About Adaptive Grids Construction

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Abstract: One of the important tasks of interpolation is the good selection of interpolation nodes. As is well known, the roots of the Chebyshev polynomial are optimal ones for interpolation with algebraic polynomials on the interval [-1,1]. Nevertheless, there are still some difficulties in constructing the grid of nodes when the number of points tend to infinity. Here we offer the formula for constructing the interpolation nodes for a rapidly increasing function or decreasing function. The formula takes into account the local behavior of the function on the previous three grid nodes, and it is based on the interpolation by local quadratic polynomial splines. Particular attention is paid to the interpolation of functions on radial-ring grids.

Key-Words: polynomial splines, non-polynomial splines, tensor product, approximation, adaptive grid of nodes

1 Introduction

As it is known, the approximation of functions of several variables are widely used in solving various problems. Such approximations can be used for: increasing the efficiency of information transfer systems; image processing; reducing the data within the table, etc. [1-10]. One of the important tasks of interpolation is good selection of interpolation nodes. To approximate fast-growing functions, often a uniform grid of nodes is not enough and it is necessary to use a special adaptive grid. Therefore, constructing an adaptive mesh of nodes that takes into account the behavior of the function is of considerable interest.

The construction of an adaptive grid of nodes was considered by many authors, for example [11-16]. As is well known, the nodes of the Chebyshev polynomials are optimal ones for interpolation by algebraic interpolating polynomials on the interval $[-1,1]$. The zeros of the Chebyshev polynomials are often used in solving various problems related to the interpolation problem (see [17-23]).

N.Brisebarre, S.-I.Filip, G.Hanrot, (see [24]) introduce a fast and efficient method, based on the computation of good nodes for polynomial interpolation and Euclidean lattice basis reduction.

In [25] the use of Chebyshev nodes with the Newton interpolation is advocated as the most efficient numerical interpolation method in terms of approximation accuracy and computational effort. However, the authors show numerically that the approximation quality obtained by the Newton interpolation with Fast Leja (FL) points is competitive in the use of Chebyshev nodes, even for extremely high degree of interpolation. This is an experimental account of the analytic result that the limited distribution of FL points and Chebyshev nodes is the same when letting the number of points go to infinity. The Fast Leja construction is easy to perform and allows us to add interpolation nodes on the fly in contrast to the use of Chebyshev nodes. Their study suggests that the Newton interpolation with FL points is currently the most efficient numerical technique for polynomial interpolation.

The use of classical interpolation polynomials often does not give a satisfactory result [5, 8-10]. The use of local approximations in solving such problems is much more preferable. Local polynomial and non-polynomial approximations, which are used here were considered earlier by the authors in papers [26-34].

This paper proposes a formula for constructing the interpolation nodes of a rapidly increasing function. In the second section we consider a construction of an adaptive grid of nodes. The formula for constructing an adaptive grid of nodes takes into account the local behavior of the function of the three previous grid intervals, and is based on interpolation by local quadratic polynomial splines. The approximation of the function on this grid is compared with the approximation, which is constructed on a grid built by the using the roots of the Chebyshev polynomials and with the approximation, which is constructed on a uniform grid. Here we also consider the approximation by local rational splines. In the third section we consider approximation of functions of two variables on a radial-circular grid.

2 Approximation on a line

Let m, r, r_1 be natural numbers, such that $q + q_1 = m + 1$, a, b real numbers such as $q \geq 1, q_1 \geq 1, a > b$. Let x_j be a sequence of distinct points, $a = x_0 < \cdots < x_{j-1} < x_j < x_{j+1} < \ldots < x_n = b$. Suppose $f(x)$ is a function such that $f \in C^{m+1}[a, b]$, and it is given in nodes x_j.

2.1 Polynomial approximation

We construct approximation $\tilde{f}(x)$ of the function $f(x)$ with local splines separately on every $[x_j, x_{j+1}]$ in the following form:

$$\tilde{f}(x) = \sum_{s=j-q+1}^{j+q_1} f(x_s)\ \omega_s(x), \qquad x \in [x_j, x_{j+1}],$$

where $\omega_s(x)$we obtain as a solution of the system of equations:

$$\tilde{f}(x) = f(x), \qquad f(x) = 1, x, \ldots, x^m.$$

It can be shown, that $\omega_j(x)$, $supp\, \omega_j = [x_{j-q_1}, x_{j+q}]$, can be written in the form:

$$\omega_j(x) = \prod_{\substack{-q+1 \leq j'-k \leq q_1 \\ j' \neq j}} \frac{x - x_{j'}}{x_j - x_{j'}}, x \in [x_k, x_{k+1}], \qquad (1)$$

$$k = j - q_1, \ldots, j + q - 1,$$

$$\omega_j(x) = 0, \qquad x \notin [x_{j-q_1}, x_{j+q}].$$

Approximation $\tilde{f}(x)$ where basic splines have form (1) are called Lagrangian type approximations or splines of the Lagrangian type, and basic splines (1) are called basic splines of the Lagrangian type. Let $h = \frac{b-a}{n}$. It was proved by Prof. Yu. Demyanovich that the following theorem is valid:

Theorem. Let $f \in C^{m+1}[a.b]$, $x \in [x_j, x_{j+1}]$. $h = const$. There is such a constant $K > 0$, that the following inequality for approximations with splines of the Lagrangian type is true:

$$\left|f(x) - \tilde{f}(x)\right| \leq K h^{m+1} \parallel f^{(m+1)} \parallel_{[x_{j-q+1},\ x_{j+q_1}]}.$$

Next, we compare the results of applying local polynomial splines of Lagrangian type and non-polynomial rational splines, investigated in earlier papers by Prof. I.G. Burova for approximating functions. In the following two examples, the constants K are given for $m = 2$ and various ways of specifying the supports of basic splines.

Example 1a. Suppose $m = 2, q_1 = 2, q = 1$. Suppose $supp\ \omega_j = [x_{j-2}, x_{j+1}]$. Knowing $f(x_j)$, $f(x_{j+1})$, $f(x_{j+2})$, we construct an approximation of the function $f(x), x \in [x_j, x_{j+1}]$, in the form:

$$\tilde{f}(x) = f(x_j)\, \omega_j(x) + f(x_{j+1})\, \omega_{j+1}(x) + f(x_{j+2})\, \omega_{j+2}(x). \quad (2)$$

When the grid of nodes is non-uniform, the basic spline $\omega_j(x)$ is the following:

$$\omega_j(x) = \begin{cases} \dfrac{(x - x_{j+1})(x - x_{j+2})}{(x_j - x_{j+1})(x_j - x_{j+2})}, & x \in [x_j, x_{j+1}], \\ \dfrac{(x - x_{j-1})(x - x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})}, & x \in [x_{j-1}, x_j], \\ \dfrac{(x - x_{j-2})(x - x_{j-1})}{(x_j - x_{j-2})(x_j - x_{j-1})}, & x \in [x_{j-2}, x_{j-1}], \end{cases}$$

and $\omega_j(x) = 0, x \notin [x_{j-2}, x_{j+1}]$.

