Sixth-Order and Fourth-Order Hybrid Boundary Value Methods for Systems of Boundary Value Problems

GRACE O. AKINLABI
Department of Mathematics
Covenant University, Canaanland, Ota, Ogun State
NIGERIA
grace.akinlabi@covenantuniversity.edu.ng

RAPHAEL B. ADENIYI
Department of Mathematics
University of Ilorin, Ilorin, Kwara State
NIGERIA
raphade@unilorin.edu.ng

Abstract: - Hybrid Boundary Value Methods (HyBVMs) are a new class of Boundary Value Methods (BVMs) proposed recently for the approximation of Ordinary Differential Equations (ODEs). These new schemes behave just like the BVMs as the HyBVMs are also based on the Linear Multistep Methods (LMMs) but utilizes data at both step and off-step points. Numerical tests on both linear and nonlinear Boundary Value Problems (BVPs) were presented using the HyBVMs of order 6 and order 4. The results were compared with the symmetric schemes: Extended Trapezoidal Rules (ETRs) of order 6 and order 4.

Key-Words: - Boundary value methods, hybrid BVMs, boundary value problems, linear multistep method, numerical methods for ODEs.

1 Introduction

The numerical approximation of Differential Equations continues to be an active research as they arise from models from Applied Sciences, Engineering and Economics. Several authors have used introduced and applied several methods for the approximation differential problems [1 – 3].

An Ordinary Differential Equation (ODE) is an equation with respect to its independent variable that involves an unknown function and its derivatives. This equation is classified into: Initial Value Problems (IVPs) and Boundary Value Problems (BVPs) based on the subsidiary conditions that accompany these problems [4].

The latter class of problem is more difficult to handle, since it is a broader class of continuous problems unlike the former and they are usually solved by the Shooting Method (SHM). This SHM works by first reducing the BVP to its equivalent system of IVPs, which makes it suffer from some numerical instability in the process of conversion [5].

A new scheme called Boundary Value Methods (BVMs) was proposed to remove this type of numerical instability and other ones familiar to the conventional methods used in approximating ODEs. The process of developing and applying this new scheme makes it suitable for solving the BVPs directly without necessarily converting them to their equivalent system of IVPs. For instance, in the derivation of the BVMs, the same continuous scheme e.g. a Linear Multistep Method (LMM) used to generate the main method is also used in generating the additional methods; these are then applied at the end points thereby avoiding some of the stability problems encountered e.g. in the application of the SHM to BVP.

Lots of BVMs have been proposed by different authors and used for the approximation of different types of differential problems. Their convergence and stability properties have also been fully discussed [5-13].

Our focus in this work is to develop new class of BVMs that utilize data at off-step points and which will be called Hybrid Boundary Value Methods (HyBVMS). In deriving these methods, we will be adopting the Adams Moulton methods, which is a LMM of the form:

$$y_{n+k} - y_{n+k-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} \tag{1}$$

This is done by using the Adam Moulton Methods at both step and off-step points. These methods are then applied as BVMs and used to solve the BVP of the form:

$$y'(x) = f(x, y(x)) \tag{2}$$

$$\begin{aligned} a_0 y(0) \pm b_0 y(0) &= \alpha_0 \\ a_1 y(1) \pm b_1 y(1) &= \alpha_1 \end{aligned} \tag{3}$$

where all $f : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions that satisfy the existence and uniqueness conditions, guaranteed by Henrici in [14].

Several authors have proposed different hybrid formulas for both LMMs and BVMs and applied them to solve different differential problems [15 – 22].

The application of BVMs for the numerical integration of BVPs was first proposed by Brugnano and Trigiante in [23] with the two symmetric schemes: Extended Trapezoidal Rule (ETRs) of order 4 and 6.

The remaining part of the paper will be structured as follows: in section 2, we derive the HyBVMs of order 4 and 6 and also discuss some of their properties. In section 3, we apply the HyBVMs to solve some BVPs (both linear and nonlinear) and compare using tables and graphs with other methods. Finally, we give a concluding remark in section 4.

2 Hybrid Boundary Value Methods

In this section, we present the HyBVMs (of order 4 and 6) with some of their properties. These methods were developed with Mathematica 9.

2.1 Derivation of HyBVMs

These HyBVMs are generalizations of the hybrid Adams-Moulton (AM) Methods. The hybrid AM can be written as:

$$y_{n+k} - y_{n+k-1} = h \sum_{i=0\left(\frac{1}{2}\right)}^{k} \beta_i f_{n+i} \tag{4}$$

These methods are normally used as IVMs but not as BVMs and have been used in the past for the approximation of ODEs and other differential problems.

We begin by constructing the continuous Adams Moulton method (1) using the interpolation and collocation technique and evaluating at off-step point, which result into (4) above.

