On General Smoothness of Minimal Splines of the Lagrange Type

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Abstract: In many cases the smoothness of splines is important (for qualitative approximation, for the calculation of a number of functionals, etc.). In the case of discontinuity of approximated functions it is difficult to use ordinary splines. It is desirable to have splines with similar properties of the approximated function. The purpose of this paper is to introduce the concept of general smoothness with the aid of linear functionals having a definite location of supports. Splines are often used for processing numerical information flows; a lot of scientific papers are devoted to these investigations. Sometimes spline treatment implies to the filtration of the mentioned flows or to their wavelet decomposition. A discrete flow often appears as a result of analog signal sampling, representing the values of a function, and in this case, the splines of the Lagrange type are used. In all cases, it is highly desirable that the generalized smoothness of the resulting spline coincides with the generalized smoothness of the original signal. Here we formulate the necessary and sufficient conditions for general smoothness of splines, and also a toolkit is being developed to build mentioned splines. The proposed scheme allows us to consider splines generated by functions from different spaces and to apply the obtained result to sources which can appear in physics, chemistry, biology, etc.

Key–Words: splines, general smoothness, chains of vectors, approximate relations

1 Introduction

The continuity of splines was considered from the moment of their appearance. Often the requirement of a certain continuity (spline or its derivatives) was included in the definition of a spline (see [1] – [6]). Basically these considerations concern polynomial splines, the last of which were defined as piecewise polynomial functions. For non-polynomial splines obtained from approximation relations (for the so-called minimal splines), necessary and sufficient continuity conditions have been obtained relatively recently (see [17]).

In many cases the continuity of splines is important (for qualitative approximation, for calculation of functional values, etc.), since the functions approximated by them are, as a rule, continuous. A study of the continuity of splines and their derivatives is devoted to a lot of work (see also [7] – [18]).

Splines are often used for processing numerical information flows; a lot of scientific papers are devoted to these investigations. Sometimes spline treatment implies to the filtration of the mentioned flows or to their wavelet decomposition. A discrete flow often appears as a result of analog signal sampling, representing the values of a function, and in this case, the splines of the Lagrange type are used.

In all cases, it is highly desirable that the generalized smoothness of the resulting spline coincides with the generalized smoothness of the original signal.

When a discontinuous function is approximated, the behavior of the function in a neighborhood of the discontinuity point is often known. Approximation of that function by ordinary splines is difficult. It is desirable to have splines with properties similar to properties of the approximated function.

Such properties can be formulated with the help of linear functionals, in particular, the usual continuity of the function can be considered as an equality of the values of two linear functionals applied to this function: one functional is the left limit of the function at this point, and the second is the right limit of it. The purpose of this paper is to introduce the concept of general smoothness with the aid of linear functionals having a definite location of supports. In what follows general smoothness are called pseudo-continuity.

Here we formulate the necessary and sufficient conditions for the pseudo-continuity of splines, and also a toolkit is being developed to build pseudo-continuous splines. The proof use properties of complete chains of vectors, local orthogonality mentioned chains, as well as the vector identities.

Thus, in this paper we propose a general scheme that allows us to consider splines generated by functions from spaces C^l, L_p, W_p^l, and handle nonsmooth data, and sources which can be generated in physics, chemistry, biology, etc.

2 Preliminaries

An ordered set $\mathbf{A} = \{\mathbf{a}_j\}_{j\in\mathbf{Z}}$ of vectors $\mathbf{a}_j \in \mathbf{R}^{m+1}$ is called *a chain of vectors*. There are different enumerations of the same chain; moreover, two enumerations can differ by only a constant term and direction of enumeration, for example, $\mathbf{A} = \{\mathbf{a}_{j'} \mid j' = -j + j_0,\ j \in \mathbf{Z}\}$ (where j_0 is an integer constant) is another enumeration of the same chain.

Chain $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$ is called *locally orthogonal to a chain* $\mathbf{B} = \{\mathbf{b}_j\}_{j\in\mathbf{Z}}$, if there exists an enumeration such that

$$\mathbf{b}_j^T \mathbf{a}_{j-p} = 0 \quad \forall j \in \mathbf{Z}, \tag{1}$$

$$p \in I_m,\ I_m = \{1, 2, \ldots, m\}.$$

The next assertion is obvious.

Lemma 1. *If chain* $\mathbf{A}$ *is locally orthogonal to chain* $\mathbf{B}$*, then chain* $\mathbf{B}$ *is locally orthogonal to chain* $\mathbf{A}$.