The error of the approximation of function $f(x)$, with spline $\tilde{f}(x), x \in [x_j, x_{j+1}]$, obtained with (2), will be the following:

$$|\tilde{f}(x) - f(x)| \leq \frac{0.385}{3!} h^3 \, \|f'''\|_{[x_j,\, x_{j+2}]}.$$

Example 1b. Suppose $m = 2, q_1 = 1, q = 2$. Suppose $supp\ \omega_j = [x_{j-1}, x_{j+2}]$. Knowing $f(x_{j-1})$, $f(x_j)$, $f(x_{j+1})$, we construct an approximation of the function $f(x)$, $x \in [x_j, x_{j+1}]$, in the form:

$$\tilde{f}(x) = f(x_{j-1})\, \omega_{j-1}(x) + f(x_j)\, \omega_j(x) + f(x_{j+1})\, \omega_{j+1}(x). \quad (3)$$

When the grid of nodes is non-uniform the basic spline $\omega_j(x)$ is the following:

$$\omega_j(x) = \begin{cases} \dfrac{(x - x_{j+1})(x - x_{j+2})}{(x_j - x_{j+1})(x_j - x_{j+2})}, & x \in [x_{j+1}, x_{j+2}], \\ \dfrac{(x - x_{j-1})(x - x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})}, & x \in [x_j, x_{j+1}], \\ \dfrac{(x - x_{j-2})(x - x_{j-1})}{(x_j - x_{j-2})(x_j - x_{j-1})}, & x \in [x_{j-1}, x_j], \end{cases}$$

and $\omega_j(x) = 0,\ x \notin [x_{j-1}, x_{j+2}]$.

The error of the approximation of function $f(x)$, with spline $\tilde{f}(x)$, $x \in [x_j, x_{j+1}]$, obtained with (3), will be the following:

$$|\tilde{f}(x) - f(x)| \le \frac{0.385}{3!} h^3 \|f'''\|_{[x_{j-1}, x_{j+1}]}.$$

Example 2. Suppose $m = 1, q_1 = 1, q = 1$. Suppose $supp\ \omega_j = [x_j, x_{j+1}]$. Knowing $f(x_{j-1})$, $f(x_j)$, $f(x_{j+1})$, we construct an approximation of the function $f(x)$, $x \in [x_j, x_{j+1}]$, in the form:

$$\tilde{f}(x) = f(x_j)\,\omega_j(x) + f(x_{j+1})\,\omega_{j+1}(x). \qquad (4)$$

When the grid of nodes is non-uniform the basic spline $\omega_j(x)$ is the following:

$$\omega_j(x) = \begin{cases} \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, x \in [x_j, x_{j+1}] \\ \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, x \in [x_{j-1}, x_j] \end{cases}$$

and $\omega_j(x) = 0$, $x \notin [x_{j-1}, x_{j+1}]$.

The error of the approximation of function $f(x)$, with spline $\tilde{f}(x), x \in [x_j, x_{j+1}]$, obtained with (4), will be the following:

$$|\tilde{f}(x) - f(x)| \le \frac{1}{8} h^2 \|f''\|_{[x_j, x_{j+1}]}.$$

2.2 Nonpolynomial approximation

Suppose $\varphi_0(x), \dots, \varphi_m(x)$ is the Chebyshev system. We construct approximation $\tilde{f}(x)$ of the function $f(x)$ with local splines separately on every $[x_j, x_{j+1}]$ in the following form:

$$\tilde{f}(x) = \sum_{s=j-q+1}^{j+q_1} f(x_s)\ \omega_s(x), \qquad x \in [x_j, x_{j+1}],$$

where $\omega_s(x)$ we obtain as a solution of the system of equations:

$$\tilde{f}(x) = f(x), \qquad f(x) = \varphi_0(x), \dots, \varphi_m(x).$$

Example 3. We construct approximation $\tilde{f}(x)$ of the function $f(x)$ with local splines separately on every $[x_j, x_{j+1}]$ in the following form:

$\tilde{f}(x) = \sum_{s=0,1,2} f(x_{j+s})\omega_{j+s}(x)$, $x \in [x_j, x_{j+1}]$(5),

where $\omega_s(x)$ we obtain as a solution of the system of equations:

$$\tilde{f}(x) = f(x), \qquad f(x) = 1,\ x,\ \frac{1}{x}.$$

We obtain:

$$\omega_j(x) = \frac{x_j\ (x - x_{j+2})(x - x_{j+1})}{x\ (x_j - x_{j+2})(x_j - x_{j+1})},$$

$$\omega_{j+1}(x) = -\frac{x_{j+1}(x - x_{j+2})(x - x_j)}{(-x_{j+2} + x_{j+1})(x_j - x_{j+1})x},$$

$$\omega_{j+3}(x) = \frac{x_{j+2}(x - x_{j+1})(x - x_j)}{(-x_{j+2} + x_{j+1})(x_j - x_{j+2})x}.$$

For $x = x_j + th_j$, $x \in [x_j, x_{j+1}]$ we get:

$$\omega_j(x_j + th_j) = \frac{x_j(x_j + th_j - x_{j+2})(x_j + th_j - x_{j+1})}{(x_j - x_{j+2})(x_j - x_{j+1})(x_j + th_j)},$$

$$\omega_{j+1}(x_j + th_j) = \frac{-x_{j+1}(x_j + th_j - x_{j+2})th_j}{(x_{j+1} - x_{j+2})(x_j - x_{j+1})(x_j + th_j)},$$

$$\omega_{j+2}(x_j + th_j) = \frac{x_{j+2}\ (x_j + th_j - x_{j+1})th_j}{(x_{j+1} - x_{j+2})(x_j - x_{j+2})(x_j + th_j)}.$$

It is not difficult to prove the following Lemma (in [31] the proof is given for a more general case).

Lemma. Let $f \in C^3[a,b]$, $x \in [x_j, x_{j+1}]$, $h = const$. There is a constant $K > 0$, such that the following inequality for approximations of type (5) is true:

$$|f(x) - \tilde{f}(x)| \le Kh^3 \max_{[x_j, x_{j+2}]} |xf^{(3)}(x) + 3f^{(2)}(x)|$$

2.3 Adaptive set of nodes construction

Now our aim is to construct an adaptive set of nodes. Here we continue to solve the problem using the results from paper [29]. Using (3) we obtain for $x \in [x_j, x_{j+1}]$:

$$\omega_j(x) = \frac{(x - x_{j-1})(x - x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})},$$

$$\omega_{j+1}(x) = \frac{(x - x_j)(x - x_{j-1})}{(x_{j+1} - x_j)(x_{j+1} - x_{j-1})},$$

$$\omega_{j-1}(x) = \frac{(x - x_j)(x - x_{j+1})}{(x_{j-1} - x_j)(x_{j-1} - x_{j+1})}.$$

Thus, for the first derivative we obtain:

$$\omega'_j(x) = \frac{(x - x_{j+1}) + (x - x_{j-1})}{(x_j - x_{j-1})(x_j - x_{j+1})},$$

$$\omega'_{j+1}(x) = \frac{(x - x_{j-1}) + (x - x_j)}{(x_{j+1} - x_j)(x_{j+1} - x_{j-1})},$$

$$\omega'_{j-1}(x) = \frac{(x - x_{j+1}) + (x - x_j)}{(x_{j-1} - x_j)(x_{j-1} - x_{j+1})}.$$

