An Efficient Method of Numerical Integration for a Class of Singularly Perturbed Two Point Boundary Value Problems

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Abstract: In this paper, a new numerical integration method on a uniform mesh is presented for the solution of singularly perturbed two-point boundary value problems having boundary layer at one end (left or right) point. The methods of Exact and Trapezoidal rule of integration with finite difference approximation of first derivatives are used to obtain a three-term recurrence relationship . The obtained tridiagonal system of equations is then solved using Thomas algorithm. Also, the stability and convergence of the proposed scheme are established. Several model example problems are solved using the proposed method. The results are presented in terms of maximum absolute errors which demonstrate the accuracy and efficiency of the method. It is observed that the proposed method is capable of producing highly accurate results with minimal computational effort for a fixed value of step size h, when perturbation parameter tends to zero.

Key–Words: Singular perturbation problems, Bounary value problems, Stability and convergence, Numerical Integration

1 Introduction

Any differential equation in which highest order derivative is multiplied by a small positive parameter ϵ greater than zero is called a singular perturbation problem. A singular perturbation problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as the perturbation parameter $\epsilon \to 0$. Singular perturbation problems have very frequent use in fluid mechanics, fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of fluid motion. Important examples include boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers [21, 16], convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers[5], etc. The solutions of these type of equations possess layer behaviour; that is, there exist some thin regions in the domain of the differential equation where the solution shows rapid change in behaviour. Because of this layer behaviour, existing standard numerical techniques fails to give uniformly convergent solutions to these problems. Thus more efficient and computationally easy methods are needed to solve singularly perturbed boundary value problems.

The survey articles [7, 13, 15] are a great source of information on singular perturbation problems.

A good number of high level monographs/books are available on the singular perturbation problems. Some of these are : O Malley [19, 20], Nayfeh [18], Kevorkian and Cole [14], Angel and Bellman [1], Bender and Orszag [2], El'sgol'ts and Norkin [4] and Hemker and Miller [6], Miller [17].

In this paper, a simple and efficient numerical integration method is proposed for solving a class of singularly perturbed two-point boundary value problems. Novelty of this method lies in the fact that it does neither depend on deviating argument [8] nor any asymptotic expansion [24] or fitted mesh [10, 23, 11].

The paper is organized as follows: Section-2, presents the description of the proposed new method to solve a class of a second order singularly perturbed two-point boundary value problem. Section-3 presents the description of the Thomas algorithm for solving obtained tridiagonal system. The stability of the proposed method is discussed in the Section-4. In the Section-5, the convergence of the proposed method is analyzed. To demonstrate the accuracy and efficiency of the proposed method, Numerical experiments are carried out for several model test problems and the results are presented in terms of maximum absolute error in tables in Section-6. Finally, the discussions and conclusions are presented in the last section-7. The paper ends with the references.

2 Statement of the problem

To describe the method proposed, consider a class of singular perturbation problems of the form:

$$\epsilon y''(x)+[a(x)y(x)]'-b(x)y(x)=f(x);\quad 0\le x\le 1 \tag{1}$$

with

$$y(0)=\alpha \ and\ y(1)=\beta \tag{2}$$

where ϵ is a small positive parameter ($0<\epsilon<<1$); $\alpha,\ \beta$ are given constants; $a(x),\ b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in [0,1]. Considering $a(x)\ge M>0$ on the whole of the interval [0,1], where M is some positive constant, the boundary layer will exist in the neighbourhood of $x=0$, while it will be present in the neighbourhood of $x=1$, if $a(x)\le M<0$ throughout the interval [0,1], where M is negative constant.

2.1 Description of the method

2.1.1 Left end layer problem

First, we divide the interval [0,1] into N equal parts with uniform mesh length h. Let $0=x_0<x_1<x_2<...<x_N=1$ be the mesh points where $x_i=x_0+ih,\ i=0,1,2,...,N$.
Integrating the Eq. (1) in $[x_i,x_{i+1}]$ $(i=1,2,...,N-1.)$, we get

$$\left[\epsilon y'(x)\right]_{x_i}^{x_{i+1}}+\left[a(x)y(x)\right]_{x_i}^{x_{i+1}}-\int_{x_i}^{x_{i+1}} b(x)\,y(x)dx = \int_{x_i}^{x_{i+1}} f(x)dx$$

$$\text{Or,}\ \epsilon\,{y'}_{i+1}-\epsilon\,{y'}_i+a_{i+1}y_{i+1}-a_iy_i - \int_{x_i}^{x_{i+1}} b(x)y(x)\,dx = \int_{x_i}^{x_{i+1}} f(x)\,dx$$