The local orthogonality is symmetric, and we can say that chains $\mathbf{A}$ and $\mathbf{B}$ are *locally orthogonal.* Chain $\mathbf{A}$ is *nondegenerate* if it does not contain zero elements and degenerate in the opposite case.

We denote by A_j a matrix with columns $\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_j$ i.e.,

$$A_j = \Big(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_j\Big).$$

Chain $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$ is called *complete* if $\det A_j \neq 0$ for all $j \in \mathbf{Z}$. It is clear that a complete chain is nondegenerate. The set of all complete chains is denoted by $\mathcal{A}$.

Lemma 2. *Let* $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$ *and* $\mathbf{B} = \{\mathbf{b}_j\}_{j\in\mathbf{Z}}$ *be locally orthogonal and nondegenerate chains. Then chain* $\mathbf{A}$ *is complete if and only if chain* $\mathbf{B}$ *is complete.*

Lemma 3. *If chains* $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$ *and* $\mathbf{B} = \{\mathbf{b}_j\}_{j\in\mathbf{Z}}$ *are complete and (1) holds, then*

$$\mathbf{b}_j^T \mathbf{a}_j \neq 0, \qquad \mathbf{b}_{j+m+1}^T \mathbf{a}_j \neq 0. \tag{2}$$

The proof of formula (2) follows by contradiction.

Lemma 4. *For any complete chain of vectors a nondegenerate locally orthogonal chain exists. The directions of vectors of this chain are uniquely determined.*

Corollary 1. *For any complete chain there exists a locally orthogonal complete chain that is uniquely determined up to a nonzero constant factor.*

The proof follows from Lemma 2 and Lemma 4.

Lemma 5. *Let* $\mathbf{A} = \{\mathbf{a}_k\}$ *be complete a chain. Suppose a chain* $\mathbf{B} = \{\mathbf{b}_k\}$ *is obtained by the formula*

$$\mathbf{b}_k^T \mathbf{x} \equiv \det(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, \ldots, \mathbf{a}_{k-1}, \mathbf{x})$$

$$\forall \mathbf{x} \in \mathbf{R^{m+1}},$$

and a chain $\mathbf{C} = \{\mathbf{c}_j\}$ *is obtained by the relation*

$$\mathbf{c}_j^T \mathbf{x} \equiv \det(\mathbf{b}_{j+1}, \mathbf{b}_{j+2}, \ldots, \mathbf{b}_{j+m}, \mathbf{x}) \tag{3}$$

$$\forall \mathbf{x} \in \mathbf{R^{m+1}}.$$

Then the relation $\mathbf{c}_j = \lambda_j \mathbf{a}_j \quad \forall j \in \mathbf{Z}$ *holds, where the constants* λ_j *are not zero.*

Proof. According to Lemma 2 chain $\mathbf{B}$ is complete. It is obvious that $\mathbf{b}_k \perp \mathbf{a}_{k-m}$, $\mathbf{b}_k \perp \mathbf{a}_{k-m+1}$, $\ldots$, $\mathbf{b}_k \perp \mathbf{a}_{k-1}$ $\forall k \in \mathbf{Z}$; the last relations are equivalent to the next ones $\mathbf{a}_j \perp \mathbf{b}_{j+1}$, $\mathbf{a}_j \perp \mathbf{b}_{j+2}$, $\ldots$, $\mathbf{a}_j \perp \mathbf{b}_{j+m}$ $\forall j \in \mathbf{Z}$. Taking into account that the chain $\mathbf{C} = \{\mathbf{c}_j\}$ is obtained by formula (3), we see that the relations $\mathbf{c}_j \perp \mathbf{b}_{j+1}, \quad \mathbf{c}_j \perp \mathbf{b}_{j+2}, \ldots, \mathbf{c}_j \perp \mathbf{b}_{j+m} \quad \forall j \in \mathbf{Z}$ are fulfilled. Comparing the last one with the relation obtained just before we get the needed result.

3 Some vector identities

Let $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{m-1} \in \mathbf{R^{m+1}}$ be vector columns. Consider multivector production

$$(\mathbf{u}_0 \times \mathbf{u}_1 \times \ldots \times \mathbf{u}_{m-1})^T \mathbf{x} = \det(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{m-1}, \mathbf{x})$$

$$\forall \mathbf{x} \in \mathbf{R^{m+1}}.$$

For shortness we denote it by symbol

$$\prod_{i=0}^{m-1} \times \mathbf{u}_i = \mathbf{u}_0 \times \mathbf{u}_1 \times \ldots \times \mathbf{u}_{m-1},$$

so that

$$(\prod_{i=0}^{m-1} \times \mathbf{u}_i)^T \mathbf{x} \equiv \det(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{m-1}, \mathbf{x}) \tag{4}$$