Using the notation

$$P_j(x) = f(x_{j-1})\omega'_{j-1}(x) + f(x_j)\omega'_j(x) + f(x_{j+1})\omega'_{j+1}(x)$$

from the following equation, where S is a real number, we obtain the next node x_{j+1}:

$$\left(\sqrt{1+P_{j-1}^{\,2}(x_j)} + \sqrt{1+P_j^{\,2}(x_{j+1})}\right)\frac{(x_{j+1}-x_j)}{2} = S. \qquad (6)$$

Formula (6) gives the rule for constructing an adaptive grid of nodes$\{x_j\}$. It is easy to see that (6) is obtained as a consequence of the formula:

$$\int_{x_j}^{x_{j+1}} \sqrt{1+(f'(x))^2}\,dx = S, \qquad (7)$$

used in [29] for the construction of a non-uniform grid. Formula (6) does not contain derivatives and integrals, therefore it is easier to use than (7). We denote:

$$F(Z, x_j, x_{j-1}, x_{j-2}) = \frac{Z-x_j}{2}\left(\sqrt{1+B_j^2} + \sqrt{1+C_j^2}\right) - S,$$

where:

$$B_j = f_{j-2}\,\frac{a_j}{b_j\ c_j} - f_{j-1}\,\frac{b_j}{a_j\ c_j} + f_j\left(\frac{1}{a_j}+\frac{1}{b_j}\right),$$

$$C_j = f_{j-1}\,\frac{Z-x_j}{(Z-x_{j-1})a_j} - f_j\,\frac{Z-x_{j-1}}{(Z-x_j)a_j} + f(Z)\left(\frac{1}{Z-x_{j-1}}+\frac{1}{Z-x_j}\right),$$

$a_j = x_j - x_{j-1}$, $b_j = x_j - x_{j-2}$, $c_j = x_{j-1} - x_{j-2}$.

Equation (6) equals to the following:

$$F(Z, x_j, x_{j-1}, x_{j-2}) = 0. \qquad (8)$$

Solving this equation with respect to Z, we put $x_{j+1} = Z$. To solve the nonlinear equation (8) we can apply Newton`s method:

$$x_{j+1} = x_j - \frac{F(x_j, x_{j-1}, x_{j-2})}{F'(x_j, x_{j-1}, x_{j-2})}.$$

For the local convergence of Newton's method we can use the following theorem (see [35], p.234).

Theorem. Let α be a simple root of the equation $F(z) = 0$ and let $I_\varepsilon = \{z \in \mathbb{R} : |z-\alpha| \le \varepsilon\}$. Assume that $F \in C^2[I_\varepsilon]$.

Define $M(\varepsilon) = \max_{s,t\in I_\varepsilon}\left|\frac{F''(s)}{2F'(t)}\right|$. If ε is so small that $2\varepsilon M(\varepsilon) < 1$, then for every $z_0 \in I_\varepsilon$, Newton's method is well defined and converges quadratically to the only root $\alpha \in I_\varepsilon$.

It is more convenient for us to apply the Newton method of convergence in the form proposed by Kantorovich (see [36]).

Let z_1 be such that

$$z_1 = z_0 - \frac{F(z_0)}{F'(z_0)}.$$

Kantorovich's Theorem: Let z_0 be a simple root of the equation $F(z) = 0$ and $|z_1 - z_0| \le \eta_0$, $|[F'(z_0)]^{-1}| \le B_0$, $|F''(z)| \le K$ in some closed ball $U(z_0, r)$. If $h_0 = B_0\eta_0 K \le \frac{1}{2}$ then the Newton sequence starting from z_0 will converge to a solution z^* of $F(z) = 0$ which exists in $U(z_0, r)$, provided $r \ge r_0 = \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0$.

Let the function $f(x)$ monotonically increase. To calculate the next node x_{j+1} by the formula (6), we need x_j, x_{j-1}, x_{j-2}, such that $x_j > x_{j-1} > x_{j-2}$. As an initial approximation of z_0 to determine the root of the equation (8) by the Newton method, we can take the point $z_0 = x_j + (x_j - x_{j-1})/2$. With the help of the Kolmogorov Theorem, one can control the process of finding the root by the Newton method. During the calculations we can control the error of the approximation on every interval $[x_j, x_{j+1}]$.

In the following examples we compare the errors of approximation obtained using the Chebyshev nodes, the errors of approximation obtained using the equidistant set of nodes and the errors of approximation obtained using nodes from (6).

Example 4. Let us construct a grid of nodes for $f(x) = x^{11/2}, x \in [0,1]$. In our example $S = 0.1$. Using (6) we get the sequence of nodes represented in the second column. In fact, if we calculate the roots of the Chebyshev polynomial of degree 34 on the interval [-1, 1] and take only non-negative ones, then we obtain a set of the Chebyshev nodes represented in the third column of Table 1.

Table 1. Set of the Chebyshev nodes and nodes (6)

x_j	Nodes (6)	Chebyshev nodes
0	0	0
1	0.09223	0.09223
2	0.1922	0.1837
3	0.2922	0.2737
4	0.3921	0.3612
5	0.4910	0.4457
6	0.5851	0.5264
7	0.6675	0.6026
8	0.7339	0.6737
9	0.7867	0.7390
10	0.8294	0.7980
11	0.8652	0.8502
12	0.8958	0.8951

13	0.9227	0.9324
14	0.9466	0.9618
15	0.9682	0.9829
16	0.9879	0.9957

We obtain the error of approximation of function $f(x) = x^{11/2}$ with splines (2) using the nodes calculated with formula (6):

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.00105.$$

If the Chebyshev nodes are chosen, we get:

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.000676.$$

For comparison we calculate the error of approximation of function $(x) = x^{11/2}$, with splines (2) for a uniform grid on [0,1] with $h = 0.0625$. We obtain:

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.00154.$$

Fig. 1 shows the plot of approximation of function $f(x) = x^{11/2}$, Fig. 2, 3 show graphs of the error of approximation of this function for a non-uniform and uniform grids. Fig. 4 shows the graph of the error of approximation of this function using the Chebyshev set of nodes.

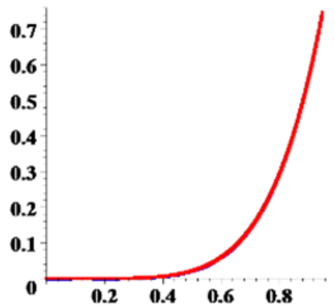

Fig.1 Plot of the function $f(x) = x^{11/2}$.

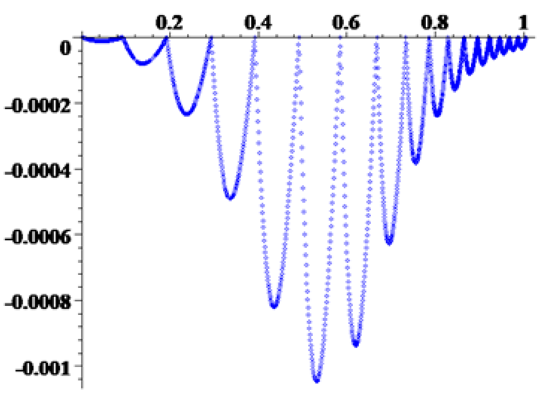

Fig.2 Plot of the error of approximation of function $f(x) = x^{11/2}$ with splines (2) using (6).