Now on applying trapezoidal rule of integration, we have

$$\epsilon\,{y'}_{i+1}-\epsilon\,{y'}_i+a_{i+1}y_{i+1}-a_iy_i -\frac{h}{2}\left[b_{i+1}y_{i+1}+b_iy_i\right]=\frac{h}{2}\left[f_{i+1}+f_i\right]$$

$$\text{Or,}\ \epsilon\,{y'}_{i+1}-\epsilon\,{y'}_i+\left[a_{i+1}-\frac{h}{2}b_{i+1}\right]y_{i+1} + \left[-a_i-\frac{h}{2}b_i\right]y_i=\frac{h}{2}\left[f_{i+1}+f_i\right] \tag{3}$$

Using the following approximations for first derivative of y,

$$y'_{i+1}=\frac{y_{i+1}-y_i}{h} \tag{4}$$

and

$$y'_i=\frac{y_i-y_{i-1}}{h} \tag{5}$$

into Eq. (3) we get

$$\epsilon\left[\frac{y_{i+1}-y_i}{h}\right]-\epsilon\left[\frac{y_i-y_{i-1}}{h}\right]+\left[a_{i+1}-\frac{h}{2}b_{i+1}\right]y_{i+1}+\left[-a_i-\frac{h}{2}b_i\right]y_i=\frac{h}{2}\left[f_{i+1}+f_i\right]$$

$$\text{Or,}\ \frac{\epsilon}{h}y_{i-1}-\left[\frac{2\epsilon}{h}+a_i+\frac{h}{2}b_i\right]y_i+\left[\frac{\epsilon}{h}+a_{i+1}-\frac{h}{2}b_{i+1}\right]y_{i+1}=\frac{h}{2}\left[f_i+f_{i+1}\right] \tag{6}$$

The Eq. (6) can be written as the three term recurrence relation of form:

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=R_i;\ i=1,2,...,N-1 \tag{7}$$

where

$$E_i=\frac{\epsilon}{h}$$

$$F_i=\frac{2\epsilon}{h}+a_i+\frac{h}{2}b_i$$

$$G_i=\frac{\epsilon}{h}+a_{i+1}-\frac{h}{2}b_{i+1}$$

$$R_i=\frac{h}{2}\left[f_i+f_{i+1}\right]$$

This tridiagonal system is solved by using Thomas algorithm. Which is described in the section-3.

2.1.2 Right end layer problem

As described the method for left end layer problem above, we first divide the interval [0,1] into N equal parts with uniform mesh length h. Let $0=x_0<x_1<x_2<...<x_N=1$ be the mesh points where $x_i=x_0+ih,\ i=0,1,2,...,N$.

Integrating the Eq. (1) in $[x_{i-1}, x_i]$ $(i = 1, 2, ..., N - 1.)$, we get

$$[\epsilon y'(x)]_{x_{i-1}}^{x_i} + [a(x)y(x)]_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} b(x)\, y(x) dx = \int_{x_{i-1}}^{x_i} f(x) dx$$

$$\text{Or, } \epsilon y'_i - \epsilon y'_{i-1} + a_i y_i - a_{i-1} y_{i-1} - \int_{x_{i-1}}^{x_i} b(x) y(x)\, dx = \int_{x_{i-1}}^{x_i} f(x)\, dx$$

Now on applying trapezoidal rule of integration, we have

$$\epsilon y'_i - \epsilon y'_{i-1} + a_i y_i - a_{i-1} y_{i-1} - \frac{h}{2}[b_i y_i + b_{i-1} y_{i-1}] = \frac{h}{2}[f_i + f_{i-1}]$$

$$\text{Or, } \epsilon y'_i - \epsilon y'_{i-1} + \left[a_i - \frac{h}{2} b_i\right] y_i + \left[- a_{i-1} - \frac{h}{2} b_{i-1}\right] y_{i-1} = \frac{h}{2}[f_i + f_{i-1}] \quad (8)$$

Using the following approximations for first derivative of y,

$$y'_{i-1} = \frac{y_i - y_{i-1}}{h} \quad (9)$$

and

$$y'_i = \frac{y_{i+1} - y_i}{h} \quad (10)$$

into Eq. (8) we get

$$\epsilon \left[\frac{y_{i+1} - y_i}{h}\right] - \epsilon \left[\frac{y_i - y_{i-1}}{h}\right] + \left[a_i - \frac{h}{2} b_i\right] y_i + \left[- a_{i-1} - \frac{h}{2} b_{i-1}\right] y_{i-1} = \frac{h}{2}[f_i + f_{i-1}]$$

$$\text{Or, } \left[\frac{\epsilon}{h} - a_{i-1} - \frac{h}{2} b_{i-1}\right] y_{i-1} - \left[\frac{2\epsilon}{h} - a_i + \frac{h}{2} b_i\right] y_i + \left[\frac{\epsilon}{h}\right] y_{i+1} = \frac{h}{2}[f_i + f_{i-1}] \quad (11)$$