However, if we choose $k = v$ in (4)

where

$$v = \begin{cases} \dfrac{k}{2}, & \text{if } k \text{ is even} \\ \dfrac{k+1}{2}, & \text{if } k \text{ is odd} \end{cases} \tag{5}$$

we then obtain the HyBVMs.

For instance, the HyBVMs with odd number (k=1) of steps have the form:

$$y_{n+v} - y_{n+v-1} = h \sum_{r=0\left(\frac{1}{2}\right)}^{2v-1} \beta_r f_{n+r} \tag{6}$$

with the polynomial of the form: $p(z) = z^{v-1}(z-1)$. They are to be used with $(v-1, v)$ boundary conditions with order *2k+2*.

Below are the two HyBVMs

Example 1: The fourth order HyBVM is given as:

$$y_{n+1} - y_n = \frac{h}{6}\left[f_n + f_{n+1} + 4 f_{n+\frac{1}{2}} \right] \tag{7}$$

which is to be used together in tandem with the final method:

$$y_{N-\frac{1}{2}} - y_N = \frac{h}{24}\left[5 f_N - f_{N-1} + 8 f_{N-\frac{1}{2}} \right] \tag{8}$$

Example 2: The sixth order HyBVM is given as:

$$y_{n+2} - y_n = \frac{h}{45}\begin{bmatrix} 7 f_n + 32 f_{n+\frac{1}{2}} + 12 f_{n+1} \\ +32 f_{n+\frac{3}{2}} + 7 f_{n+2} \end{bmatrix} \tag{9}$$

which is to be used together in tandem with the initial methods:

$$y_{\frac{1}{2}} - y_0 = \frac{h}{1440}\begin{bmatrix} 251 f_0 + 646 f - 264 f_1 \\ +106 f_{\frac{3}{2}} - 19 f_2 \end{bmatrix} \tag{10}$$

and the final methods

$$y_{N-1} - y_N = -\frac{h}{180}\begin{bmatrix} 29 f_N + 124 f_{N-\frac{1}{2}} + 24 f_{N-1} \\ +4 f_{N-\frac{3}{2}} - f_{N-2} \end{bmatrix} \tag{11}$$

$$y_{N-\frac{3}{2}} - y_N = -\frac{h}{160}\begin{bmatrix} 27 f_N + 102 f_{N-\frac{1}{2}} + 72 f_{N-1} \\ +42 f_{N-\frac{3}{2}} - 3 f_{N-2} \end{bmatrix} \tag{12}$$

2.2 Analysis of BVMs [24]

Here, we highlight some of the properties associated with BVMs.

Definition 1: A polynomial $p(z)$ of degree $k = k_1 + k_2$ is an S_{k_1,k_2}- polynomial if its roots are

such that $|z_1| \le |z_2| \le \ldots \le |z_{k_1}| < 1 < |z_{k_1+1}| \le \ldots \le |z_k|$
.

Definition 2: A polynomial $p(z)$ of degree $k = k_1 + k_2$ is an N_{k_1,k_2} - polynomial if its roots are such that $|z_1| \le |z_2| \le \ldots \le |z_{k_1}| \le 1 < |z_{k_1+1}| \le \ldots \le |z_k|$ with simple roots of unit modulus.

Definition 3: A BVM with (k_1, k_2) - boundary conditions is $0_{k_1,k_2}$ - stable if its corresponding polynomial $p(z)$ is an N_{k_1,k_2} - polynomial.

Remark: The obtained formulas (7) and (9) have their first characteristic polynomials $p(z) = z - 1$ and $p(z) = (z-1)(z+1)$, respectively, which are both N_{k_1,k_2} polynomials.
Hence, they are $0_{k_1,k_2}$ stable.

3 Numerical Tests and Discussion

In this section, we apply the HyBVMs of order 4 and 6 stated in the section above to three (3) first order systems of BVPs using Mathematica software to generate the approximate values. Figures 1, 2 and 3 show the relationship between the exact and the approximate solutions for the three cases respectively. Their maximum errors and Rate of Convergence (ROC) are also compared with the ETRs of order 4 and 6 in the tables below.