Example 4 shows that for the function $x^{11/2}$ the error of approximation obtained with Chebyshev nodes is less compared to the error of approximation obtained using an equidistant set of nodes or a set of nodes obtained from (6).

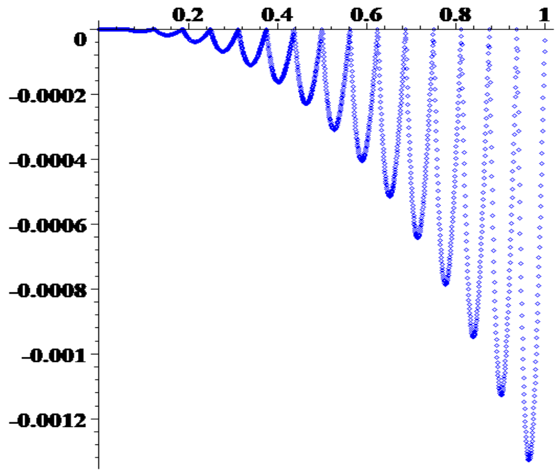

Fig.3 Plot of the error of approximation of function $f(x) = x^{11/2}$ with splines (2) using the equidistant set of nodes.

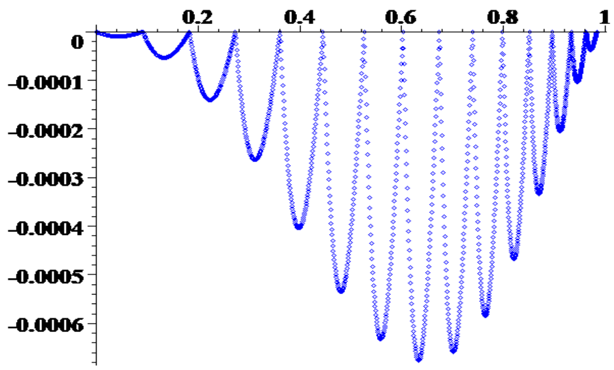

Fig.4 Plot of the error of approximation of function $f(x) = x^{11/2}$ with splines (2) using the Chebyshev set of nodes.

Example 5. Let us construct a grid of nodes for $f(x) = \frac{1}{x+1}, x \in [0,1]$. In our example $S = 0.1$. Using (6) we get the sequence of nodes represented in the second column. If we calculate the roots of the Chebyshev polynomial of degree 34 on the interval
[-1,1] and take only non-negative ones, then we obtain a set. The first 11 of them are represented in the right column of Table 2.

We obtain the error of approximation of function $f(x) = \frac{1}{x+1}$ using the nodes calculated by formula (6):

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.000206.$$

For comparison, we note that for a uniform grid with $h = 0.0625$ we obtain the error of approximation of the function :

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.000076.$$

If the Chebyshev nodes are chosen, we get:

$$\max_{[0,1]}\left|\tilde{f}(x) - f(x)\right| \leq 0.000224.$$

Table 2 Set of the Chebyshev nodes and nodes (6)

x_j	Nodes (6)	Chebyshev nodes
0	0	0
1	0.0922	0.0922
2	0.1709	0.1837
3	0.2535	0.2737
4	0.3394	0.3612
5	0.4280	0.4457
6	0.5188	0.5264
7	0.6113	0.6026
8	0.7053	0.6737
9	0.8003	0.7390
10	0.8963	0.7980
11	0.9930	0.8502

Fig. 5 shows the plot of the approximation of function $f(x) = \frac{1}{x+1}, x \in [0,1]$. Fig. 6-8 show graphs of the error of approximation of this function for non-uniform and uniform grids. Thus, for the function, the smallest in absolute value error of approximation is found using a uniform grid, and the greatest in absolute value error of approximation is found using the Chebyshev set of nodes.

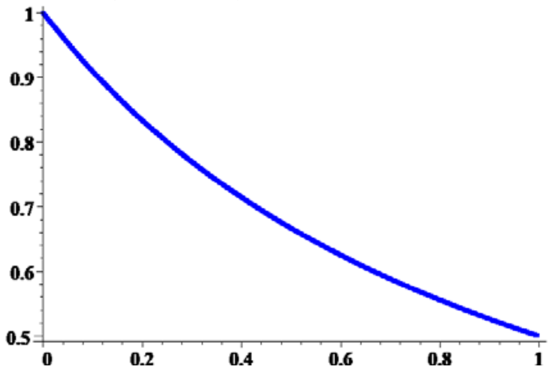

Fig.5 Plot of function $f(x) = \frac{1}{x+1}$.

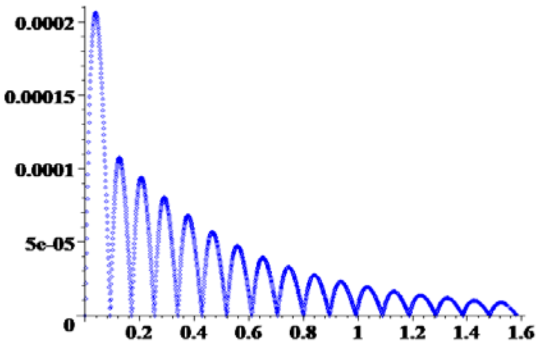

Fig.6. Plot of the error of approximation of function $f(x) = \frac{1}{x+1}$ with splines (2) using (6).

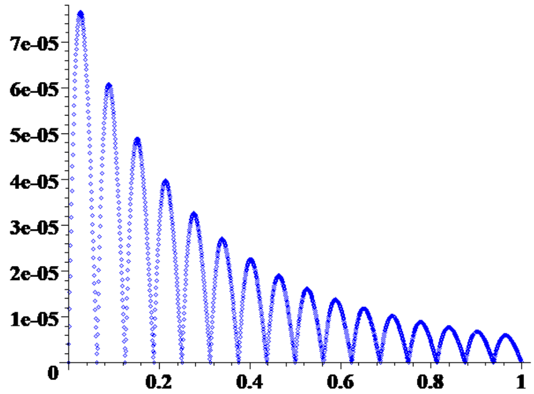

Fig.7 Plot of the error of approximation of function $f(x) = \frac{1}{x+1}$ with splines (2) using the equidistant set of nodes.

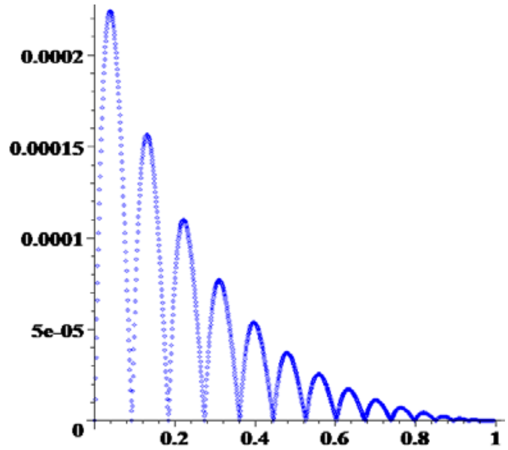

Fig.8 Plot of the error of approximation of function $f(x) = \frac{1}{x+1}$ with splines (2) using the Chebyshev set of nodes.