The Eq. (11) can be written as the three term recurrence relation of form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = R_i;\ i = 1, 2, ..., N - 1 \quad (12)$$

where

$$E_i = \frac{\epsilon}{h} - a_{i-1} - \frac{h}{2} b_{i-1}$$
$$F_i = \frac{2\epsilon}{h} - a_i + \frac{h}{2} b_i$$
$$G_i = \frac{\epsilon}{h}$$
$$R_i = \frac{h}{2}[f_i + f_{i-1}]$$

This tridiagonal system is solved by using Thomas algorithm. Which is described in the section-3.

3 Thomas Algorithm

We briefly discuss the Thomas algorithm to solve the tridiagonal system [3]:-

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = R_i;\ \ i = 1, 2, ..., N - 1 \quad (13)$$

subject to the boundary conditions

$$y_0 = y(0) = \alpha,\ \ y_N = y(1) = \beta \quad (14)$$

First, we set

$$y_i = W_i y_{i+1} + T_i \quad for\ \ i = N - 1, N - 2, ..., 2, 1 \quad (15)$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$, which are to be determined. From Eq. (15), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \quad (16)$$

Substituting Eq. (16) in Eq. (13), we have

$$E_i (W_{i-1} y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = R_i$$
$$y_i (E_i W_{i-1} - F_i) + G_i y_{i+1} = R_i - E_i T_{i-1}$$
$$y_i (F_i - E_i W_{i-1}) = (E_i T_{i-1} - R_i) + G_i y_{i+1}$$
$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - R_i}{F_i - E_i W_{i-1}} \quad (17)$$

By comparing Eqs. (15) and (17), we get the recurrence relations

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \quad (18)$$

$$T_i = \frac{E_i T_{i-1} - R_i}{F_i - E_i W_{i-1}} \quad (19)$$

T_0 solve these recurrence relations for $i = 0, 1, 2, ..., N - 1$, we need to know the initial conditions for W_0 and T_0. This can be done by considering (14):-

$$y_0 = \alpha = W_0 y_1 + T_0$$

If we choose $W_0 = 0$ then we set $T_0 = \alpha$.
Using these initial values, we compute W_i and T_i for $i = 0, 1, 2, ..., N-1$ from Eqs. (18) and (19) in the y_i in the backward process from the Eqs. (15) and (14). Now, we will show that the method is computationally stable.

4 Stability

By stability we mean that the effect of an error made in one stage of calculation is not propagated into larger errors at latter stages of computation. In other words, local errors are not magnified by further computation [9, 12].

Let us now examine the recurrence relation given by Eq. (13). Suppose a small error τ_i, has been made in the calculation of W_i, then we have

$$\widetilde{W}_i = W_i + \tau_i \tag{20}$$

and we are actually calculating

$$\widetilde{W}_i = \frac{G_i}{F_i - E_i\widetilde{W}_{i-1}} \tag{21}$$

From Eqs. $(18)$and (21), we have

$$\begin{aligned}\tau_i &= \tilde{W}_i - W_i \\ &= \frac{G_i}{F_i - E_i\tilde{W}_{i-1}} - \frac{G_i}{F_i - E_iW_{i-1}} \\ &= \frac{G_i}{F_i - E_i\left(W_{i-1}+\tau_{i-1}\right)} - \frac{G_i}{F_i - E_iW_{i-1}} \\ &= E_iG_i\tau_{i-1}\left[F_i - E_i\left(W_{i-1} + \tau_{i-1}\right)\right]^{-1}\left[F_i - E_iW_{i-1}\right]^{-1} \\ &= \frac{E_i\tau_{i-1}}{G_i}\left[\frac{G_i}{F_i - E_i\left(W_{i-1}+\tau_{i-1}\right)}\right]\left[\frac{G_i}{F_i - E_iW_{i-1}}\right] \\ &= W_i^2\frac{E_i}{G_i}\tau_{i-1} \end{aligned} \tag{22}$$

under the assumption that the error is small initially.

