectors belong to hyperplane $\mathcal{L}\{\mathbf{a}_{2k-1}, \mathbf{a}_{2k}\}$. Analogously vectors Φ_{k-1} and Φ'_{k-1} are linear independent, and belong to hyperplane $\mathcal{L}\{\mathbf{a}_{2k-3}, \mathbf{a}_{2k-2}\}$. Taking into account supposition (H_1) for $i = k$, the two hyperplanes have trivial intersection. Therefore vectors Φ_{k-1}, Φ'_{k-1}, Φ_k, Φ'_k are linear independent.

It is clear that system (37) has unique solution $\widetilde{\omega}_i(t)$, and $\operatorname{supp}\widetilde{\omega}_i = \operatorname{supp}\omega_i$, $i = 2k-3$, $2k-2$, $2k-1$, $2k$.

Previuous discussion demonstrates that the coordinate splines ω_j coincide to coordinate splines $\widetilde{\omega}_i$ by relations (38) – (41) for arbitrary system of vectors satisfying conditions (H_1) and (31).

Thus, coinciding hulls of the mentioned spline systems be the same. This completes the proof.

It is clear to see that the space mentioned in Theorem 6 is defined by grid X, vector function $\varphi(t)$ and by the family $\Phi_{(1)}$; this space we denote by $H^1(X, \varphi, \Phi_{(1)})$.

5.2 Pseudo-smoothness of the Hermite type splines of the second height

Let F_k, F'_k, F''_k and F_{k+1}, F'_{k+1}, F''_{k+1} be linear functionals in $\mathcal{X}$ with supports in adjacent cells $\mathcal{C}_k = (x_k, x_{k+1})$ and $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$ accordingly.

Discussing the situation of subsection 4.2, we introduce a condition

(D_2)

$$F_k\varphi = F_{k+1}\varphi, \quad F'_k\varphi = F'_{k+1}\varphi, \quad F''_k\varphi = F''_{k+1}\varphi.$$

Under condition (D_2) we define

$$\Phi_k = F_k\varphi = F_{k+1}\varphi, \qquad \Phi'_k = F'_k\varphi = F'_{k+1},$$

$$\Phi''_k = F''_k\varphi = F''_{k+1}\varphi. \qquad (42)$$

We are interested in relations

$$F_k\omega_j = F_{k+1}\omega_j, \qquad F'_k\omega_j = F'_{k+1}\omega_j, \qquad (43)$$

$$F''_k\omega_j = F''_{k+1}\omega_j \qquad \forall j \in \mathbf{Z}. \qquad (44)$$

If condition (D_2) is fulfilled then according to Theorem 4 and Corollary 2 relation (43) is equivalent to equalities

$$F_k\omega_j = 0 \qquad \forall \mathbf{a}_j \in A_k \backslash A_{k+1}. \qquad (45)$$

By formula (19) we see that (45) is equivalent to relations

$$F_k\omega_{3k-5} = 0 \iff$$

$$\iff \det(\Phi_k, \mathbf{a}_{3k-4}, \mathbf{a}_{3k-3}, \ldots, \mathbf{a}_{3k}) = 0, \qquad (46)$$

$$F_k\omega_{3k-4} = 0 \iff$$

$$\iff \det(\mathbf{a}_{3k-5}, \Phi_k, \mathbf{a}_{3k-3}, \ldots, \mathbf{a}_{3k}) = 0, \qquad (47)$$

$$F_k\omega_{3k-3} = 0 \iff$$

$$\iff \det(\mathbf{a}_{3k-5}, \mathbf{a}_{3k-4}, \Phi_k, \mathbf{a}_{3k-2}, \ldots, \mathbf{a}_{3k}) = 0. \quad (48)$$

According to formulas (46) – (48) we have

$$\Phi_k \in \mathcal{L}\{\mathbf{a}_{3k-2}, \mathbf{a}_{3k-1}, \mathbf{a}_{3k}\}. \quad (49)$$

Analogously by (44) we obtain

$$\Phi'_k, \Phi''_k \in \mathcal{L}\{\mathbf{a}_{3k-2}, \mathbf{a}_{3k-1}, \mathbf{a}_{3k}\}. \quad (50)$$

Suppose the next condition is fulfilled

(E_2) the conditions (D_2) is true, and vectors, defined by relations (42), are linear independent (for each fixed $k \in \mathbf{Z}$).