Example 6. Let us construct a grid of nodes for $f(x) = \frac{1}{1-x}, x \in [0,1]$. In our example $S = 0.1$, $n = 65, h_0 = 0.0826$. Using (6) we get the sequence of nodes. Fig. 9 shows the plot of approximation of the function $f(x)$, Fig. 10 shows the graphs of the error of approximation of this function with splines (5) using (6). Thus, for the function, the smallest in absolute value error of approximation we receive using the grid constructed with splines (5) using (6).

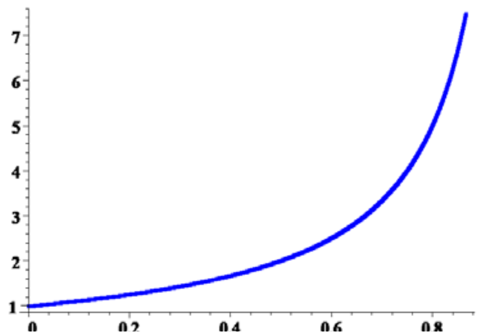

Fig.9 Plot of approximation of function $f(x) = \frac{1}{1-x}$ with splines (5) using (6).

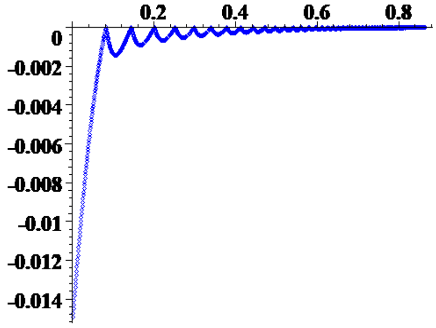

Fig.10 Plot of the error of approximation of function $f(x) = \frac{1}{1-x}$ with splines (5) using (6).

Fig. 11 shows the graphs of the approximation of this function with splines (2) using roots of the Chebyshev polynomials, $2n = 38$. Fig. 12 shows graph of the error of approximation of the function obtained using the Chebyshev set of nodes. The greatest error of approximation gives the grid constructed using the Chebyshev set of nodes.

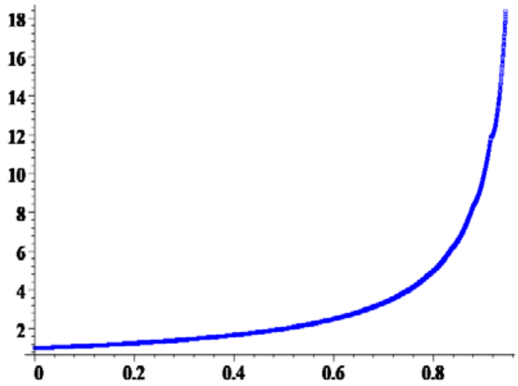

Fig.11 Plot of approximation of function $f(x) = \frac{1}{1-x}$ with splines (2) using the Chebyshev set of nodes.

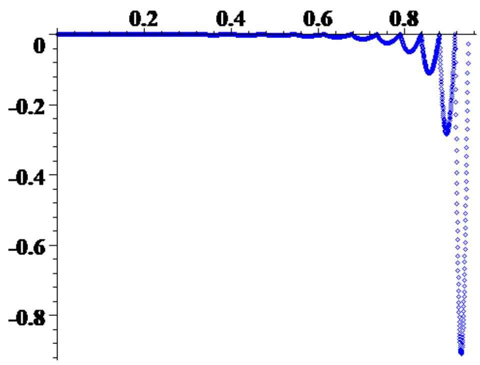

Fig.12 Plot of the error of approximation of function $f(x) = \frac{1}{1-x}$ with the Chebyshev set of nodes, $x \in [0, 0.91], n = 19$.

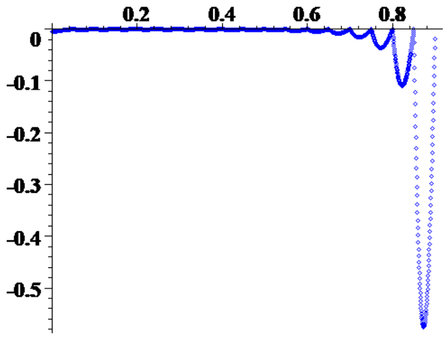

Fig.13 Plot of the error of approximation of function $f(x) = \frac{1}{1-x}$ with splines (2) using the equidistant set of nodes, $h = 0.05$, $x \in [0,\ 0.9]$.

3 Approximation of functions of two variables in a circle

In this section we consider the application of the construction of the non-uniform grid for interpolating a function of two variables in a circle D. Suppose the function $u(\rho, \varphi)$ depends on two variables: one is of the radial variable ρ and another is of the angular variable φ. Let $m,\ n_1,\ n_2, r,\ r_1$ be natural numbers, such that $r + r_1 = m + 1$. We consider the approximation of the function u in the circle. Let x_j, y_k be sequences of distinct points:

$$0 = r_0 < \dots < r_{j-1} < r_j < r_{j+1} < \ldots < r_{n_1} = 1,$$

$$0 = \varphi_0 < \dots < \varphi_{j-1} < \varphi_j < \varphi_{j+1} < \ldots < \varphi_{n_2} == 2\pi.$$

We construct the approximation to the function separately in each elementary segment $S_{j,k}$ bounded by two arcs and two line segments using the tensor product of one-dimensional spline approximations

$$\tilde{u}(\rho, \varphi) = \sum_{s,q} u(r_s, \varphi_q)\, \omega_s(\rho) \omega_q(\varphi),\ (\rho, \varphi) \in S_{j,k},$$

We will compare the approximations constructed on a uniform grid with the step $h = 0.1$ along the radial variable and on the approximations constructed non-uniformly condensed grid of nodes with ten interpolation points ($n_1 = n_2 = 10$).

If we use piecewise linear splines, then the approximation $\tilde{u}_1(\rho, \varphi)$ of function $u(\rho, \varphi)$ can be constructed separately in each elementary segment $S_{j,k}$ as follows:

$$\begin{aligned} \tilde{u}_1(\rho, \varphi) &= u(r_j, \varphi_k) w_j(\rho) v_k(\varphi) + \\ &+ u(r_{j+1}, \varphi_k) w_{j+1}(\rho) v_k(\varphi) + \\ &+ u(r_j, \varphi_{k+1}) w_j(\rho) v_{k+1}(\varphi) + \\ &+ u(r_{j+1}, \varphi_{k+1}) w_{j+1}(\rho) v_{k+1}(\varphi), \quad (\rho, \varphi) \in S_{j,k}, \end{aligned} \tag{9}$$

where

$$w_j(\rho) = \frac{r_{j+1} - \rho}{r_{j+1} - r_j}, w_{j+1}(\rho) = \frac{\rho - r_j}{r_{j+1} - r_j},$$

and

$$v_k(\varphi) = \frac{\varphi_{k+1} - \varphi}{\varphi_{k+1} - \varphi_k}, \ v_{k+1}(\varphi) = \frac{\varphi_k - \varphi}{\varphi_k - \varphi_{k+1}}.$$