$$\forall \mathbf{x} \in \mathbf{R^{m+1}}.$$

Lemma 6. *The next assertions are right:*

1) a transposition of neighbor factors changes the sign of result,

2) two collinear factors result in zero, *3) the multivector production has a distributive property; in particular, if* $i \in \{1, 2, \dots, m-2\}$ *then*

$$\mathbf{u}_0 \times \mathbf{u}_1 \times \dots \times \mathbf{u}_{i-1} \times (\mathbf{u}'_i + \mathbf{u}''_i) \times \mathbf{u}_{i+1} \times \dots \times \mathbf{u}_{m-1} =$$

$$= \mathbf{u}_0 \times \mathbf{u}_1 \times \dots \times \mathbf{u}_{i-1} \times \mathbf{u}'_i \times \mathbf{u}_{i+1} \times \dots \times \mathbf{u}_{m-1} +$$

$$+ \mathbf{u}_0 \times \mathbf{u}_1 \times \dots \times \mathbf{u}_{i-1} \times \mathbf{u}''_i \times \mathbf{u}_{i+1} \times \dots \times \mathbf{u}_{m-1},$$

4) if a vector $\mathbf{b} \in \mathbf{R^{m+1}}$ *is a factor in multivector production* $\prod_{i=0}^{m-1} \times \mathbf{u}_i$ *then the relation* $\left(\prod_{i=0}^{m-1} \times \mathbf{u}_i\right)^T \mathbf{b} = 0$ *is right.*

The proof is followed by formula (4).

Theorem 1. *Let* $\mathbf{v}_0, \dots, \mathbf{v}_{2m-2}, \mathbf{f} \in \mathbf{R^{m+1}}$ *be column vectors and let*

$$\mathcal{C} = \Big(\prod_{i=0}^{m-1} \times \mathbf{v}_i, \dots, \prod_{i=m-1}^{2m-2} \times \mathbf{v}_i, \mathbf{f} \Big)$$

be matrix with column $\mathbf{f}$*. Then the relation*

$$\det \mathcal{C} = (-1)^m \prod_{i=0}^{m-2} \det(\mathbf{v}_i, \dots, \mathbf{v}_{i+m}) \cdot \mathbf{f}^T \mathbf{v}_{m-1} \tag{5}$$

is correct.

Proof. By definition put

$$\mathbf{V}_j = \Big((\prod_{i=0}^{m-1} \times \mathbf{v}_i)^T \mathbf{v}_j, (\prod_{i=1}^{m} \times \mathbf{v}_i)^T \mathbf{v}_j, \dots$$

$$\dots, (\prod_{i=m-1}^{2m-2} \times \mathbf{v}_i)^T \mathbf{v}_j, \mathbf{f}^T \mathbf{v}_j, \Big)^T, j = 0, 1, \dots, m.$$

Consider the production of matrix $\mathcal{C}^T$ and $(\mathbf{v}_0, \dots, \mathbf{v}_m)$. We have

$$\mathcal{C}^T (\mathbf{v}_0, \dots, \mathbf{v}_m) = (\mathbf{V}_0, \mathbf{V}_1, \dots, \mathbf{V}_m).$$

Now we discuss the vectors

$$\mathbf{W}_k = \Big(\underbrace{0, 0, \dots, 0}_{k+1}, (\prod_{i=j+1}^{m+j} \times \mathbf{v}_i)^T \mathbf{v}_k,$$

$$(\prod_{i=j+2}^{m+j+1} \times \mathbf{v}_i)^T \mathbf{v}_k, \dots, (\prod_{i=m-1}^{2m-2} \times \mathbf{v}_i)^T \mathbf{v}_k, \mathbf{f}^T \mathbf{v}_k \Big)^T,$$

$$k = 0, 1, \dots, m-1,$$

and vector

$$\mathbf{W}_m = \Big((\prod_{i=0}^{m-1} \times \mathbf{v}_i)^T \mathbf{v}_m, 0, 0, \dots, 0, \mathbf{f}^T \mathbf{v}_m \Big)^T.$$

Taking into account formula (4) and point 4) of Lemma 6, we obtain

$$\mathbf{V}_j = \mathbf{W}_j \qquad \forall j \in \{0, 1, \dots, m\};$$

therefore we have

$$\det \mathcal{C}^T (\mathbf{v}_0, \dots, \mathbf{v}_m) = \det(\mathbf{W}_0, \mathbf{W}_1, \dots, \mathbf{W}_m).$$