Then from the definition of $E_i,\ F_i$ and G_i from (7) with the assumption that $E_i > 0$ and $G_i > 0$ for $i = 1, 2, ..., N-1$. and it can be shown that $|F_i| > |E_i + G_i|$; provided $|a_i + \frac{h}{2}b_i| > |a_{i+1} - \frac{h}{2}b_{i+1}|$; $\forall\, i = 1, 2, ..., N-1$.
From the initial condition on W_0, it is clear that $|W_0| < 1$, We now make use of the assumptions on E_i and G_i to show $|W_i| < 1$ for $i = 0, 1, 2, ..., N-1$. from Eq. (18)

$$W_1 = \frac{G_1}{F_1} < 1 \quad (\text{Since } F_1 > G_1)$$

$$\begin{aligned} W_2 &= \frac{G_2}{F_2 - E_2W_1} < \frac{G_2}{F_2 - E_2} \ (\text{Since } W_1 < 1) \\ &< \frac{G_2}{E_2 + G_2 - E_2} = 1 \ (\text{Since } F_2 \geq E_2 + G_2) \end{aligned}$$

Successively it follows that $|W_i| < 1$ for $i = 0, 1, 2, ..., N-1$.
Then it follows from the Eq. (22) that

$$\begin{aligned} |\tau_i| &= |W_i^2\frac{E_i}{G_i}\tau_{i-1}| \\ &= |W_i|^2|\frac{E_i}{G_i}||\tau_{i-1}| \\ &< |\tau_{i-1}|;\ \text{provided } |G_i| \geq |E_i|, \end{aligned} \tag{23}$$

and thus the recurrence relation (18) is stable.
Similarly suppose a small error $\widetilde{\tau_i}$ has been made in the calculation of T_i, then we have

$$\widetilde{T}_i = T_i + \tau_i$$

similar argument give

$$\widetilde{\tau_i} = W_i\frac{G_i}{E_i}\widetilde{\tau_{i-1}}$$

Making use of the condition $W_i < 1$ for $i = 0, 1, 2, ..., N-1$; it follows that

$$\begin{aligned} |\widetilde{\tau_i}| &= |W_i||\frac{G_i}{E_i}||\widetilde{\tau_{i-1}}| \\ &< |\widetilde{\tau_{i-1}}| \end{aligned}$$

and thus the recurrence relations (18) and (19) are stable. Hence, we conclude that the discrete imvarient imbedding algorithm is computationally stable. Finally, the convergence of this discrete invariant imbedding method is ensured by the condition $|W_i| < 1$ for $i = 0, 1, 2, ..., N-1$.

5 Convergence Analysis

Writing the tri-diagonal system (13) in matrix-vector form, we obtain

$$AY = C \tag{24}$$

where $A = (m_{i,j})$, $1 \leq i, j \leq N-1$ is a tri-diagonal matrix of order N-1, with

$$\begin{aligned} m_{i,i+1} &= -\epsilon - ha_{i+1} + \frac{h^2}{2}b_{i+1} \\ &= \text{coefficient of } y_{i+1} \text{ in } (6);\ i = 1(1)N-2 \end{aligned}$$

$$\begin{aligned} m_{i,i} &= 2\epsilon + ha_i + \frac{h^2}{2}b_i \\ &= \text{coefficient of } y_i \text{ in } (6);\ i = 1(1)N-1 \end{aligned}$$

$$\begin{aligned} m_{i,i-1} &= -\epsilon \\ &= \text{coefficient of } y_{i-1} \text{ in } (6);\ i = 2(1)N-1 \end{aligned}$$

and $C = (d_i)$ is a column vector with $d_i = -\frac{h^2}{2}[f_i + f_{i+1}]$, where $i = 1, 2, ..., N-1$ with local truncation error

$$\tau_i(h) = h^2 \left[\left(\frac{a''_i}{2} - \frac{b'_i}{2} \right) y_i + \left(a'_i - \frac{b_i}{2} \right) y'_i + \frac{a_i}{2} y''_i - \frac{f'_i}{2} \right] + o(h^3) \quad (25)$$

We also have

$$A\overline{Y} - \tau(h) = C \quad (26)$$

where $\overline{Y} = (\overline{Y}_0, \overline{Y}_1, \overline{Y}_2, ..., \overline{Y}_N)^t$ and $\tau(h) = (\tau_1(h), \tau_2(h), ..., \tau_N(h))^t$ denote the actual solution and the local truncation error respectively.
From Eqs. (24) and (26), we have

$$A(\overline{Y} - Y) = \tau(h) \quad (27)$$

Thus, the error equation is

$$AE = \tau(h) \quad (28)$$

where $E = \overline{Y} - Y = (e_0, e_1, e_2, ..., e_N)^t$.
Let S_i be the sum of elements of the i^{th} row of A, then we have

$$S_1 = \sum_{j=1}^{N-1} m_{1,j} = \epsilon + h(a_1 - a_2) + \frac{h^2}{2}(b_1 + b_2)$$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j} = \epsilon + ha_{N-1} + \frac{h^2}{2} b_{N-1}$$

$$S_i = \sum_{j=1}^{N-1} m_{1,j} = h[a_i - a_{i+1}] + \frac{h^2}{2}[b_i + b_{i+1}] = hB_i;\ i = 2(1)N-2.$$

where $B_i = [a_1 - a_{i+1}] + \frac{h}{2}[b_i + b_{i+1}]$
Since $0 < \epsilon << 1$, the matrix A is irreducible and monotone. Then, it follows that A^{-1} exists and its elements are non-negative. Hence, from Eq.(28) we have

$$E = A^{-1}\tau(h) \quad (29)$$

$$\|E\| \leq \|A^{-1}\|\|\tau(h)\| \quad (30)$$

Let $\bar{m}_{k,i}$ be the $(k,i)^{th}$ element of A^{-1}. Since $\bar{m}_{k,i} \geq 0$, from the operations of matrices we have,