If we use piecewise quadratic splines, then the approximation $\widetilde{u}_2(\rho,\varphi)$ of function $u(\rho,\varphi)$ can be constructed separately in each elementary segment $S_{j,k}$ as follows:

$$\begin{aligned}\widetilde{u}_2(\rho,\varphi) = u(r_j,\varphi_k)\omega_j(\rho)\omega_k(\varphi) + \\ + u(r_{j+1},\varphi_{k+1})\omega_{j+1}(\rho)\omega_{k+1}(\varphi) + \\ + u(r_{j+1},\varphi_k)\omega_{j+1}(\rho)\omega_k(\varphi) + \\ + u(r_j,\varphi_{k+1})\omega_j(\rho)\omega_{k+1}(\varphi) + \\ + u(r_{j+1},\varphi_{k-1})\omega_{j+1}(\rho)\omega_{k-1}(\varphi) + \\ + u(r_j,\varphi_{k-1})\omega_j(\rho)\omega_{k-1}(\varphi) + \\ + u(r_{j-1},\varphi_{k-1})\omega_{j-1}(\rho)\omega_{k-1}(\varphi) + \\ + u(r_{j-1},\varphi_k)\omega_{j-1}(\rho)\omega_k(\varphi) + \\ + u(r_{j-1},\varphi_{k+1})\omega_{j-1}(\rho)\omega_{k+1}(\varphi),\end{aligned} \tag{10}$$

where

$$\omega_j(\rho) = \frac{(\rho - r_{j-1})(\rho - r_{j+1})}{(r_j - r_{j-1})(r_j - r_{j+1})},$$

$$\omega_{j+1}(\rho) = \frac{(\rho - r_j)(\rho - r_{j-1})}{(r_{j+1} - r_j)(r_{j+1} - r_{j-1})},$$

$$\omega_{j-1}(\rho) = \frac{(\rho - r_j)(\rho - r_{j+1})}{(r_{j-1} - r_j)(r_{j-1} - r_{j+1})},$$

and

$$\omega_k(\varphi) = \frac{(\varphi - \varphi_{k-1})(\varphi - \varphi_{k+1})}{(\varphi_k - \varphi_{k-1})(\varphi_k - \varphi_{k+1})},$$

$$\omega_{k+1}(\varphi) = \frac{(\varphi - \varphi_k)(\varphi - \varphi_{k-1})}{(\varphi_{k+1} - \varphi_k)(\varphi_{k+1} - \varphi_{k-1})},$$

$$\omega_{k-1}(\varphi) = \frac{(\varphi - \varphi_k)(\varphi - \varphi_{k+1})}{(\varphi_{k-1} - \varphi_k)(\varphi_{k-1} - \varphi_{k+1})}$$

Example 7. Let us take function $u(\rho,\varphi) = \rho^{11/2}$ and build 11 grid points along the radius applying formula (5): $r_0 = 0$, $r_1 = 0.25$, $r_2 = 0.41$, $r_3 = 0.565$, $r_4 = 0.694$, $r_5 = 0.786$, $r_6 = 0.851$, $r_7 = 0.9$, $r_8 = 0.94$, $r_9 = 0.976$, $r_{10} = 1.0$, $r_{11} = 1.03$. For applying formula (8) we need one extra point r_{11}. Also we take 11 equidistant grid points along the border of the circle D. Now we can construct the set of nodes inside the circle in a known way.

In case of approximation the function $u(\rho,\varphi) = \frac{1}{1+25\rho^2}$ we choose the nodes of interpolation: $r_0 = 0$, $r_1 = 0.06$, $r_2 = 0.11$, $r_3 = 0.16$, $r_4 = 0.22$, $r_5 = 0.2$, $r_6 = 0.40$, $r_7 = 0.54$, $r_8 = 0.69$, $r_9 = 0.85$, $r_{10} = 1.01$, $r_{11} = 1.09$.

Note that formula (9) in error uses the second derivative, and formula (10) uses the third derivative of the approximated function (see Examples 1 and 2).

The maximum of theoretical errors for $u(\rho,\varphi) = \frac{1}{1+25\rho^2}$ are presented in Table 3 to control the errors of the approximation on the calculated intervals $[\boldsymbol{r_j}, \boldsymbol{r_{j+1}}]$.

Table 3. The maximum in absolute value of the theoretical errors of the approximations (9) and (10) for $\frac{1}{1+25\rho^2}$ over the calculated intervals

Interval $[r_j, r_{j+1}]$	approximation (9)	approximation (10)
$[r_0, r_1]$	0.013	0.0080
$[r_1, r_2]$	0.00065	0.0047
$[r_2, r_3]$	0.0033	0.0047
$[r_3, r_4]$	0.0056	0.0055
$[r_4, r_5]$	0.0075	0.0011
$[r_5, r_6]$	0.013	0.0024
$[r_6, r_7]$	0.011	0.0020
$[r_7, r_8]$	0.0051	0.00088
$[r_8, r_9]$	0.0012	0.00043
$[r_9, r_{10}]$	0.00016	0.00020

Table 4 shows the approximation errors in absolute values of the function $u(\rho,\varphi) = \rho^{11/2}$ and other functions $u(\rho,\varphi)$. Approximations were constructed using the values of the function on the nodes of a uniform grid and the values of the function on the nodes of the adaptive grid constructed with formula (6) and using formulae (9) and (10).

Table 4. The errors of approximation $\max_{\rho,\varphi\in D}|\tilde{u}_1 - u|$ and $\max_{\rho,\varphi\in D}|\tilde{u}_2 - u|$

$u(\rho,\varphi)$	$\max_{\rho,\varphi}|\tilde{u}_1 - u|$ using (9)		$\max_{\rho,\varphi}|\tilde{u}_2 - u|$ using (10)	
	$h = 0.1$	Formula (6)	$h = 0.1$	Formula (6)
$\rho^{11/2}\varphi^2$	1.52	0.42	0.27	0.16
$\frac{1}{1+25\rho^2}$	0.0136	0.00963	0.0225	0.00485
$\frac{\varphi}{1+25\rho^2}$	0.26	0.12	0.15	0.035
$\frac{\cos(\varphi)}{1+25\rho^2}$	0.056	0.034	0.0225	0.015

We construct surfaces using the MAPLE package with *pointplot3d* command and the option *coords* = *cylindrical*. Points forming graphs are calculated by the formulae (9) and (10). Fig.14, 15 show the approximations of the function $\cos(\varphi)/(1+25\rho^2)$ constructed with formulae (9) and (10) and equidistant set of nodes. Fig.16 shows the error of the approximation of the function $\cos(\varphi)/(1+25\rho^2)$ constructed with formulae (10) and equidistant set of nodes. Fig.17 shows the error of the approximation of the function $\varphi/(1+25\rho^2)$ constructed with formula (10) and set of nodes (6). Fig.17 shows the error of the approximation of the function
$\varphi/(1+25\rho^2)$ constructed with formulae (10) and set of nodes (6).