Calculating the right determinant by the "deleting" of the first row and of the last column, we obtain the determinant of the triangular matrix:

$$\det \mathcal{C}^T \cdot \det(\mathbf{v}_0, \dots, \mathbf{v}_m) =$$

$$= (-1)^m (\prod_{i=0}^{m-1} \times \mathbf{v}_i)^T \mathbf{v}_m \cdot (\prod_{i=1}^{m} \times \mathbf{v}_i)^T \mathbf{v}_0 \cdot \dots$$

$$\dots \cdot (\prod_{i=m-1}^{2m-2} \times \mathbf{v}_i)^T \mathbf{v}_{m-2} \cdot \mathbf{f}^T \mathbf{v}_{m-1}.$$

Using Lemma 6 and relation (4), we deduce

$$\det \mathcal{C}^T \cdot \det(\mathbf{v}_0, \dots, \mathbf{v}_m) =$$

$$= (-1)^m \det(\mathbf{v}_0, \dots, \mathbf{v}_m) \cdot (-1)^m \det(\mathbf{v}_0, \dots, \mathbf{v}_m) \cdot$$

$$\cdot (-1)^m \det(\mathbf{v}_1, \dots, \mathbf{v}_{m+1}) \cdot \dots$$

$$\dots \cdot (-1)^m \det(\mathbf{v}_{m-2}, \dots, \mathbf{v}_{2m-2}) \cdot \mathbf{f}^T \mathbf{v}_{m-1}.$$

Here we have m factors $(-1)^m$. Because the relation $(-1)^{m^2} = (-1)^m$ holds, we get the equality (5).

Corollary 2. *Let* $\mathbf{v}_0, \dots, \mathbf{v}_{2m-1} \in \mathbf{R^{m+1}}$ *be column vectors. Then the relation*

$$\det \Big(\prod_{i=0}^{m-1} \times \mathbf{v}_i, \dots, \prod_{i=m}^{2m-1} \times \mathbf{v}_i \Big) =$$

$$= \prod_{i=0}^{m-1} \det(\mathbf{v}_i, \dots, \mathbf{v}_{i+m}) \tag{6}$$

is fulfilled.

Proof. If we put $\mathbf{f} = \prod_{i=m}^{2m-1} \times \mathbf{v}_i$ in relation (5) then we get formula (6).

4 Classification of complete chains

Consider a complete chain $\mathbf{A} = \{\mathbf{a}_j\}_{j \in \mathbf{Z}}$ of column vectors $\mathbf{a}_j \in \mathbf{R}^{m+1}$.

It is clear that the multiplication of vectors for a complete chain by nonzero numbers gives a new complete chain. The aforementioned multiplication generates the relation of equivalency between chains. Two nondegenerate chains $\mathbf{A} = \{\mathbf{a}_j\}_{j \in \mathbf{Z}}$ and $\mathbf{A}' = \{\mathbf{a}'_j\}_{j \in \mathbf{Z}}$ are called equivalent chains, if there are nonzero numbers λ_j such $\mathbf{a}'_j = \lambda_j \mathbf{a}_j$. The introduced equivalency is denoted by $\sim$; it divides the set $\mathcal{A}$ on classes of equivalent chains.

Consider a complete chain $\mathbf{A} \in \mathcal{A}$. Let $\overline{\mathcal{A}} \subset \mathcal{A}$ be the class of equivalent complete chains such that $\overline{\mathcal{A}} =$

$\{\mathbf{A}' \mid \mathbf{A}' \sim \mathbf{A}\}$. The set of all classes is denoted by $\mathcal{K}$.

Let us return to the discussion of the local orthogonality of complete chains. Let $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$ be a complete chain. By definition put $\mathbf{B} = \{\mathbf{b}_j\}_{j\in\mathbf{Z}}$, where

$$\mathbf{b}_j = \prod_{s=1}^{m} \times \mathbf{a}_{j-s} \qquad \forall j \in \mathbf{Z}. \tag{7}$$

We want to prove that chains $\mathbf{A}$ and $\mathbf{B}$ are locally orthogonal. For any $p \in \{1, 2, \dots, m\}$ we have $j - p \in \{j-1, j-2, \dots, j-m\}$, so that the vector $\mathbf{a}_{j-p}$ locates among the factors of multivector production in formula (7). Let us verify the relation (1). Using Lemma 6 (see its point 4), we get relation (1). This completes the proof.

