A Weighted Generalized Prime Geodesic Theorem

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Abstract: Prime geodesic theorem gives an asymptotic estimate for the number of prime geodesics over underlying symmetric space counted by their lengths. In any setting, the search for the optimal error term is widely open. Our objective is to derive a weighted, generalized form of the prime geodesic theorem for compact, even-dimensional, locally symmetric Riemannian manifolds of strictly negative sectional curvature. We base our methodology on an application of the integrated, Chebyshev-type counting function of appropriate order. The obtained error term improves the corresponding, and best known one in the case of classical prime geodesic theorem. Our conclusion in the case at hand is that a weighted sense yields a better result.

Key–Words: Weighted prime geodesic theorem, counting functions, zeta functions, topological singularities, spectral singularities

1 Introduction

Recently, the authors in [2], improved the error term in DeGeorge's prime geodesic theorem [4], for compact, n-dimensional, locally symmetric Riemannian manifolds Y with strictly negative sectional curvature (see, [8] for yet another proof of the same result in the even-dimensional case).

The prime geodesic theorem [2], [8] states that

$$\begin{aligned} &\pi_\Gamma(x) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1\times \\ &\times \sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]} \operatorname{li}\left(x^{s^{p,\tau,\lambda}}\right) + \\ &O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}} (\log x)^{-1}\right) \end{aligned} \tag{1}$$

as $x \to +\infty$, where $\pi_\Gamma(x)$ is the number of prime geodesics over Y of the length at most $\log x$, and $\operatorname{li}(x) = \int_2^x \frac{dt}{\log t}$.

Let us explain this into more details.

Y can be represented as a double coset space $\Gamma\backslash G/K = \Gamma\backslash X$, where G is a connected, linear, semi-simple Lie group of real rank one, K is a maximal compact subgroup of G, and Γ is a discrete, co-compact, torsion-free subgroup of G.

Since Γ is co-compact and torsion-free, there are only two types of conjugacy classes: the class of the identity $e \in \Gamma$ and classes of hyperbolic elements.

It is known that every hyperbolic element $g \in G$ is conjugated to some element $a_g m_g \in A^+M$, where $A^+ = \exp(\mathfrak{a}^+)$, $\mathfrak{a}^+$ is the half line in $\mathfrak{a}$ on which the positive roots take positive values, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{p}$ is given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, $\Phi^+(\mathfrak{g},\mathfrak{a}) \subset \Phi(\mathfrak{g},\mathfrak{a})$ is a system of positive roots, $\Phi(\mathfrak{g},\mathfrak{a})$ is the root system, and M is the centralizer of $\mathfrak{a}$ in K (see, [3], and e.g., [7]).

As it is also known, a prime geodesic over Y corresponds to a conjugacy class of a primitive hyperbolic element in Γ.

If $\gamma \in \Gamma$ is a primitive hyperbolic element, then the corresponding prime geodesic over Y is denoted by C_γ.

Now, for $g \in \Gamma$, the number $l(g) = l(a_g m_g) = |\log a_g|$ is the length of the closed geodesic over Y determined by g.

One can see that $\pi_\Gamma(x)$ is the number of prime geodesics C_γ over Y, whose norm $N(\gamma) = e^{l(\gamma)}$ is not larger than x.

ρ is defined by

$$\rho = \frac{1}{2} \sum_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})} \dim(\mathfrak{n}_\alpha)\,\alpha,$$

where

$$\mathfrak{n} = \sum_{\alpha \in \Phi^{+}(\mathfrak{g},\mathfrak{a})} \mathfrak{n}_{\alpha}$$

is the sum of the root spaces.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \rho$ resp. $\mathrm{Re}(s) > 2\rho$, the Selberg Zeta function $Z_{S,\chi}(s,\sigma)$ resp. the Ruelle zeta function $Z_{R,\chi}(s,\sigma)$ is defined by the infinite product (see, [3, pp. 96-97])

$$\begin{aligned} &Z_{S,\chi}(s,\sigma) \\ &= \prod_{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \prod_{k=0}^{+\infty} \det\Bigg(1 - \Big(\sigma(m_{\gamma_0}) \otimes \chi(\gamma_0) \otimes \\ &\quad S^{k}\left(\mathrm{Ad}(m_{\gamma_0} a_{\gamma_0})_{\bar{\mathfrak{n}}}\right)\Big) e^{-(s+\rho)l(\gamma_0)}\Bigg), \end{aligned}$$

resp.

$$\begin{aligned} &Z_{R,\chi}(s,\sigma) \\ &= \prod_{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \det\Bigg(1 - \Big(\sigma(m_{\gamma_0}) \otimes \chi(\gamma_0)\Big) \times \\ &\quad \times e^{-sl(\gamma_0)}\Bigg)^{-1}, \end{aligned}$$

where S^k is the $k-$th symmetric power of an endomorphism, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, θ is the Cartan involution of $\mathfrak{g}$, σ and χ are finite-dimensional unitary representations of M and Γ, respectively, and Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ is the set of $\Gamma-$conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

By Fried [6], the Ruelle zeta function can be expressed as a product of Selberg zeta functions.