Problem 1: Consider the nonlinear second order BVP [25]:

$$y'' = \frac{(y')^2 + y^2}{2e^x} \quad , \quad x \in (0,1)$$

with boundary conditions:

$$y(0) - y'(0) = 0$$

$$y(1) + y'(1) = 2e$$

and with exact solution: $y(x) = e^x$

To solve, we first recast to its equivalent first order system:

$$y_1' = y_2$$

$$y_2' = \frac{(y_2)^2 + (y_1)^2}{2e^x}$$

for $x \in (0,1)$

with boundary conditions:

$$y_1(0) - y_2(0) = 0$$

$$y_1(1) + y_2(1) = 2e$$

with exact solutions: $y_1(x) = e^x$, $y_2(x) = e^x$

Problem 2: Consider the linear second order BVP [23]:

$$y'' - 4y = 16x + 12x^2 - 4x^4 \quad , \quad x \in (0,1)$$

with boundary conditions:

$$y(0) = y'(1) = 0$$

with exact solution: $y(x) = x^4 - 4x$

To solve, we first recast to its equivalent first order system:

$$y_1' = y_2$$

$$y_2' = 4y_1 + 16x + 12x^2 - 4x^4$$

for $x \in (0,1)$

with boundary conditions:

$$y_1(0) = 0, \quad y_2(1) = 0$$

with exact solutions: $y_1(x) = e^x$, $y_2(x) = e^x$

Problem 3: Consider the nonlinear BVP [25]:

$$y'' = \frac{e^{2y} + (y')^2}{2} \quad , \quad x \in (0,1)$$

with boundary conditions:

$$y(0) - y'(0) = 1$$

$$y(1) + y'(1) = -ln\,2 - \frac{1}{2}$$

with exact solution: $y(x) = \log \frac{1}{1+x}$

To solve, we first recast to its equivalent first order system:

$$y_1' = y_2$$

$$y_2' = \frac{e^{2y_1} + (y_2)^2}{2}$$

for $x \in (0,1)$

with boundary conditions:

$$y_1(0) - y_2(0) = 1$$

$$y_1(1) + y_2(1) = -ln\,2 - \frac{1}{2}$$

with exact solutions:

$$\begin{cases} y_1(x) = \log \dfrac{1}{1+x} \\ y_2(x) = -\dfrac{1}{1+x} \end{cases}$$