Suppose that chain $\mathbf{B} \in \mathcal{A}$, $\mathbf{B} = \{\mathbf{b}_j\}_{j\in\mathbf{Z}}$, is local orthogonal to chain $\mathbf{A} \in \mathcal{A}$, $\mathbf{A} = \{\mathbf{a}_i\}_{i\in\mathbf{Z}}$, i.e. relation (1) is fulfilled. Consider classes $\overline{\mathcal{A}} = \{\mathbf{A}' \mid \mathbf{A}' \sim \mathbf{A}\}$ and $\overline{\mathcal{B}} = \{\mathbf{B}' \mid \mathbf{B}' \sim \mathbf{A}\}$.

Discuss chains $\mathbf{A}' \in \overline{\mathcal{A}}$, $\mathbf{A}' = \{\mathbf{a}'_j\}_{i\in\mathbf{Z}}$, and $\mathbf{B}' \in \overline{\mathcal{B}}$, $\mathbf{B}' = \{\mathbf{b}'_j\}_{j\in\mathbf{Z}}$. According to the definition of equivalence we have $\mathbf{a}'_i = \lambda_i \mathbf{a}_i$, $\mathbf{b}'_j = \mu_j \mathbf{b}_j$, where λ_i, μ_j are nonzero numbers. Multiplying relation (1) by $\mu_j \lambda_{j-p}$, we obtain a relation

$$(\mathbf{b}'_j)^T \mathbf{a}'_{j-p} = 0 \quad \forall j \in \mathbf{Z}, \quad p \in I_m.$$

The last one is equivalence to the local orthogonality of chains $\mathbf{A}'$ and $\mathbf{B}'$.

Thus, a chain of class $\overline{\mathcal{A}}$ is local orthogonal to each chain of class $\overline{\mathcal{B}}$, and otherwise, a chain of the class $\overline{\mathcal{B}}$ is local orthogonal to each chain of class $\overline{\mathcal{A}}$. In the discussed case the classes $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are called *local orthogonal classes*. Applying Lemma 5, we see that each class has the unique local orthogonal class.

Using formula (5), it is possible to make Lemma 5 more precise: the next assertion is right.

Lemma 7. *Let $\mathbf{A} = \{\mathbf{a}_k\}$ be complete chain. Suppose chain $\mathbf{B} = \{\mathbf{b}_k\}$ is obtained by formula*

$$\mathbf{b}_k = \mathbf{a}_{k-m} \times \mathbf{a}_{k-m+1} \times \ldots \times \mathbf{a}_{k-1}, \tag{8}$$

and chain $\mathbf{C} = \{\mathbf{c}_j\}$ is obtained by formula

$$\mathbf{c}_j = \mathbf{b}_{j+1} \times \mathbf{b}_{j+2} \times \ldots \times \mathbf{b}_{j+m}. \tag{9}$$

Then the relation

$$\mathbf{c}_j = (-1)^m \prod_{i=0}^{m-2} \det(\mathbf{a}_{i-m+j+1}, \dots, \mathbf{a}_{i+j+1})\mathbf{a}_j \tag{10}$$

is right.

Proof. By formula (9) we have

$$\mathbf{c}_j^T \mathbf{x} = \det(\mathbf{b}_{j+1}, \dots, \mathbf{b}_{j+m)}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathbf{R^{m+1}}.$$

Taking into account formula (8), we obtain

$$\mathbf{c}_j^T \mathbf{x} = \det\Big(\prod_{i=j+1-m}^{j} \times \mathbf{a}_i, \prod_{i=j+2-m}^{j+1} \times \mathbf{a}_i, \dots \prod_{i=j}^{j+m-1} \times \mathbf{a}_i, \mathbf{x}\Big).$$

By substitution $i' = i + m - j - 1$ we have

$$\mathbf{c}_j^T \mathbf{x} = \det\Big(\prod_{i'=0}^{m-1} \times \mathbf{a}_{i'-m+j+1}, \prod_{i'=1}^{m} \times \mathbf{a}_{i'-m+j+1}, \dots, \prod_{i'=m-1}^{2m-2} \times \mathbf{a}_{i'-m+j+1}, \mathbf{x}\Big).$$

Using (5) for $\mathbf{v}_{i'} = \mathbf{a}_{i'-m+j+1}$, $\mathbf{f} = \mathbf{x}$, $\forall \mathbf{x} \in \mathbf{R^{m+1}}$, we deduce relation (10).

The passage from class $\overline{\mathcal{A}}$ to the orthogonal class $\overline{\mathcal{B}}$ is denoted by *(star), so that for discussed classes we have $\overline{\mathcal{B}} = \overline{\mathcal{A}}^\star$, $\overline{\mathcal{A}} = \overline{\mathcal{B}}^\star$. It is evident that $\overline{\mathcal{A}} = (\overline{\mathcal{A}}^\star)^\star$. Thus, the passage from class $\overline{\mathcal{A}}$ to the local orthogonal class is involution in the set $\mathcal{K}$.