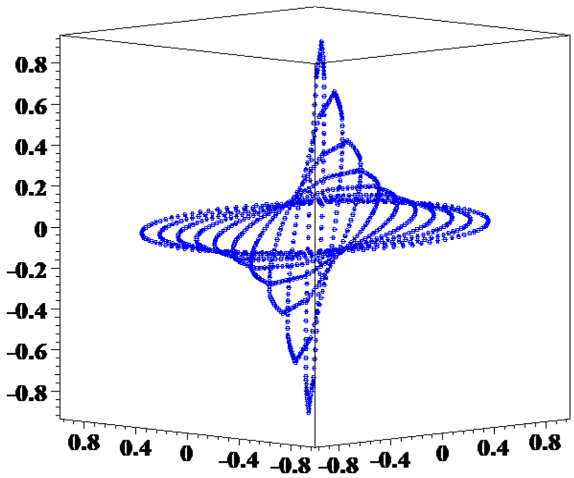

Fig.14 The approximation of $\cos(\varphi)/(1+25\rho^2)$ constructed with formula (9) and equidistant set of nodes ($h=0.1$).

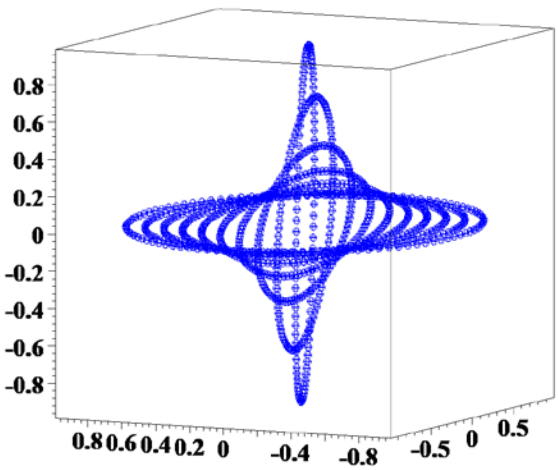

Fig.15 The approximation of the function $\cos(\varphi)/(1+25\rho^2)$ constructed with formula (10) and equidistant set of nodes ($h=0.1$).

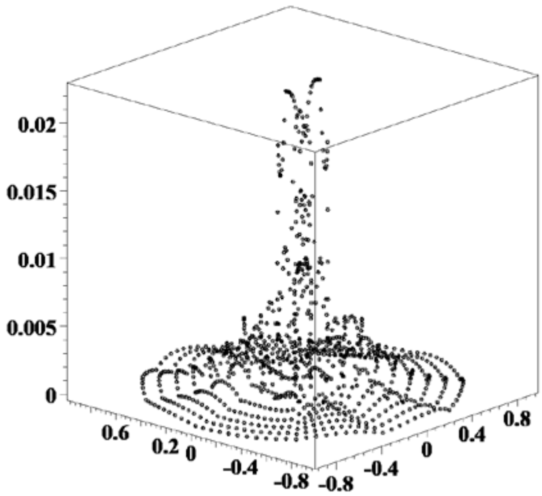

Fig.16 The error of the approximation of $\cos(\varphi)/(1+25\rho^2)$ constructed with formula (9) and equidistant set of nodes ($h=0.1$).

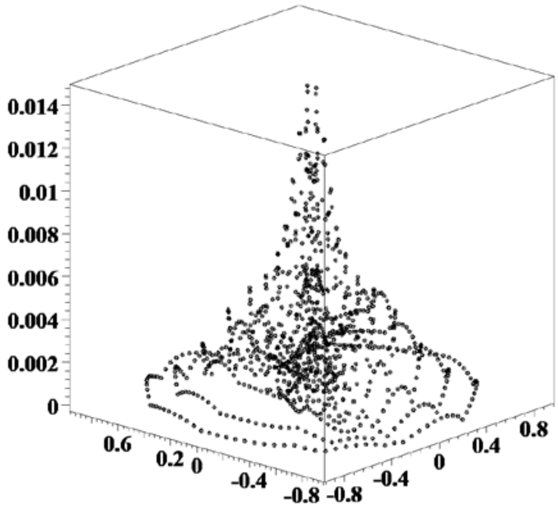

Fig.17 The error of the approximation of $\cos(\varphi)/(1+25\rho^2)$ constructed with formula (10) and set of nodes (6).

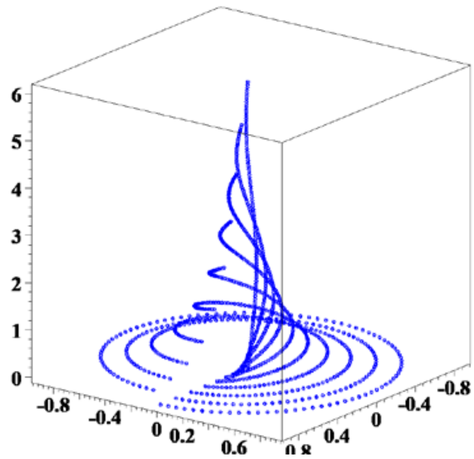

Fig.18 The approximation of $\varphi/(1+25\rho^2)$ constructed with formula (10) and set of nodes (6).

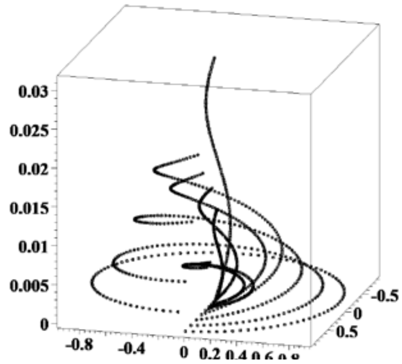

Fig.19 The error of the approximation of $\varphi/(1+25\rho^2)$ constructed with formula (10) and set of nodes (6).

4 Conclusion

The approximation based on a uniform grid of nodes does not always give a satisfactory result. The use of non-uniform grids opens great opportunities for increasing accuracy without increasing the number of nodes if there is preliminary information about the behavior of a solution to the problem. For the values of an argument, where it the behavior of the function which changes rapidly, the chosen step of the table should be smaller in contrast to the values of the argument where the function changes more slowly. Various approaches to the construction of an uneven mesh are known. The most well known and commonly used method is the use of nodes of the Chebyshev polynomial as interpolation nodes. The approximation on a grid of nodes constructed with the help of the roots of the Chebyshev polynomials is often (but not always) much more useful. The main limitation is often that this grid of nodes is difficult to continue without changing the previous nodes. The grid of nodes (6) in many cases gives a good result. It is easy to build and can be continued. The method of constructing an adaptive grid proposed in this article can be applied to two-dimensional and multidimensional cases. In the following papers we will look in more detail at the theoretical aspects of constructing this grid. The approximation by local rational splines in a number of cases gives a much better approximation. In the following papers some more complicated local splines and grids similar to those constructed in this paper will be examined in the region of a plane.

References:

[1] J.H.Mathews, R.W.Howell, *Complex Analysis for Mathematics and Engineering*, 5th ed., Sudbury, MA: Jones and Bartlett Publishers,, 2006.

[2] M. Eisenberg, J.M.David, Jr.Park, Visualizing complex functions with the Cardano3 application, in Applied Mathematica, *Electronic Proceedings of the Eighth International Mathematica Symposium (IMS'06)*, Avignon, France (Y. Papegay, ed.), Sophia Antipolis, France: INRIA, 2006.