In our case, there are sets

$$I_p = \left\{(\tau,\lambda) \mid \tau \in \hat{M}, \lambda \in \mathbb{R}\right\}$$

such that $\wedge^{p}\mathfrak{n}_{\mathbb{C}}$, $p \geq 0$ (considered as a representation of MA) decomposes with respect to MA as

$$\wedge^{p}\mathfrak{n}_{\mathbb{C}} = \sum_{(\tau,\lambda)\in I_p} V_{\tau} \otimes \mathbb{C}_{\lambda},$$

where A follows from the Iwasawa decomposition $G = KAN$ that corresponds to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, V_{τ} is the space of the representation τ, $\mathfrak{n}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{n}$ of N (available since G is a linear group), and $\mathbb{C}_{\lambda}$ is the one-dimensional representation of A given by $A \ni a \to a^{\lambda} \cong \mathfrak{a}_{\mathbb{C}}^{*}$ (see, [3, p. 99]).

Now,

$$\begin{aligned} &Z_{R,\chi}(s,\sigma) \\ &= \prod_{p=0}^{n-1} \prod_{(\tau,\lambda)\in I_p} Z_{S,\chi}(s+\rho-\lambda, \tau\otimes\sigma)^{(-1)^{p}}. \end{aligned}$$

Finally, $s^{p,\tau,\lambda}$ in (1) denotes a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$, where $Z_S(s+\rho-\lambda,\tau)$ is obtained from $Z_{S,\chi}(s+\rho-\lambda,\tau\otimes\sigma)$ by fixing $\chi \in \hat{\Gamma}$, $\sigma \in \hat{M}$ and omitting them in the notation.

The relation (1) follows easily from the relation

$$\begin{aligned} &\psi_0(x) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1 \times \\ &\quad \times \sum_{s^{p,\tau,\lambda} \in \left(2\rho\frac{n+\rho-1}{n+2\rho-1}, 2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} + \\ &\quad O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right), \end{aligned} \tag{2}$$

where the counting function $\psi_0(x)$ is defined by

$$\psi_0(x) = \sum_{\gamma\in\Gamma_{\mathrm{h}}, N(\gamma)\leq x} \Lambda_0(\gamma),$$

and

$$\Lambda_0(\gamma) = \log N(\gamma_0)$$

for $\gamma \in \Gamma_{\mathrm{h}}$, $\gamma = \gamma_0^{n_{\Gamma}(\gamma)}$ (see, e.g., [12, p. 102]).

Here, we use the fact that $\gamma \in \Gamma_{\mathrm{h}}$ is of the form $\gamma = \gamma_0^{n_{\Gamma}(\gamma)}$ for some $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$, where $n_{\Gamma}(g) = \#\left(\Gamma_g / \langle g \rangle\right)$, Γ_g is the centralizer of g in Γ, and $\langle g \rangle$ is the group generated by g.

Therefore, it is enough to prove (2) to have (1) (see, [2, p. 317], [1, p. 368]), i.e., it is enough to prove that

$$\begin{aligned} &\sum_{\gamma\in\Gamma_{\mathrm{h}}, N(\gamma)\leq x} \Lambda_0(\gamma) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} 1 \times \end{aligned} \tag{3}$$

$$\times \sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}+ O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right),$$

holds true.

The relation (1) is widely known as a prime geodesic theorem. Since it follows from the relation (3), we shall call the relation (3) by the same name.

In order to obtain (3) (or (2)), one usually uses a higher order counting function $\psi_j(x)$, $j \in \mathbb{N}$, where $\psi_j(x)$ is defined recursively by

$$\psi_j(x) = \int_0^x \psi_{j-1}(t)\,dt, \quad j \in \mathbb{N}.$$

By [11, p. 18, Th. A],

$$\psi_j(x) = \frac{1}{j!} \sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x} \Lambda_0(\gamma)(x - N(\gamma))^j.$$

In this paper we pay attention to function

$$\psi_1(x) = \sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x} \Lambda_0(\gamma)(x - N(\gamma)).$$

Since $\psi_1(x)$ is integrated $\psi_0(x)$, we pay our particular attention to function $\frac{\psi_1(x)}{x}$.

We shall prove that the counting function

$$\frac{\psi_1(x)}{x} = \sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x} \Lambda_0(\gamma)\left(1 - \frac{N(\gamma)}{x}\right).$$

yields the error term

$$O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}-\frac{\rho}{n+2\rho-1}}\right)$$

in a variant of (3), i.e., a better result than the one obtained by an application of the counting function

$$\psi_0(x) = \sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x} \Lambda_0(\gamma).$$

Such result is usually called a weighted, generalized prime geodesic theorem.

2 Motivation and main ideas overview

In this section we review the main steps that have led us to the result (1) in [2].

In order to obtain (1), i.e., (2), we have used the higher order counting function $\psi_{2n}(x)$.

Consequently, to move from the level $\psi_{2n}(x)$ to the level $\psi_0(x)$, we applied the order $2n$ differential operator Δ_{2n}^{+}.

There, we assumed that $1 \leq h \leq \frac{x}{2}$.