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} S_i = 1;\ \ k = 1, 2, ..., N-1 \quad (31)$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \leq \frac{1}{\underset{0 \leq i \leq N-1}{min} S_i} = \frac{1}{hB_{i_0}} \leq \frac{1}{h|B_{i_0}|} \quad (32)$$

for some i_0 between 1 and N-1.
We define $\|A^{-1}\| = \underset{0 \leq k \leq N-1}{max} \sum_{i=1}^{N-1} |\bar{m}_{k,i}|$ and $\|\tau(h)\| = \underset{0 \leq k \leq N-1}{max} |\tau(h)|$.
Therefore, from Eqs. (25),(29) and (31) we obtain

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k,i} \tau_i(h);\ \ j = 1(1)N-1$$

and therefore

$$|e_j| \leq \frac{kh^2}{h|B_{i_0}|};\ \ j = 1(1)N-1 \quad (33)$$

where $k = \left[\left(\frac{a''_i}{2} - \frac{b'_i}{2} \right) |y_i| + \left(a'_i - \frac{b_i}{2} \right) |y'_i| + \frac{a_i}{2} |y''_i| - \frac{f'_i}{2} \right]$ is constant independent of h.
Therefore, using the definitions and Eq.(33)

$$\|E\| = o(h)$$

This implies that our method is first order convergence on uniform mesh.
As above, we can apply the same procedure for showing the method is of first order convergence on uniform mesh for right layer problem.

6 Numerical examples

6.1 Numerical examples: left-end boundary layer problems

In this subsection, we have applied the proposed method on four linear singular perturbation problems and presented the computational results in the tables in terms of the maximum absolute errors. These examples have been chosen because they have been widely discussed in literature.

Example 1:- Consider the following non-homogeneous SPP from fluid dynamics for fluid of small viscosity, [22, Example 2]

$$\epsilon y''(x) + y'(x) = 1 + 2x; \text{ for } 0 \leq x \leq 1$$
$$\text{with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\epsilon) + (2\epsilon - 1)\frac{(1 - exp(-x/\epsilon))}{(1 - exp(-1/\epsilon))}$$

Table 1: The Maximum absolute error for the Example-1 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	1.332803E-01	1.945177E-01	1.770201E-01	1.034513E-01	6.082416E-02
10^{-3}	1.571649E-02	3.094596E-02	6.002969E-02	1.128439E-01	1.833405E-01
10^{-4}	1.597345E-03	3.189027E-03	6.357253E-03	1.263452E-02	2.495885E-02
10^{-5}	1.598597E-04	3.201365E-04	6.384850E-04	1.279414E-03	2.551436E-03
10^{-6}	1.591444E-05	3.230572E-05	6.490906E-05	1.255870E-04	2.585649E-04
10^{-7}	1.668930E-06	3.059442E-06	5.304813E-06	1.424551E-05	2.050400E-05
10^{-8}	2.980232E-07	7.599592E-07	1.370907E-06	2.777204E-06	5.662441E-06
10^{-9}	0.000000E+00	4.172325E-07	1.456589E-06	1.192093E-07	2.384186E-07
10^{-10}	0.000000E+00	0.000000E+00	0.000000E+00	1.164153E-10	1.164153E-10
10^{-11}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-15}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-20}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

It can be easily observed from the Table 1 that the maximum absolute error for the example problem-1 is becoming zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error to converge to zero is fast with respect to larger values of N.

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 1.

Example 2:- Consider the following homogeneous SPP from Kevorkian and Cole [14, p. 33, Eqs. (2.3.26) and (2.3.27)] with $\alpha = 0$:

$$\epsilon y''(x) + y'(x) = 0; \text{ for } 0 \leq x \leq 1$$
$$\text{with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = \frac{1 - exp(-x/\epsilon)}{1 - exp(-1/\epsilon)}$$

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 2.