If condition (E_2) is true then by (49) – (50) we have

$$\mathcal{L}\{\Phi_k, \Phi'_k, \Phi''_k\} = \mathcal{L}\{\mathbf{a}_{3k-2}, \mathbf{a}_{3k-1}, \mathbf{a}_{3k}\} \qquad \forall k \in \mathbf{Z}.$$

Let $\Phi_{(2)}$ be a sequence of triple $(\Phi_k, \Phi'_k, \Phi''_k)$ so that $\Phi_{(2)} = \{(\Phi_k, \Phi'_k, \Phi''_k)\}_{k\in\mathbf{Z}}$.

If the conditions (H_2) and (E_2) are true, then the space $H^2(X, A, \varphi)$ is named *a space of the Hermite type splines with psuedo-smoothness* $\Phi_{(2)}$.

Theorem 7. *If grid X and vector function $\varphi(t)$ are fixed, then the space of the Hermite type splines with psuedo-smoothness $\Phi_{(2)}$ is unique.*

Proof of Theorem 7 is similar to proof of Theorem 6.

The space mentioned in Theorem 7 is defined by grid X, vector function $\varphi(t)$ and by the family $\Phi_{(2)}$; this space we denote by $H^2(X, \varphi, \Phi_{(2)})$.

5.3 On smoothness of the Hermite type splines of the third height

Here we give a short discussion of the Hermite splines of the third height (see subsection 4.3).

Let F_k, F'_k, F''_k, F'''_k and F_{k+1}, F'_{k+1}, F''_{k+1}, F'''_{k+1} be linear functionals in $\mathcal{X}$ with supports in adjacent cells $\mathcal{C}_k = (x_k, x_{k+1})$ and $\mathcal{C}_{k+1} = (x_{k+1}, x_{k+2})$ accordingly.

Now we discuss a condition

(D_3)

$$F_k\varphi = F_{k+1}\varphi, \quad F'_k\varphi = F'_{k+1}\varphi,$$

$$F''_k\varphi = F''_{k+1}\varphi, \quad F'''_k\varphi = F'''_{k+1}\varphi,$$

and define vectors

$$\Phi_k = F_k\varphi = F_{k+1}\varphi,\ \Phi'_k = F'_k\varphi = F'_{k+1}, \quad (51)$$

$$\Phi''_k = F''_k\varphi = F''_{k+1}\varphi,$$
$$\Phi'''_k = F'''_k\varphi = F'''_{k+1}\varphi. \quad (52)$$

We are interested in relations

$$F_k\omega_j = F_{k+1}\omega_j, \quad (53)$$

$$F'_k\omega_j = F'_{k+1}\omega_j, \quad F''_k\omega_j = F''_{k+1}\omega_j,$$
$$F'''_k\omega_j = F'''_{k+1}\omega_j \quad \forall j \in \mathbf{Z}.$$

If condition (D_3) is fulfilled then according to Theorem 4 and Corollary 2, relation (53) is equivalent to equalities

$$F_k\omega_j = 0 \qquad \forall \mathbf{a}_j \in A_k \backslash A_{k+1}. \quad (54)$$

Taking into account formula (24), we see that by (20) and (54) we have

$$\Phi_k \in \mathcal{L}\{\mathbf{a}_{4k-3}, \mathbf{a}_{4k-2}, \mathbf{a}_{4k-1}, \mathbf{a}_{4k}\}. \quad (55)$$

Analogously by (24) we obtain

$$\Phi'_k, \Phi''_k, \Phi'''_k \in \mathcal{L}\{\mathbf{a}_{4k-3}, \mathbf{a}_{4k-2}, \mathbf{a}_{4k-1}, \mathbf{a}_{4k}\}. \quad (56)$$

Suppose the next condition is fulfilled

(E_3) the conditions (D_3) are true, and vectors defined by relations (51) – (52) are linear independent (for each fixed $k \in \mathbf{Z}$).

If condition (E_3) is true then by (55) – (56) we have

$$\mathcal{L}\{\Phi_k, \Phi'_k, \Phi''_k, \Phi''_k\} = \mathcal{L}\{\mathbf{a}_{4k-3}, \mathbf{a}_{4k-2}, \mathbf{a}_{4k-1}, \mathbf{a}_{4k}\}$$
$$\forall k \in \mathbf{Z}.$$

Let $\Phi_{(3)}$ be a sequence of quadruple $(\Phi_k, \Phi'_k, \Phi''_k, \Phi'''_k)$ so that $\Phi_{(3)} = \{(\Phi_k, \Phi'_k, \Phi''_k, \Phi'''_k, \Phi'''_k)\}_{k\in\mathbf{Z}}$.