RESULTS AND DISCUSSION

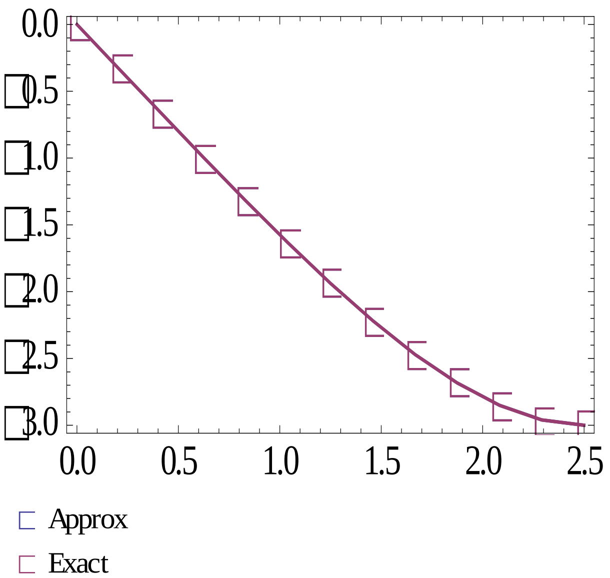

Fig. 1: Exact and Approximate Solutions for Problem 1

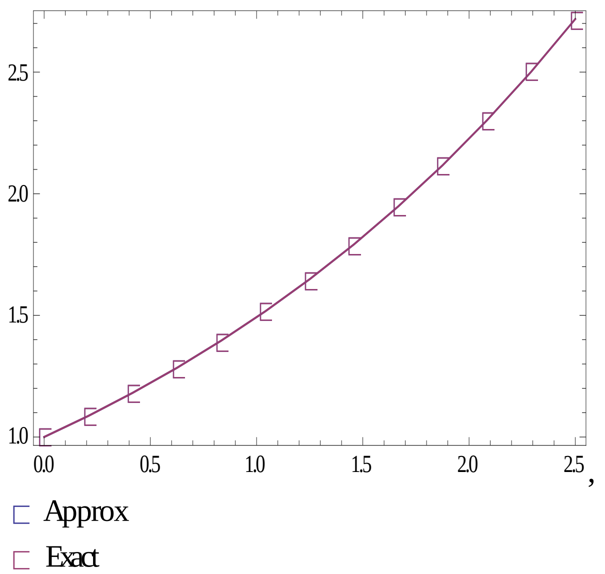

Fig. 2: Exact and Approximate Solutions for Problem 2

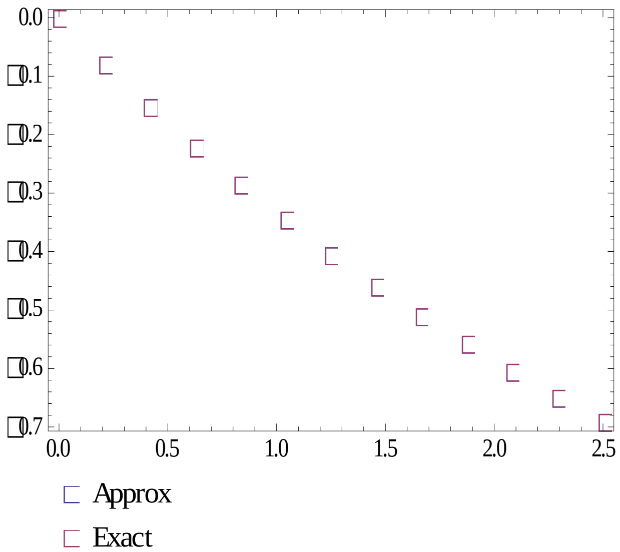

Fig. 3: Exact and Approximate Solutions for Problem 3

Table 1: Maximum errors and ROC for HyBVM of order 4 for Problem 1

N	HyBVM of Order 4	
	$\|e\|_\infty$	ROC
20	$4.286e-08$	–
40	$2.734e-09$	3.97
80	$1.727e-10$	3.98
160	$1.085e-11$	3.99

Table 2: Maximum errors and ROC for HyBVM of order 6 for Problem 1

N	HyBVM of Order 6	
	$\|e\|_\infty$	ROC
20	$1.592e-11$	–
40	$2.495e-13$	6.00
80	$4.082e-15$	5.93
160	$6.280e-16$	2.70

Table 3: Maximum errors and ROC of ETR of order 6 for Problem 1

N	ETR of Order 6	
	$\|e\|_\infty$	ROC
20	$7.448e-08$	–
40	$1.480e-09$	5.65
80	$2.576e-11$	5.84
160	$4.234e-13$	5.93

Table 4: Maximum errors and ROC of HyBVM of order 4 for Problem 2

N	HyBVM Order 4	
	$\|e\|_\infty$	ROC
4	$4.385e-4$	–
8	$2.848e-5$	3.94
16	$1.812e-6$	3.97
32	$1.143e-7$	3.99
64	$7.171e-9$	3.99

Table 5: Maximum errors and ROC of HyBVM of order 6 for Problem 2

N	HyBVM Order 6	
	$\|e\|_\infty$	ROC
4	$1.332e-16$	–
8	$6.280e-16$	1.08
16	$4.965e-16$	0.34
32	$4.578e-16$	0.12
64	$1.831e-15$	2.00

Table 6: Maximum errors and ROC of ETR of order 4 for Problem 2

N	ETR Order 4	
	$\|e\|_\infty$	ROC
4	$2.628e-3$	–
8	$1.955e-4$	3.75
16	$1.359e-5$	3.85
32	$8.989e-7$	3.92
64	$5.785e-8$	3.96

Table 7: Maximum errors and ROC of HyBVM of order 4 for Problem 3

N	HyBVM Order 4	
	$\|e\|_\infty$	ROC
4	$1.642e-5$	–
8	$1.074e-6$	3.93
16	$6.797e-8$	3.98
32	$4.264e-9$	3.99
64	$2.669e-10$	4.00

Table 8: Maximum errors and ROC of HyBVM of order 6 for Problem 3

N	HyBVM Order 6	
	$\|e\|_\infty$	ROC
4	$2.189e-5$	–
8	$4.762e-7$	5.52
16	$9.261e-9$	5.68
32	$1.633e-10$	5.83
64	$2.719e-12$	5.91

Tables 1, 2 and 3 concern problem 1 and they show the Rate of Convergence (ROC) with the maximum error for ETRs of order 6, HyBVM of order 4 and HyBVM of order 6.
As seen from these tables, the ROC for HyBVM of order 4, 6 and the ETR of order 6 are all consistent with their order.
Table 4 – 6 concern problem 2 and they show the ROC with the maximum error for ETRs of order 4, HyBVM of order 4 and HyBVM of order 6.
These tables revealed that the ROC for HyBVM of order 4 and the ETR of order 6 are consistent with their order while the ROC for HyBVM of order 6 does not appear to be consistent with its order. Moreover, the numerical tests show that the HyBVM of order 4 is more accurate than the other two methods.
Table 7 – 8 concern problem 3 and they show the ROC with the maximum error for HyBVM of order 4 and HyBVM of order 6. As seen from these tables,

the ROC for HyBVM of order 4 and order 6 are consistent with their orders.

4 Conclusion

In this work, we have applied a sixth-order HyBVM to two systems of BVPs and compared the maximum error and rate of convergence of the solutions with other two BVMs: ETR and TOM called symmetric schemes. In constructing these methods, we have adopted the Adams Moulton methods derived through interpolation and collocation procedure by utilizing data at both step and off-step points and implemented them as BVMs.

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Conflict of Interests

The authors declare no conflict of interest with regards to the publication of this paper.

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