5 Space of minimal splines of the Lagrange type

On a finite or infinite interval $(\alpha, \beta) \subset \mathbf{R^1}$ we consider a grid

$$X: \; \ldots < x_{-1} < x_0 < x_1 < \ldots; \tag{11}$$

here $\alpha = \lim_{j\to -\infty} x_j$, $\beta = \lim_{j\to +\infty} x_j$.

Let $\mathbf{A} = \{\mathbf{a}_j\}_{j\in\mathbf{Z}}$ be a complete chain of column vectors $\mathbf{a}_j \in \mathbf{R}^{m+1}$.

Introduce some notation

$$G = \cup_{j\in\mathbf{Z}}(x_j, x_{j+1}), \quad S_j = [x_j, x_{j+m+1}],$$

$$J_k = \{k-m, k-m+1, \dots, k-1, k\} \quad \forall k, j \in \mathbf{Z}.$$

Let $U = U(G)$ be a linear space of functions defined on the set G; the last one depends on grid (11).

Consider $m+1$-component vector function $\varphi(t)$ with components in the space $U(G)$. Discuss the next supposition

(L) *The component of vector function $\varphi(t)$ are the linear independent system on the set $G \cap (c, d)$ for any interval $(c, d) \subset (\alpha, \beta)$.*

We define functions $\omega_j(t)$, $t \in G$, $j \in \mathbf{Z}$, by approximate relations

$$\sum_{j'} \mathbf{a}_{j'} \omega_{j'}(t) \equiv \varphi(t) \quad \forall t \in G, \tag{12}$$

$$\omega_j(t') \equiv 0 \quad \forall t' \in G \setminus S_j. \qquad (13)$$

According to (12) – (13) we obtain

$$\sum_{j'=k-m}^{k} \mathbf{a}_{j'} \omega_{j'}(t) \equiv \varphi(t) \quad \forall t \in (x_k, x_{k+1}) \quad \forall k \in \mathbf{Z}. \qquad (14)$$

By (14) we have $\text{supp}\ \omega_j \subset S_j$, $\omega_j \in U(G)$ $\forall j \in \mathbf{Z}$, and

$$\omega_j(t) = \frac{\det\Big(\{\mathbf{a}_{j'}\}_{j' \in J_k, j' \neq j} \ \| \ ^{\prime j} \varphi(t)\Big)}{\det\Big(\{\mathbf{a}_{j'}\}_{j' \in J_k}\Big)} \qquad (15)$$

$$\forall t \in (x_k, x_{k+1}) \ \forall j \in J_k,$$

where symbol $\| \ ^{\prime j}$ denotes that the determined of the numerator is deduced from the denominator by the replacement of the column $\mathbf{a}_j$ with column $\varphi(t)$ (with preservation of the previous order of columns). Thus, the linear space

$$\widetilde{\mathbf{S}} = \{\widetilde{u} \mid \widetilde{u}(t) = \sum_{j \in \mathbf{Z}} c_j \omega_j(t) \quad \forall t \in G, \ \forall c_j \in \mathbf{R}^1\}, \qquad (16)$$

is contained in the space $U(G)$.

By (12) – (16) we see that the set of elements of the space $\widetilde{\mathbf{S}}$ does not change if vectors $\mathbf{a}_j$ are multiplied with the nonzero coefficients λ_j. Let $\overline{\mathcal{A}}$ be a class of equivalent chains, which contains the chain $A = \{\mathbf{a}_j\}_{j \in \mathbf{Z}}$. It is clear that space $\mathbf{S}$ does not depend on the choice of a chain in class $\overline{\mathcal{A}}$, but that it depends on the choice of class $\overline{\mathcal{A}}$. Therefore we will denote the space (16) by $\widetilde{\mathbf{S}}_{(X, \overline{\mathcal{A}}, \varphi)}$. This space is called the *space of minimal splines of the Lagrange type*.