We have derived the equation (12) in [2], where $A^{p,\tau,\lambda}$ is the set of poles of

$$(\log Z_S(s+\rho-\lambda,\tau))' \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)},$$

and $c_z(p,\tau,\lambda)$ is the residue at $s = z$.

Having in mind the singularity pattern [3, p. 113, Th. 3.15] of $Z_S(s+\rho-\lambda,\tau)$, we divided $A^{p,\tau,\lambda}$ into:

1. $I_{p,\tau,\lambda}$, the set of singularities z of $Z_S(s+\rho-\lambda,\tau)$ such that $z \in I_{-2n} = \{0,-1,...,-2n\}$ (note that these singularities are the only poles of $(\log Z_S(s+\rho-\lambda,\tau))' \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)}$ of order two),

2. $I'_{p,\tau,\lambda} = I_{-2n} \setminus I_{p,\tau,\lambda}$,

3. $S^{p,\tau,\lambda}$, the remaining singularities of $Z_S(s+\rho-\lambda,\tau)$.

Hence, following Hejhal [9], we calculated $c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$ for $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}$, $c_{-j}(p,\tau,\lambda)$ for $-j \in I_{p,\tau,\lambda}$ and $-j \in I'_{p,\tau,\lambda}$ (see, (13)-(15) in [2]).

There, $o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}$ is the order of the singularity $s^{p,\tau,\lambda}$, and $a^{p,\tau,\lambda}_{1,-j}$ is given by the series expansion of $(\log Z_S(s+\rho-\lambda,\tau))'$ at $-j$.

Note the following:

(a) the error term in (2) is $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$,

(b) $\psi_0(x) \leq h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)$,

(c) $h^{-2n}\Delta_{2n}^{+}x^{s^{p,\tau,\lambda}+2n} = \left(s^{p,\tau,\lambda}+2n\right)\left(s^{p,\tau,\lambda}+2n-1\right)\ldots\left(s^{p,\tau,\lambda}+1\right)\cdot \tilde{x}^{s^{p,\tau,\lambda}}$ for some $\tilde{x} \in [x, x+2nh]$.

In other words: $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ is the error term we were looking for in the explicit expression (2) for $\psi_0(x)$, the final form (2) for $\psi_0(x)$ follows from $h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)$, and $h^{-2n}\Delta_{2n}^{+}$ acts on $x^{s^{p,\tau,\lambda}+2n}$ very similarly to the classical differential operator.

We further divided $A^{p,\tau,\lambda}$, by introducing the sets: $B_{p,\tau,\lambda}$, $B'_{p,\tau,\lambda}$, $C^k_{p,\tau,\lambda}$, $1 \leq k \leq 4$, and $S^{p,\tau,\lambda}_{-\rho+\lambda} = S^{p,\tau,\lambda} \setminus S^{p,\tau,\lambda}_{\mathbb{R}}$, where $S^{p,\tau,\lambda}_{\mathbb{R}} = S^{p,\tau,\lambda} \cap \mathbb{R}$.

The corresponding equation in [2] is (16).

Clearly, an analogous formula for

$$h^{-2n}\Delta_{2n}^{+}\sum_{z\in A^{p,\tau,\lambda}} c_z(p,\tau,\lambda)$$

holds also true.

Let us explain our motivation to introduce the aforementioned sets.

Since (a) and (b) hold true, we required that either the corresponding sum in $\psi_{2n}(x)$ or the corresponding sum in $h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)$ be of the size $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$.

In particular, we required that the sum in $h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)$ that corresponds to $C^4_{p,\tau,\lambda}$ be of the form: an explicit term larger than $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ plus the error term $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$. More precisely, $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ size summands in $\psi_{2n}(x)$ are replaced by $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$, the operator $h^{-2n}\Delta_{2n}^{+}$ is applied to the newly-acquired $\psi_{2n}(x)$, and, finally, $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ size summands in $h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)$ are replaced by $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$. The obtained formula is the formula (2).

X is the universal covering of Y. It is a Riemannian symmetric space of rank one. Therefore, it is a real, complex or quaternionic hyperbolic space or the hyperbolic Cayley plane.

We assume that the Riemannian metric over Y induced from the Killing form is normalized such that the sectional curvature of Y varies between -4 and -1. Therefore, $n = k$, $2m$, $4l$, 16 and $\rho = \frac{1}{2}(k-1)$, m, $2l+1$, 11 for $X = H\mathbb{R}^k$ ($k \geq 2$, k even), $H\mathbb{C}^m$ ($m \geq 2$), $H\mathbb{H}^l$ ($l \geq 2$), $H\mathbb{C}a^2$, respectively. Here we assume that $m \geq 2$ and $l \geq 2$ since $H\mathbb{C}^1 \cong H\mathbb{R}^2$ and $H\mathbb{H}^1 \cong H\mathbb{R}^4$ (see, e.g., [10]).

Note that, $c_{-2n}(p,\tau,\lambda)$ is $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$. On the other side, $2n > 2\rho$ and $\frac{n+\rho-1}{n+2\rho-1} < 1$ yield that $c_0(p,\tau,\lambda)$ is not $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$. In other words, the definition of the set $B_{p,\tau,\lambda}$ is a natural one. It determines the complement $B'_{p,\tau,\lambda}$ of $B_{p,\tau,\lambda}$ in I_{-2n} uniquely.