Example 3:- Consider the following SPP with variable coefficients from Kevorkian and Cole [14, p. 33, Eqs. (2.3.26) and (2.3.27)] with $\alpha = -1/2$:

$$\epsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0; \text{ for } 0 \leq x \leq 1$$
$$\text{with } y(0) = 0 \text{ and } y(1) = 1$$

First we rewrite above equation in the form of main equation, i.e., as

$$\epsilon y''(x) + \left[(1 - \frac{x}{2}) y(x)\right]' = 0$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [18, p. 148, Eq. (4.2.32)]) as our exact solution,

$$y(x) = \frac{1}{2 - x} - \frac{1}{2} exp\left(-\left(x - \frac{x}{4}\right)^2 / \epsilon\right)$$

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 3.

Example 4:- Consider the following singular perturbation problem:

$$\epsilon y''(x) + y'(x) = 2; \text{ for } 0 \leq x \leq 1$$
$$\text{with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = 2x + \frac{(1 - exp(-x/\epsilon))}{(exp(-1/\epsilon) - 1)}$$

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 4.

Table 2: The Maximum absolute error for the Example-2 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	1.360003E-01	1.984887E-01	1.806326E-01	1.055667E-01	6.207365E-02
10^{-3}	1.574641E-02	3.100961E-02	6.015033E-02	1.130707E-01	1.837178E-01
10^{-4}	1.597524E-03	3.186345E-03	6.359279E-03	1.263821E-02	2.497596E-02
10^{-5}	1.599193E-04	3.218055E-04	6.321669E-04	1.285970E-03	2.553523E-03
10^{-6}	1.597404E-05	3.385544E-05	6.401539E-05	1.128316E-04	2.711415E-04
10^{-7}	2.503395E-06	3.218651E-06	7.390979E-06	2.038479E-05	3.027916E-05
10^{-8}	1.072884E-06	3.576279E-06	4.410744E-06	1.502037E-05	1.776218E-05
10^{-9}	0.000000E+00	1.907349E-06	7.380976E-06	1.192093E-07	2.384186E-07
10^{-10}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-11}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-15}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-20}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

As like Table-1 for example problem-1, one can easily observed from the Table 2 that the maximum absolute error for the example problem-2 is becoming zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error to converge to zero is fast with respect to larger values of N.

6.2 Numerical examples: right-end boundary layer problems

In this subsection, we have applied the proposed method on two model linear singular perturbation problems having boundary layer at the right-end. The computational results are presented in terms of the maximum absolute errors in tables. These examples

Table 3: The Maximum absolute error for the Example-3 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	3.362869E-01	3.774481E-01	4.179145E-01	4.369655E-01	4.468246E-01
10^{-3}	4.811954E-02	2.736307E-01	4.062337E-01	4.300662E-01	4.629640E-01
10^{-4}	7.537007E-04	5.097389E-04	1.235052E-01	3.484314E-01	4.464494E-01
10^{-5}	7.575750E-05	1.541972E-04	3.157854E-04	1.550600E-02	2.106749E-01
10^{-6}	7.987022E-06	1.579523E-05	2.950430E-05	6.616116E-05	3.427267E-05
10^{-7}	8.344650E-07	1.251698E-06	4.768372E-06	4.231930E-06	1.829863E-05
10^{-8}	3.576279E-07	1.072884E-06	1.907349E-06	3.039837E-06	1.847744E-06
10^{-9}	4.768372E-07	7.152557E-07	1.192093E-07	7.152557E-07	7.569790E-06
10^{-10}	5.960464E-08	1.788139E-07	2.384186E-07	4.072884E-07	6.198883E-07
10^{-11}	5.960464E-08	1.192093E-07	1.192093E-07	3.576279E-07	5.364418E-07
10^{-15}	5.960464E-08	1.192093E-07	1.192093E-07	3.576279E-07	2.980232E-07
10^{-20}	5.960464E-08	1.192093E-07	1.192093E-07	3.576279E-07	2.980232E-07

It can be easily observed from the Table-3 that the maximum absolute errors tends to zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error to become uniform is fast with respect to larger values of N.

have been chosen because they have been widely discussed in literature.

Example 5:- Consider the following singular perturbation problem:

$$\epsilon y''(x) - y'(x) = 0; \text{ for } 0 \le x \le 1$$
$$\text{with } y(0) = 1 \text{ and } y(1) = 0$$

Clearly, this problem has a boundary layer at $x = 1$ i.e., at the right end of the underlying interval.
The exact solution is given by

$$y(x) = \frac{(e^{(x-1)/\epsilon} - 1)}{(e^{-1/\epsilon} - 1)}$$

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 5.

Example 6:- Consider the following singular perturbation problem:

$$\epsilon y''(x) - y'(x) - (1+\epsilon)y(x) = 0; \text{ for } 0 \le x \le 1$$

with

$$y(0) = 1 + exp(-(1+\epsilon)/\epsilon) \ \ and \ \ y(1) = 1 + 1/e$$

Clearly, this problem has a boundary layer at $x = 1$ i.e., at the right end of the underlying interval. The exact solution is given by

$$y(x) = e^{(1+\epsilon)(x-1)/\epsilon} + e^{-x}$$

The maximum absolute errors for different values of N and singular perturbation parameter ϵ are presented in Table 6.