[3] M. Eisenberg, J.M.David, Jr.Park, Visualizing Complex Functions with the Presentations Application, *The Mathematica Journal*, 2011, pp. 226-251.

[4] G.Opfer, M.L.Puri, Complex Chebyshev polynomials and generalizations with an application to the optimal choice of interpolating knots in complex planar splines, *Journal of Approximation Theory*, 37 (1), 1983, pp. 89-101.

[5] S. Chen, X. Hong, E.Khalaf, F.E. Alsaadi, C.J. Harris, Complex-valued B-spline neural network and its application to iterative frequency-domain decision feedback equalization for Hammerstein communication systems, *Proceedings of the International Joint Conference on Neural Networks*, 2016, pp. 4097-4104.

[6] S.Chen Chen, X.Hong, J.Gao, C.J.Harris, Complex-valued B-spline neural networks for modeling and inverting hammerstein systems, *IEEE Transactions on Neural Networks and Learning Systems*, Vol.25 (9), 2014, pp. 1673-1685.

[7] X.Hong, J.Gao, C.J.Harris, Modelling and inverting complex-valued wiener systems, *Proceedings of the International Joint Conference on Neural Networks*, 2012.

[8] C. de Boor, The error in polynomial tensor-product, and in ChungYao, interpolation, *Vanderbilt University Press*, 1997, pp. 35-50.

[9] C.de Boor, A multivariate divided difference, *Approximation Theory*, VIII, Vol. 1: Approximation and Interpolation, Charles K. Chui and Larry L. Schumaker (eds), World Scientific Publishing Co., Inc., Singapore, 1995, pp. 87–96.

[10] Möner Bernhard, Reif Ulrich, Error bounds for polynomial tensor product interpolation, *Computing*, Vol.86 (2-3), 2009, pp. 185-197.

[11] N.Overton, X.Gao, S.Guzik, Spline interpolation for a fourth-order adaptive mesh refinement algorithm on arbitrary mapped grids, *23rd AIAA Computational Fluid Dynamics Conference*, 2017.

[12] S.Iqbal, A.R.Ansari, A.Javed, A.M.Siddiqui, An adaptive finite element formulation for the solution of second order obstacle problems

using quadratic Lagrange polynomials, *Journal of Numerical Analysis, Industrial and Applied Mathematics*, Vol.9-10 (3-4) , 2016, pp. 37-45.

[13] I.Ukhov, P.Eles, Z.Peng, Probabilistic Analysis of Electronic Systems via Adaptive Hierarchical Interpolation, *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, Vol.36(11), 2017, pp. 1883-1896.

[14] C.Hofreither, B.Jüttler, G.Kiss, W.Zulehner, Multigrid methods for isogeometric analysis with THB-splines, *Computer Methods in Applied Mechanics and Engineering*, Vol.308, 2016, pp. 96-112.

[15] A.Pawar, Y.Zhang, X.Wei, C.L.Chan, C.Anitescu, An adaptive non-rigid image registration technique using hierarchical B-splines, *Computational Vision and Medical Image Processing V - Proceedings of 5th Eccomas Thematic Conference on Computational Vision and Medical Image Processing*, VipIMAGE 2015, pp. 3-8.

[16] S.Rostami, A.Shenfield, A multi-tier adaptive grid algorithm for the evolutionary multi-objective optimization of complex problems, *Soft Computing*, Vol.21(17), 2017, pp. 4963-4979.

[17] M.Revers, Extremal Polynomials and Entire Functions of Exponential Type, *Results in Mathematics*,Vol.73(3), 2018.

[18] M.Kuian, L.Reichel, S.V.Hiyanovskii, Fast factorization of rectangular Vandermonde matrices with Chebyshev nodes, *Numerical Algorithms*, 2018, pp.1-13.

[19] F.Fakhar-Izadi, M. Dehghan, Modal spectral element method in curvilinear domains, *Applied* Numerical Mathematics, Vol.128, 2018, pp.157-182.

[20] T.Hasegawa, H.Sugiura, Uniform approximation to Cauchy principal value integrals with logarithmic singularity, *Journal of Computational and Applied Mathematics*, Vol.327, 2018, pp.1-11.

[21] K.Xu, The Chebyshev points of the first kind, *Applied Numerical Mathematics*, Vol.102, 2016, pp.17-30.

[22] D.Chen, Generalization of polynomial interpolation at chebyshev nodes, *Springer Proceedings in Mathematics*, Vol.13, 2012, pp.17-35.

[23] M.I.Ganzburg, Lagrange interpolation and new asymptotic formulae for the riemann zeta function, *Springer Proceedings in Mathematics*,Vol.13, 2012, pp.79-93.

[24] N.Brisebarre, S.-I.Filip, G.Hanrot, A.Lattice, Basis Reduction Approach for the Design of Finite Wordlength FIR Filters, *IEEE Transactions on Signal Processing*, Vol. 66(10), 2018, pp.2673-2684.

[25] M.Breuss, F.Kemm, O.Vogel, A numerical study of Newton interpolation with extremely high degrees, *Kybernetika*, Vol.54(2), 2018, pp.279-288.

[26] I.G.Burova, Approximation by complex splines, *Vestnik Leningradskogo Universiteta, Seriya Matematika, Mekhanika Astronomiya*, Vol.2, 1986, pp. 3-9.

[27] I.G.Burova, O.V.Rodnikova, Integro-differential polynomial and trigonometrical splines and quadrature formulae, *WSEAS Transactions on Mathematics*, Vol.16, 2017, pp. 11-18.

[28] I.G.Burova, A.G.Doronina, I.D.Miroshnichenko, A Comparison of Approximations with left, right and middle Integro-Differential Polynomial Splines of the Fifth Order, *WSEAS Transactions on Mathematics*, Vol.6, 2017, pp. 339-349.

[29] I.G.Burova, S.V.Poluyanov, On approximations by polynomial and trigonometrical integro-differential splines, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.10, 2016, pp. 190-199.

[30] I.G.Burova, A.G.Doronina, On approximations by polynomial and nonpolynomial integro-differential splines, *Applied Mathematical Sciences*, Vol.10 (13-16), 2016, pp. 735-745.

[31] I.G.Burova, On left integro-differential splines and Cauchy problem, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.9, 2015, pp. 683-690.

[32] I.G.Burova, O.V.Rodnikova, Application of integro-differential splines to solving an interpolation problem, *Computational Mathematics and Mathematical Physics*, Vol.54 (12), 2014, pp. 1903-1914.

[33] I.G.Burova, S.V.Poluyanov, Construction of meansquare approximation with integro-differential splines of fifth order and first level, *Vestnik St. Petersburg University: Mathematics*, Vol.47(2), 2014, pp. 57-63.

[34] I.G.Burova, T.O.Evdokimova, On construction third order approximation using values of integrals, *WSEAS Transactions on Mathematics*, Vol.13, 2014, pp. 676-683.

[35] Walter Gautschi, *Numerical Analysis: an Introduction*, Boston, Basel, Berlin, 1997.

[36] L.R. Rall, *Computational solution of nonlinear Operator Equations*, Wiley, New York, 1969.