Recall Theorem 3.15 in [3, pp. 113-115]. The topological singularities of $Z_S(s+\rho-\lambda,\tau)$ are less than $-\rho+\lambda$. There are infinite many of them since they are generated via the lattice $L(\sigma)$ introduced in [3, p. 47, Def. 1.13]. The spectral singularities of $Z_S(s+\rho-\lambda,\tau)$ are contained in the union of the interval $[-2\rho+\lambda,\lambda]$ with the line $-\rho+\lambda+i\,\mathbb{R}$. There may be an overlap of the topological and the spectral singularities at finite many points inside $[-2\rho+\lambda,-\rho+\lambda)$.

Note that $z \leq -2n-1 < -2\rho+\lambda$ for $z \in C^1_{p,\tau,\lambda}$. Moreover, $z \in S^{p,\tau,\lambda}_{\mathbb{R}} \subset S^{p,\tau,\lambda}$ for $z \in C^1_{p,\tau,\lambda}$, i.e., $C^1_{p,\tau,\lambda} \cap I_{-2n} = \emptyset$. In other words, we introduced $C^1_{p,\tau,\lambda}$ to deal with infinite many simple poles of $(\log Z_S(s+\rho-\lambda,\tau))' \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)}$ given as topological singularities of $Z_S(s+\rho-\lambda,\tau)$.

The set $C^2_{p,\tau,\lambda} \subset S^{p,\tau,\lambda}_{\mathbb{R}} \subset S^{p,\tau,\lambda}$ is introduced to agree with $c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$, $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}$, and thus automatically enable a cancellation of exponents, i.e., to produce the estimate $c_z(p,\tau,\lambda) = O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ for $z \in C^2_{p,\tau,\lambda}$.

Similarly, the set $C^3_{p,\tau,\lambda} \subset S^{p,\tau,\lambda}$ is introduced to agree with $c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$, $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}$ and (c), i.e., to automatically produce the estimate $h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda) = O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ for $z \in C^3_{p,\tau,\lambda}$.

Having in mind what we said above about the spectral singularities of $Z_S(s+\rho-\lambda,\tau)$, the definition of $S^{p,\tau,\lambda}_{-\rho+\lambda}$ is also natural one.

The remaining poles of $(\log Z_S(s+\rho-\lambda,\tau))' \frac{x^{s+2n}}{\prod_{k=0}^{2n}(s+k)}$ are denoted by $C^4_{p,\tau,\lambda}$.

$C^4_{p,\tau,\lambda}$ also agrees with $c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$, $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}$ and (c).

Consequently, (see, e.g., [13, p. 246], [12, p. 101], [2, p. 316]), the equation (32) in [2] holds true.

Furthermore, the equation (35) in [2] holds true.

There, $O\left(x^{-\rho+\lambda}M^{n-1}\right)$ and $O\left(h^{-2n}x^{-\rho+\lambda+2n}M^{-n-1}\right)$ follow from the estimates (34) and (33) in [2], respectively, where $M > 2\rho$.

We notice that the sets $C^4_{p,\tau,\lambda}$ and $S^{p,\tau,\lambda}_{-\rho+\lambda}$ are responsible for achieving the error term $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ in (2).

Namely, $O\left(h^{2\rho}\right)$, $O\left(x^{-\rho+\lambda}M^{n-1}\right)$ and $O\left(h^{-2n}x^{-\rho+\lambda+2n}M^{-n-1}\right)$ all become $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$ for the choice $h = x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M = x^{\frac{\rho}{n+2\rho-1}}$. This actually means that once, the estimates (32) and (35) in [2] are established, and the corresponding error term is determined, the aforementioned sets may be defined. Note that the lower bound of the interval in [2, (32)] does not have to be explicitly known immediately (it is known after the error term is known). This fact, however, does not prevent us from establishing this relation. It is enough to temporarily replace the set $C^4_{p,\tau,\lambda}$ with some set

$$O^{\varepsilon}_{2\rho} = \left\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid 2\rho - \varepsilon < s^{p,\tau,\lambda} \leq 2\rho \right\},$$

where $\varepsilon > 0$ is fixed and small.

The corresponding relation in the case at hand is the following.

$$\sum_{z \in O^{\varepsilon}_{2\rho}} h^{-2n}\Delta^{+}_{2n} c_z\left(p,\tau,\lambda\right) = \sum_{z \in O^{\varepsilon}_{2\rho}} \frac{x^z}{z} + O\left(h^{2\rho}\right).$$

The goal of this paper is to derive a formula analogous to the formula (3) by moving from the level $\psi_{2n}(x)$ to the level $\psi_1(x)$. The method described in this section will be followed and adapted when necessary. In particular, the operator $h^{-2n}\Delta^{+}_{2n}$ will be replaced by $h^{-(2n-1)}\Delta^{+}_{2n-1}$.

3 Main Result

The following theorem is the main result of this paper.