If the conditions (H_3) and (E_3) are true, then the space $H^3(X, A, \varphi)$ is named *a space of the Hermite type splines with psuedo-smoothness* $\Phi_{(3)}$.

The previous discussion proves the next assertion (see also the proof of Theorem 6).

Theorem 8. *If grid X and vector function $\varphi(t)$ are fixed, then the space of the Hermite type splines of the third height with psuedo-smoothness $\Phi_{(3)}$ is unique.*

The space mentioned in Theorem 8 is defined by grid X, vector function $\varphi(t)$ and by the family $\Phi_{(3)}$; this space we denote by $H^3(X, \varphi, \Phi_{(3)})$.

6 Conclusion

Returning to the definition of basic splines, we note that the approximate relations consist of identities that include a priori given vector sequence $A = \{\mathbf{a}_j \mid \mathbf{a}_j \in$

$\mathbf{R^{m+1}}\}$ and $m+1$-dimensional vector-valued function $\varphi(t)$ (that is named generating vector function) defined on the interval (α, β):

$$\sum_j \mathbf{a}_j \omega_j(t) = \varphi(t);$$

here m is a nonnegative integer. In addition, location of coordinate spline supports (relatively to the grid X) is indicated at the mentioned interval.

The smoothness of coordinate functions inside of cells are defined by the smoothness of generating vector function in approximate relations, but the smoothness of coordinate functions on the boundary of adjacent cells required additional discussion.

The location of supports determines the type of splines: for example, if the supports are "ladder", $\operatorname{supp}\omega_j \subset [x_j, x_{j+m+1}]$, then splines of the Lagrange type are obtained. The case of nested supports leads to the splines of the Hermite type.

Different variants of choice of vector sequence A lead to different sets of coordinate functions with various types of the grid pseudo-smoothness. In general, the corresponding linear hulls of these functions form different linear spaces.

In this paper we considered the splines of the Hermite type of the first, second and third height. Here we took $m = 3, 5, 7$ and arranged the supports of the coordinate splines like two, three and four identical "stairs". If we discuss $k+1$ identical "stairs" we'll obtain the Hermite splines of k-th height $k \in \mathbf{Z}$. Moreover, various positions of the supports can lead to splines of mixed type. For approximation of functions with singularities, the singular splines are required; the last ones can be obtained by the choice of the generating vector function $\varphi(t)$ and using the corresponding sequence of vectors $\mathbf{a}_j$. Questions of generalized smoothness for some of these splines are supposed to be discussed in the future.

This paper discusses general smoothness as a coincidence of values of two linear functionals on the appropriate functions where mentioned functionals have their supports in adjacent cells. It gives the opportunity to discuss different sorts of smoothness.

Simplest example for adjacent cells $\mathcal{C}_k = (x_k, x_{k+1})$ and $\mathcal{C}_{k'} = (x_{k'}, x_{k'+1})$, $k' = k+1$, is the next one

$$F_k u = \frac{1}{a_k} \lim_{\tau \to -0} u(x_{k+1} + \tau)$$

and

$$F'_k u = \frac{1}{a'_k} \lim_{\tau \to +0} u(x_{k+1} + \tau),$$

where $a_k a'_k \neq 0$, $a_k, a'_k \in \mathbf{R}^1$. The equality $F_k u = F_{k'} u$ allow us to discuss discontinuities of the first kind.

More complicated example is

$$F_k u = \lim_{\tau \to -0} \int_\tau^0 \psi(\xi) u(x_{k+1} + \xi) d\xi,$$

$$F_{k'} u = \lim_{\tau \to +0} \int_0^\tau \psi(\xi) u(x_{k+1} + \xi) d\xi,$$

where $\psi(\xi)$ is a weight function; now the equality $F_k u = F_{k'} u$ illustrates "weighted smoothness" (see also [28] – [29]).

Here we have only discussed the spaces of the Hermite type spline with the first, second and third heights. The obtained result gives the opportunity to prove the uniqueness of the Hermite spline spaces of the highest smoothness with arbitrary height.

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