Table 4: The Maximum absolute error for the Example-4 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	1.360005E-01	1.984872E-01	1.806326E-01	1.055616E-01	6.206429E-02
10^{-3}	1.574832E-02	3.100765E-02	6.015033E-02	1.130707E-02	1.837056E-01
10^{-4}	1.597524E-03	3.190279E-03	6.359279E-03	1.263821E-02	2.496111E-02
10^{-5}	1.599193E-04	3.199577E-04	6.399751E-04	1.278341E-03	2.553582E-03
10^{-6}	1.597404E-05	3.194809E-05	6.407499E-05	1.283288E-04	2.558827E-04
10^{-7}	1.549721E-06	3.278255E-06	6.794930E-06	1.281500E-05	2.598763E-05
10^{-8}	2.607703E-07	1.065433E-06	1.212582E-06	3.932044E-06	5.027744E-06
10^{-9}	0.000000E+00	5.774200E-07	2.022833E-06	1.192093E-07	2.384186E-07
10^{-10}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-11}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-15}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-20}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

As like Table-1 and 2 for example problem-1 and 2, one can easily observed from the Table-4 that the maximum absolute error for the example problem-4 is becoming zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error to converge to zero are fast with respect to larger values of N.

7 Conclusion

In this paper, we have proposed a new method of numerical integration for a class of singularly perturbed two-point boundary value problems. Using the methods of Exact and Trapezoidal rule of integration with finite difference approximation of first derivatives, a three-term recurrence relationship is obtained. Thomas algorithm is used to solve the system. Also, the stability and convergence of the proposed scheme is established. The applicability of the proposed method is demonstrated by performing numerical experiments on six model problems (four linear problems with left-end boundary layer and two problems with right-end boundary layer) by taking different values of $N = \frac{1}{h}$ and perturbation parameter ϵ. The computational results are presented in Tables 1-6. It is easily observed from the tables that the presented method is capable of producing highly accurate results for fixed value of step size $h = 1/N$, when perturbation parameter ϵ tends to zero. Further, for smaller values of N the tendency of maximum absolute error to converge to zero is fast with respect to larger values of N. The maximum absolute errors are becoming either zero or uniform for any fixed values of N when $\epsilon \to 0$. Novelty of the method lies in the fact that it does neither depend on deviating argument nor any asymptotic expansion or fitted mesh.

Table 5: The Maximum absolute error for the Example-5 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	1.360003E-01	1.984891E-01	1.806325E-01	1.055671E-01	6.206626E-02
10^{-3}	1.574701E-02	3.100961E-02	6.015027E-02	1.130705E-02	1.837207E-01
10^{-4}	1.597524E-03	3.187954E-03	6.359279E-03	1.263821E-02	2.497619E-02
10^{-5}	1.599193E-04	3.218065E-04	6.357431E-04	1.285970E-03	2.553523E-03
10^{-6}	1.597404E-05	3.385544E-05	6.401539E-05	1.203418E-04	2.717415E-04
10^{-7}	2.503395E-06	3.218651E-06	3.695488E-06	2.038479E-05	1.513958E-05
10^{-8}	1.072884E-06	3.337860E-06	4.410744E-06	1.382828E-05	1.776218E-05
10^{-9}	0.000000E+00	1.907349E-06	7.390976E-06	1.192093E-07	2.384186E-07
10^{-10}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-11}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-15}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
10^{-20}	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

As like Table-1,2 and 4 for left layer model example problem-1,2 and 4, we can easily observe from the Table-5 that the maximum absolute error for the model example problem-5 is tending to zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error to converge to zero is fast with respect to larger values of N.

Table 6: The Maximum absolute error for the Example-6 for different values of N and singular perturbation parameter ϵ.

ϵ	N=16	N=32	N=64	N=128	256
10^{-2}	1.311816E-01	1.951567E-01	1.796666E-01	1.049874E-01	6.194413E-02
10^{-3}	1.513460E-02	3.048119E-02	5.964637E-02	1.125763E-01	1.833011E-01
10^{-4}	1.428306E-03	3.110558E-03	6.301254E-03	1.258498E-02	2.491739E-02
10^{-5}	1.189113E-04	2.849996E-04	6.271303E-04	1.271814E-03	2.549857E-03
10^{-6}	1.189411E-04	2.965331E-05	5.590916E-05	1.216531E-04	2.549291E-04
10^{-7}	1.188815E-04	2.998114E-05	6.943941E-06	1.019239E-05	2.786517E-05
10^{-8}	1.195073E-04	2.968311E-05	7.808208E-06	7.152557E-07	1.907349E-06
10^{-9}	1.194775E-04	3.018975E-05	7.390976E-06	1.251698E-06	1.430511E-06
10^{-10}	1.195967E-04	2.989173E-05	7.361174E-06	1.192093E-06	2.056360E-06
10^{-11}	1.195967E-04	2.989173E-05	7.510185E-06	1.609325E-06	1.192093E-07
10^{-15}	1.195967E-04	2.989173E-05	7.539988E-06	1.668930E-06	5.364418E-07
10^{-20}	1.195967E-04	2.989173E-05	7.539988E-06	1.668930E-06	5.364418E-07