Theorem 1. *(Weighted Generalized Prime Geodesic Theorem) Let Y be as above. Then,*

$$\sum_{\gamma \in \Gamma_h, N(\gamma) \leq x} \Lambda_0(\gamma)\left(1 - \frac{N(\gamma)}{x}\right)$$
$$= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} 1 \times$$
$$\times \sum_{s^{p,\tau,\lambda} \in \left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1} - 1, 2\rho\right]} \frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} +$$
$$O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1} - \frac{\rho}{n+2\rho-1}}\right)$$

as $x \to +\infty$, where $s^{p,\tau,\lambda}$ is a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$.

Proof. If f is at least $2n-1$ times differentiable function, then

$$\Delta^{+}_{2n-1} f(x) = \int_{x}^{x+h} \int_{t_{2n-1}}^{t_{2n-1}+h} \ldots \int_{t_2}^{t_2+h} f^{(2n-1)}(t_1) \times \times dt_1 \ldots dt_{2n-1}. \tag{4}$$

Hence, by the mean value theorem

$$\begin{aligned} &\Delta^{+}_{2n-1} f(x) \\ &= h \int_{x}^{x+h} \int_{t_{2n-1}}^{t_{2n-1}+h} \ldots \int_{t_3}^{t_3+h} f^{(2n-1)}(t_2+\alpha_1) \times \\ &\quad \times dt_2 \ldots dt_{2n-1} \\ &= \ldots \\ &= h^{2n-1} f^{(2n-1)}\left(x+\alpha_{2n-1}\right), \end{aligned}$$

where $\alpha_i \in [0, ih]$. Put $\tilde{x} = x + \alpha_{2n-1}$. We conclude,

$$\Delta^{+}_{2n-1} f(x) = h^{2n-1} f^{(2n-1)}(\tilde{x}), \tag{5}$$

where $\tilde{x} \in [x, x+(2n-1)h]$.

Consider the set

$$O^{\varepsilon}_{2\rho} = \left\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid 2\rho - \varepsilon < s^{p,\tau,\lambda} \leq 2\rho \right\},$$

where $\varepsilon > 0$ is fixed and small. Let $s^{p,\tau,\lambda} \in O^{\varepsilon}_{2\rho}$. By (5),

$$\begin{aligned} &h^{-(2n-1)}\Delta^{+}_{2n-1} c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right) \\ &= o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}} \frac{\tilde{x}_{s^{p,\tau,\lambda}}^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} \end{aligned}$$

for some $\tilde{x}_{s^{p,\tau,\lambda}} \in [x, x+(2n-1)h]$. Now, reasoning as in [13, p. 246] or [12, p. 101], one obtains

$$\begin{aligned} &\sum_{s^{p,\tau,\lambda} \in O^{\varepsilon}_{2\rho}} h^{-(2n-1)}\Delta^{+}_{2n-1} c_{s^{p,\tau,\lambda}}\left(p,\tau,\lambda\right) \\ &= \sum_{s^{p,\tau,\lambda} \in O^{\varepsilon}_{2\rho}} \frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} + O\left(h^{2\rho+1}\right). \end{aligned} \tag{6}$$

Consider the set $S^{p,\tau,\lambda}_{-\rho+\lambda}=S^{p,\tau,\lambda}\backslash S^{p,\tau,\lambda}_{\mathbb{R}}$. Let $z\in S^{p,\tau,\lambda}_{-\rho+\lambda}$.

Since,

$$\begin{aligned}&\Delta^{+}_{2n-1}f(x)\\&=\sum_{i=0}^{2n-1}(-1)^{i}\binom{2n-1}{i}\times\\&\times f(x+(2n-1-i)h),\end{aligned}$$

we have that

$$\begin{aligned}&h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\\&=h^{-(2n-1)}\frac{o_z^{p,\tau,\lambda}}{z(z+1)\ldots(z+2n)}\times\\&\times\sum_{i=0}^{2n-1}(-1)^{i}\binom{2n-1}{i}\times\\&\times(x+(2n-1-i)h)^{z+2n}\\&=O\left(h^{-(2n-1)}|z|^{-2n-1}x^{-\rho+\lambda+2n}\right)\\&=O\left(h^{-(2n-1)}|z|^{-2n-1}x^{\rho+2n}\right).\end{aligned}\tag{7}$$

On the other side, by (4)

$$\begin{aligned}&h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\\&=h^{-(2n-1)}\frac{o_z^{p,\tau,\lambda}}{z(z+1)}\times\\&\times\int_{x}^{x+h}\int_{t_{2n-1}}^{t_{2n-1}+h}\ldots\int_{t_2}^{t_2+h}t_1^{z+1}dt_1\ldots dt_{2n-1}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\right|\\&\le h^{-(2n-1)}\left|o_z^{p,\tau,\lambda}\right||z|^{-1}|z+1|^{-1}\times\\&\times\int_{x}^{x+h}\int_{t_{2n-1}}^{t_{2n-1}+h}\ldots\int_{t_2}^{t_2+h}t_1^{-\rho+\lambda+1}dt_1\ldots dt_{2n-1}.\end{aligned}$$