6 On representation of minimal splines

Let $\{\mathbf{d}_j\}_{j \in \mathbf{Z}}$ be complete chain. In formulas (12), (15) – (16) we put

$$\mathbf{a}_j = \prod_{s=j+1}^{j+m} \times \mathbf{d}_s. \qquad (17)$$

Using (15) for $j = k$, we have

$$\omega_k(t) = \frac{\det(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, \ldots, \mathbf{a}_{k-1}, \varphi(t))}{\det(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_k)} \qquad (18)$$

$$\forall t \in (x_k, x_{k+1}).$$

By (17) equality (18) can be rewritten:

$$\omega_k(t) = \frac{N}{D},$$

where

$$N = \det\Big(\prod_{s=k-m+1}^{k} \times \mathbf{d}_s, \prod_{s=k-m+2}^{k+1} \times \mathbf{d}_s, \ldots$$

$$\ldots, \prod_{s=k}^{k+m-1} \times \mathbf{d}_s, \varphi(t)\Big), \qquad (19)$$

$$D = \det\Big(\prod_{s=k-m+1}^{k} \times \mathbf{d}_s, \prod_{s=k-m+2}^{k+1} \times \mathbf{d}_s, \ldots$$

$$\ldots, \prod_{s=k}^{k+m-1} \times \mathbf{d}_s, \prod_{s=k+1}^{k+m} \times \mathbf{d}_s\Big) \qquad (20)$$

$$\forall t \in (x_k, x_{k+1}).$$

Using Theorem 1 for $\mathbf{v}_i = \mathbf{d}_{i+k-m+1}$, $\mathbf{f} = \varphi(t)$ we transform (19) – (20). As a result we obtain

$$N =$$

$$= (-1)^m \prod_{i=0}^{m-2} \det(\mathbf{d}_{i+k-m+1}, \ldots, \mathbf{d}_{i+k+1}) \cdot \varphi^T(t) \mathbf{d}_k,$$

$$D = \prod_{i=0}^{m-1} \det(\mathbf{d}_{i+k-m+1}, \ldots, \mathbf{d}_{i+k+1}).$$

Thus, we have

$$\omega_k(t) = (-1)^m \cdot \frac{\mathbf{d}_k^T \varphi(t)}{\det(\mathbf{d}_k, \ldots, \mathbf{d}_{k+m})}. \qquad (21)$$

7 Pseudo-continuity of minimal splines

We assume that for integer k there exists a pair of linear functionals F_k^- and F_k^+ in the adjoint space U^* with supports in segments $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ accordingly. We assume that the action of a functional on a vector function is understood as componentwise, so that we obtain a constant vector.

A function $u \in U$ is called pseudo-continuous in x_k if $F_k^- u = F_k^+ u$. We denote by $C_U(x_k)$,

$$C_U(x_k) = \{u \mid u \in U, \ F_k^- u = F_k^+ u\}.$$

It is obvious that $C_U(x_k)$ is a linear space and $C_U(x_k) \subset U$. The linear space of all (m + 1)-dimensional vector-valued functions with components in $C_U(x_k)$ is denoted by $\mathbf{C}_U(x_k)$.

We consider the conditions under which ω_j, $j \in \mathbf{Z}$, belong to $C_U(x_k)$.

Lemma 8. *Assume that $j, k \in \mathbf{Z}$ are such that $k-j \in J_m$ and the functional $F \in U^*$ has support in $[x_k, x_{k+1}]$. Then $F\omega_j = 0$, if and only if*

$$\det\Big(\{\mathbf{a}_{j'}\}_{j'\in J_k, j'\neq j} \;\; \| \;\; {}'^{j}F\varphi\Big) = 0.$$

The proof follows from (15).

Lemma 9. *Let $\varphi \in \mathbf{C}_U(x_k)$ and $\mathbf{A} \in \mathcal{A}$. If*

$$F_k^-\omega_{k-m-1} = 0, \qquad\qquad F_k^+\omega_k = 0, \tag{22}$$

then

$$F_k^-\omega_j = F_k^+\omega_j \tag{23}$$

If $\mathbf{a}_{k-m-1}$ and $\mathbf{a}_k$ are not collinear, then (23) is a sufficient condition for the validity of (22).

Proof. Necessity. Replacing k with $k-1$ in (14), we get

$$\sum_{j=k-m-1}^{k-1} \mathbf{a}_j\omega_j(t) \equiv \varphi(t), \quad t \in (x_{k-1}, x_k), \tag{24}$$

which implies

$$\sum_{j=k-m}^{k-1} \mathbf{a}_j F_k^-\omega_j = F_k^-\varphi \tag{25}$$

in view of the first relation in (22).

Similarly, by (14) and the second relation in (22),

$$\sum_{j=k-m}^{k-1} \mathbf{a}_j F_k^+\omega_j = F_k^+\varphi. \tag{26}$$

Since the vectors $\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, \ldots, \mathbf{a}_{k-1}$ are linearly independent and $F_k^-\varphi = F_k^+\varphi$ by assumption, from (25) and (26) we obtain (23). Necessity is proved.