As like Table-3 for left layer problem-3, one can easily observed from the Table-6 that the maximum absolute errors tends to zero, when singular perturbation parameter $\epsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. Further, for smaller values of N the tendency of maximum absolute error is becoming uniform is fast with respect to larger values of N.

References:

[1] E. Angel, R. Bellman, *Dynamic Programming and Partial Differential Equations,* Academic Press– New York 1972.

[2] C. M. Bender, S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill– New York 1978.

[3] P. P. Chakravarthy, S. D. Kumar, R. N. Rao, An exponentially fitted finite difference scheme for a class of singularly perturbed delay differential equations with large delays, *Ain Shams Engineering Journal,* xxx, 2015, pp. xxx-xxx.

[4] L. E. El'sgol'ts, S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments,* Academic Press– New York 1973.

[5] R. R. Gold, Magnetohydrodynamic pipe flow, *Part I. Journal of Fluid Mechanics,* 13, 1962, pp. 505-512.

[6] P. W. Hemker, J. J. H. Miller, *Numerical Analysis of Singular Perturbation Problems,* Academic Press– New York 1979.

[7] M. K. Kadalbajoo, Y. N. Reddy, Asymptotic, numerical analysis of singular perturbation problems: a survey, *Applied Mathematics and Computation,* 30, 1989, pp. 223-259.

[8] M. K. Kadalbajoo, Y. N. Reddy, An Approximate Method for Solving a Class of Singular Perturbation Problems, *Journal of Mathematical Analysis and Applications,* 133, 1988, pp. 306-323.

[9] M. K. Kadalbajoo, Y.-N. Reddy, Numerical Treatment of Singularly Perturbed Two Point Boundary Value Problems, *Applied Mathematics and Computation,* 21, 1987, pp. 93-110.

[10] M. K. Kadalbajoo, V. K. Aggarwal, Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems, *Applied Mathematics and Computation,* 161, 2005, pp. 973987.

[11] M. K. Kadalbajoo, K. K. Sharma, Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior, *Electronic Transactions on Numerical Analysis,* 23, 2006, pp. 180-201.

[12] M. K. Kadalbajoo, P. Arora, B-spline collocation method for the singular-perturbation problem using artificial viscosity, *Computers and Mathematics with Applications,* 57, 2009, pp. 650-663.

[13] M. K. Kadalbajoo, V. Gupta, A brief survey on numerical methods for solving singularly perturbed problems, *Applied Mathematics and Computation,* 217, 2010, pp. 3641-3716.

[14] J. Kevorkian, J. D. Cole, *Perturbation Methods in Applied Mathematics,* Springer– New York 1981.

[15] M. Kumar, A. K. Singh, Singular Perturbation Problems in Nonlinear Elliptic Partial Differential Equations: A Survey, *International Journal of Nonlinear Science,* 17, 3, 2014, pp. 195-214.

[16] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems, Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions,* World Scientific– Singapore 1996.

[17] J. J. H. Miller, *Singular Perturbation Problems in Chemical Physics Analytic and Computational Methods,* John Wiley– New York 1997.

[18] A. H. Nayfeh, *Perturbation Methods,* Wiley– New York 1979.

[19] R. E. O'Malley, *Introduction to Singular Perturbations,* Academic Press– New York 1974.

[20] R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations,* Springer– New York 1991.

[21] H. G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations,* Springer– Berlin 1996.

[22] H. J. Reinhardt, Singular Perturbations of difference methods for linear ordinary differential equations, *Applicable Analysis,* 10, 1980, pp. 53-70.

[23] V. Shanthi, N. Ramanujam, S. Natesan, Fitted mesh method for singularly perturbed reaction-convection-diffusion problems with boundary and interior layers, *J. Appl. Math. and Computing,* 22, 1, 2006, pp. 49-65.

[24] V. Subburayan, N. Ramanujam, Asymptotic Initial Value Technique for singularly perturbed convectiondiffusion delay problems with boundary and weak interior layers, *Applied Mathematics Letters,* 25, 2012, pp. 2272-2278.