Applying the mean value theorem as well as the fact that $h\le\frac{x}{2}$, we obtain

$$\begin{aligned}&h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\\&=O\left(|z|^{-2}x^{-\rho+\lambda+1}\right)=O\left(|z|^{-2}x^{\rho+1}\right).\end{aligned}\tag{8}$$

Let $M>2\rho$. Now, using (7) and (8), we deduce

$$\begin{aligned}&\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\\&=\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|-\rho+\lambda|<|z|\le M}}h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)+\\&\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|z|>M}}h^{-(2n-1)}\Delta^{+}_{2n-1}c_z(p,\tau,\lambda)\\&=O\left(x^{\rho+1}\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|-\rho+\lambda|<|z|\le M}}|z|^{-2}\right)+\\&O\left(h^{-(2n-1)}x^{\rho+2n}\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|z|>M}}|z|^{-2n-1}\right)\\&=O\left(x^{\rho+1}\int_{|-\rho+\lambda|}^{M}t^{-2}dN_{p,\tau,\lambda}(t)\right)+\\&O\left(h^{-(2n-1)}x^{\rho+2n}\int_{M}^{+\infty}t^{-2n-1}dN_{p,\tau,\lambda}(t)\right)\\&=O\left(x^{\rho+1}M^{n-2}\right)+\\&O\left(h^{-(2n-1)}x^{\rho+2n}M^{-n-1}\right),\end{aligned}\tag{9}$$

where $N_{p,\tau,\lambda}(t)$ is the number of singularities of $Z_S(s+\rho-\lambda,\tau)$ on the interval $-\rho+\lambda+\mathrm{i}\,x$, $0<x\le t$.

Notice that $N_{p,\tau,\lambda}(t)=At^n+O\left(t^{n-1}(\log t)^{-1}\right)$ for some explicitly known constant A (see, e.g., [5, p. 89, Th. 9.1.]). However, the estimate $N_{p,\tau,\lambda}(t)=O(t^n)$ is sufficient for our needs.

The error terms $O\left(h^{2\rho+1}\right)$, $O\left(x^{\rho+1}M^{n-2}\right)$ and $O\left(h^{-(2n-1)}x^{\rho+2n}M^{-n-1}\right)$ that appear in (6) and (9) all become $O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right)$ for the choice $h=x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M=x^{\frac{\rho}{n+2\rho-1}}$.

Thus, we are able to introduce the sets $B_{p,\tau,\lambda}$, $B'_{p,\tau,\lambda}$, $C^{i}_{p,\tau,\lambda}$, $i\in\{1,2,3,4\}$.

We define,

$$B_{p,\tau,\lambda}$$
$$= \Big\{ -j \in I_{-2n} \mid c_{-j}(p,\tau,\lambda) =$$
$$O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right)\Big\},$$
$$B'_{p,\tau,\lambda} = I_{-2n} \backslash B_{p,\tau,\lambda}.$$

Obviously, $c_{-2n}(p,\tau,\lambda)$ is $O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right)$. Note that $2n = 2k$, $4m$, $8l$, 32 and $2\rho+1 = k$, $2m+1$, $4l+3$, 23 for $X = H\mathbb{R}^k$, $H\mathbb{C}^m$, $H\mathbb{H}^l$, $H\mathbb{C}a^2$, respectively. Hence, $2n > 2\rho+1$ and $\frac{n+\rho-1}{n+2\rho-1} < 1$ yield that $c_0(p,\tau,\lambda)$ is not $O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right)$. In this way, the definition of the set $B_{p,\tau,\lambda}$ is justified. The set $B_{p,\tau,\lambda}$ determines the set $B'_{p,\tau,\lambda}$ automatically.

We introduce the set

$$C^1_{p,\tau,\lambda} = \left\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid s^{p,\tau,\lambda} \le -2n-1 \right\}$$

in the same way as in the previous section. Namely, the sum that corresponds to this set in $\psi_{2n}(x)$ is $O\left(x^{-1}\right)$ (see, [2, p. 345, (17)]). This is satisfactory since the new error term $O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right)$ is not smaller than the old one $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)$.

We adapt the set

$$C^2_{p,\tau,\lambda}$$
$$= \Big\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid -2n-1 < s^{p,\tau,\lambda} \le$$
$$-2n + (2\rho+1)\frac{n+\rho-1}{n+2\rho-1}\Big\},$$

to the new error term.

In the previous section we introduced the set $C^3_{p,\tau,\lambda}$ to be in accordance with $c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$, $s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}$ and (c). Now, we adapt $C^3_{p,\tau,\lambda}$ to be in line with (5).

$$C^3_{p,\tau,\lambda}$$
$$= \Big\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid -2n + (2\rho+1)\frac{n+\rho-1}{n+2\rho-1} <$$
$$s^{p,\tau,\lambda} \le -1 + (2\rho+1)\frac{n+\rho-1}{n+2\rho-1}\Big\}.$$

The definition of the adapted set $C^4_{p,\tau,\lambda}$ follows automatically.

$$C^4_{p,\tau,\lambda}$$
$$= \Big\{ s^{p,\tau,\lambda} \in S^{p,\tau,\lambda}_{\mathbb{R}} \mid -1 + (2\rho+1)\frac{n+\rho-1}{n+2\rho-1} <$$
$$s^{p,\tau,\lambda} \le 2\rho\Big\}.$$

Note that,

$$-1 + (2\rho+1)\frac{n+\rho-1}{n+2\rho-1} < 2\rho$$

holds obviously true.