Sufficiency. Using (24), we find

$$\sum_{j=k-m-1}^{k-1} \mathbf{a}_j F_k^-\omega_j = F_k^-\varphi, \tag{27}$$

and (14) implies

$$\sum_{j=k-m}^{k} \mathbf{a}_j F_k^+\omega_j = F_k^+\varphi. \tag{28}$$

Subtracting (28) from (27) and taking into account (23) and the condition $\varphi \in \mathbf{C}_U(x_k)$, we find $\mathbf{a}_{k-m-1}F_k^-\omega_{k-m-1} = \mathbf{a}_k F_k^+\omega_k$. Since $\mathbf{a}_{k-m-1}$ and $\mathbf{a}_k$ are linearly independent, we get (22).

Theorem 2. *Let $\varphi \in \mathbf{C}_U(x_k)$ and $\mathbf{A} \in \mathcal{A}$. Then the functions $\omega_j(t) \quad (\forall j \in \mathbf{Z})$, in the point x_k are pseudo-continuous if and only if (22) holds.*

Proof. Sufficiency. If (22) holds, then it is obvious that

$$F_k^-\omega_{k-m-1} = F_k^+\omega_{k-m-1}, \qquad F_k^+\omega_k = F_k^-\omega_k,$$

since the right-hand sides of the last identities vanish because of the location of supports of the functionals. If k is such that x_k lies outside the support of ω_j, then $F_k^-\omega_j = F_k^+\omega_j$ since the right-hand and left-hand sides of these identities vanish because the supports of functionals do not intersect the supports of functions to which these functionals are applied. By the relations (23), which are valid in view of Lemma 9, sufficiency is established. Necessity is obvious.

For the equal (by assumption) $F_k^-\varphi$ and $F_k^+\varphi$ we set Φ_k: $\Phi_k = F_k^-\varphi = F_k^+\varphi$.

Theorem 3. *Let $\varphi \in \mathbf{C}_U(x_k)$, $k \in \mathbf{Z}$, and let $\mathbf{A} \in \mathcal{A}$. Then $\omega_j(t)$ $(\forall j \in \mathbf{Z})$ are pseudo-continuous on the grid X if and only if*

$$\mathbf{d}_j^T\Phi_j = 0 \quad \forall j \in \mathbf{Z}. \tag{29}$$

Proof. By (21) equalities (29) are equivalent to relations

$$F_j^+\omega_j = 0, \qquad\qquad \forall j \in \mathbf{Z}. \tag{30}$$

Consider a chain $\widetilde{\mathbf{d}}_j = \prod_{s=j-m}^{j-1} \times \mathbf{a}_s$. According to Lemma 7, the chain is equivalent to $\{\mathbf{d}_j\}$. Therefore formula (29) can be written in equivalent form

$$\det\Big(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, \Phi_j\Big) = 0 \qquad \forall j \in \mathbf{Z}.$$

Replacing j with $j+m+1$ in the last formula, we find

$$\det\Big(\mathbf{a}_{j+1}, \mathbf{a}_{j+2}, \ldots, \mathbf{a}_{j+m}, \Phi_{j+m+1}\Big) = 0 \quad \forall j \in \mathbf{Z},$$

which implies $F_{j+m+1}^-\omega_j = 0 \; \forall j \in \mathbf{Z}$, in view of (15). Replacing $k = j+m+1$, we arrive at the relation

$$F_k^-\omega_{k-m-1} = 0 \qquad \forall k \in \mathbf{Z}. \tag{31}$$

Thus, the identities (29) are equivalent to (30) and (31). It remains to use Lemma 9.

8 Conclusion

This paper discusses continuity of a function as a coincidence of values of two linear functionals on the function where mentioned functionals have their supports in adjacent segments. It gives the opportunity to discuss different sorts of continuity.

For example, for adjacent segments (x_{k-1}, x_k) and (x_k, x_{k+1}) we put

$$F_k^- u = \lim_{\tau \to -0} \int_\tau^0 \psi(\xi) u(x_k + \xi) d\xi,$$

$$F_k^+ u = \lim_{\tau \to +0} \int_0^\tau \psi(\xi) u(x_k + \xi) d\xi,$$

where $\psi(\tau)$ is a weight function. In that case the equality $F_k^- u = F_k^+ u$ is "mean weighted continuity".

Consider another example:

$$F_k^- u = \lim_{\tau \to -0} \psi(\tau) u'(x_k + \tau),$$

$$F_k^+ u = \lim_{\tau \to +0} \psi(\tau) u'(x_k + \tau).$$

Now the equality $F_k^- u = F_k^+ u$ illustrates "weighted continuity of derivative".

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