Now, reasoning in exactly the same way as in [2, p. 315], we deduce that

$$\sum_{z \in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{-1}\right). \tag{10}$$

By the very definition of the set $B_{p,\tau,\lambda}$,

$$\sum_{z \in B_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{11}$$

Finally, by the definition of $C^2_{p,\tau,\lambda}$,

$$\sum_{z \in C^2_{p,\tau,\lambda}} c_z(p,\tau,\lambda) = O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{12}$$

Combining (10)-(12), we obtain

$$\begin{aligned} &\psi_{2n}(x) \\ &= \sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}} 1\times \\ &\times c_z(p,\tau,\lambda) + \\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^3_{p,\tau,\lambda}} 1\times \\ &\times c_z(p,\tau,\lambda) + \\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^4_{p,\tau,\lambda}} 1\times \\ &\times c_z(p,\tau,\lambda) + \\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}} 1\times \end{aligned} \tag{13}$$

$$\times c_z\left(p,\tau,\lambda\right)+ O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

Substituting $C^4_{p,\tau,\lambda}$ into (6), we get

$$\begin{aligned}&\sum_{z\in C^4_{p,\tau,\lambda}} h^{-(2n-1)}\Delta^+_{2n-1}c_z\left(p,\tau,\lambda\right)\\ &=\sum_{s^{p,\tau,\lambda}\in\left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1}-1,2\rho\right]} 1\times\\ &\times\frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)}+O\left(h^{2\rho+1}\right).\end{aligned}\tag{14}$$

Now, by (13), (14) and (9)

$$\begin{aligned}&h^{-(2n-1)}\Delta^+_{2n-1}\psi_{2n}\left(x\right)\\ &=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}}1\times\\ &\times h^{-(2n-1)}\Delta^+_{2n-1}c_z\left(p,\tau,\lambda\right)+\\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^3_{p,\tau,\lambda}}1\times\\ &\times h^{-(2n-1)}\Delta^+_{2n-1}c_z\left(p,\tau,\lambda\right)+\\ &\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times\\ &\times\sum_{s^{p,\tau,\lambda}\in\left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1}-1,2\rho\right]}1\times\\ &\times\frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)}+O\left(h^{2\rho+1}\right)+\\ &O\left(x^{\rho+1}M^{n-2}\right)+\\ &O\left(h^{-(2n-1)}x^{\rho+2n}M^{-n-1}\right)+\\ &O\left(h^{-(2n-1)}x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned}\tag{15}$$

Recall the following formulas:

$$\begin{aligned}\left(x^n\log x\right)^{(n)}&=n!\log x+n!\sum_{l=1}^{n}\frac{1}{l},\\ \left(x^n\right)^{(n)}&=n!,\\ \left(x^k\log x\right)^{(n)}&=k!\left(-1\right)^{n-k-1}\frac{(n-k-1)!}{x^{n-k}},\\ &\quad 0\le k<n,\\ \left(x^k\right)^{(n)}&=0,\ 0\le k<n,\\ \left(x^n\log x\right)^{(k)}&=\frac{n!}{(n-k)!}x^{n-k}\log x+\\ &\quad\frac{n!}{(n-k)!}x^{n-k}\sum_{l=n-k+1}^{n}\frac{1}{l},\\ &\quad 1\le k<n,\\ \left(x^n\right)^{(k)}&=\frac{n!}{(n-k)!}x^{n-k},\ 1\le k<n.\end{aligned}$$

Consider the sum over $B'_{p,\tau,\lambda}$ on the right hand side of (15).

Let $z\in B'_{p,\tau,\lambda}$, $z=0$.

Suppose that $0\in I_{p,\tau,\lambda}$. Now,

$$\begin{aligned}c_0\left(p,\tau,\lambda\right)&=\frac{o_0^{p,\tau,\lambda}}{(2n)!}x^{2n}\log x\\ &+\frac{o_0^{p,\tau,\lambda}}{(2n)!}\left(-\sum_{l=1}^{2n}\frac{1}{l}+a_{1,0}^{p,\tau,\lambda}\right)x^{2n}.\end{aligned}$$

Applying the aforementioned formulas and (5), we deduce

$$\begin{aligned}&h^{-(2n-1)}\Delta^+_{2n-1}c_0\left(p,\tau,\lambda\right)\\ &=h^{-(2n-1)}\frac{o_0^{p,\tau,\lambda}}{(2n)!}\Delta^+_{2n-1}\left(x^{2n}\log x\right)+\\ &h^{-(2n-1)}\frac{o_0^{p,\tau,\lambda}}{(2n)!}\left(-\sum_{l=1}^{2n}\frac{1}{l}+a_{1,0}^{p,\tau,\lambda}\right)\Delta^+_{2n-1}\left(x^{2n}\right)\\ &=\frac{o_0^{p,\tau,\lambda}}{(2n)!}\left(\tilde{x}_1^{2n}\log\tilde{x}_1\right)^{(2n-1)}+\\ &\frac{o_0^{p,\tau,\lambda}}{(2n)!}\left(-\sum_{l=1}^{2n}\frac{1}{l}+a_{1,0}^{p,\tau,\lambda}\right)\left(\tilde{x}_2^{2n}\right)^{(2n-1)}\\ &=o_0^{p,\tau,\lambda}\left(\tilde{x}_1\log\tilde{x}_1+\tilde{x}_1\sum_{l=2}^{2n}\frac{1}{l}\right)+\\ &o_0^{p,\tau,\lambda}\left(-\sum_{l=1}^{2n}\frac{1}{l}+a_{1,0}^{p,\tau,\lambda}\right)\tilde{x}_2\end{aligned}$$

for some $\tilde{x}_1,\tilde{x}_2\in\left[x,x+(2n-1)h\right]$. Hence, the fact that $h\le\frac{x}{2}$ yields

$$h^{-(2n-1)}\Delta^+_{2n-1}c_0\left(p,\tau,\lambda\right)=O\left(x\log x\right).\tag{16}$$

Now, suppose that $0 \in I'_{p,\tau,\lambda}$. We have,

$$c_0(p,\tau,\lambda) = \frac{Z'_S(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)}\frac{x^{2n}}{(2n)!}.$$

Therefore,

$$\begin{aligned}
&h^{-(2n-1)}\Delta^{+}_{2n-1}c_0(p,\tau,\lambda)\\
&= h^{-(2n-1)}\frac{Z'_S(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)}\frac{1}{(2n)!}\Delta^{+}_{2n-1}\left(x^{2n}\right)\\
&= \frac{Z'_S(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)}\frac{1}{(2n)!}\left(\tilde{x}_3^{2n}\right)^{(2n-1)}\\
&= \frac{Z'_S(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)}\tilde{x}_3
\end{aligned}$$

for some $\tilde{x}_3 \in [x, x+(2n-1)h]$. Since $h \le \frac{x}{2}$, we obtain

$$h^{-(2n-1)}\Delta^{+}_{2n-1}c_0(p,\tau,\lambda) = O(x). \tag{17}$$

Let $z \in B'_{p,\tau,\lambda}$, $z=-1$.
Suppose that $-1 \in I_{p,\tau,\lambda}$. Now,

$$\begin{aligned}
&c_{-1}(p,\tau,\lambda)\\
&= \frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}x^{2n-1}\log x+\\
&\frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}\left(-\sum_{\substack{l=0\\l\neq1}}^{2n}\frac{1}{-1+l}+a_{1,-1}^{p,\tau,\lambda}\right)x^{2n-1}.
\end{aligned}$$

We deduce,

$$\begin{aligned}
&h^{-(2n-1)}\Delta^{+}_{2n-1}c_{-1}(p,\tau,\lambda)\\
&= h^{-(2n-1)}\frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}\Delta^{+}_{2n-1}\left(x^{2n-1}\log x\right)+\\
&h^{-(2n-1)}\frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}\times\\
&\times\left(-\sum_{\substack{l=0\\l\neq1}}^{2n}\frac{1}{-1+l}+a_{1,-1}^{p,\tau,\lambda}\right)\Delta^{+}_{2n-1}\left(x^{2n-1}\right)\\
&= \frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}\left(\tilde{x}_4^{2n-1}\log\tilde{x}_4\right)^{(2n-1)}+\\
&\frac{o_{-1}^{p,\tau,\lambda}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}\left(-\sum_{\substack{l=0\\l\neq1}}^{2n}\frac{1}{-1+l}+a_{1,-1}^{p,\tau,\lambda}\right)\times\\
&\times\left(\tilde{x}_5^{2n-1}\right)^{(2n-1)}\\
&= \frac{o_{-1}^{p,\tau,\lambda}}{-(2n-1)!}\left(\tilde{x}_4^{2n-1}\log\tilde{x}_4\right)^{(2n-1)}+\\
&\frac{o_{-1}^{p,\tau,\lambda}}{-(2n-1)!}\left(-\sum_{\substack{l=0\\l\neq1}}^{2n}\frac{1}{-1+l}+a_{1,-1}^{p,\tau,\lambda}\right)\times\\
&\times\left(\tilde{x}_5^{2n-1}\right)^{(2n-1)}\\
&= -o_{-1}^{p,\tau,\lambda}\left(\log\tilde{x}_4+\sum_{l=1}^{2n-1}\frac{1}{l}\right)-\\
&o_{-1}^{p,\tau,\lambda}\left(-\sum_{\substack{l=0\\l\neq1}}^{2n}\frac{1}{-1+l}+a_{1,-1}^{p,\tau,\lambda}\right),
\end{aligned}$$

where $\tilde{x}_4 \in [x, x+(2n-1)h]$. Now, $h \le \frac{x}{2}$ yields

$$h^{-(2n-1)}\Delta^{+}_{2n-1}c_{-1}(p,\tau,\lambda) = O(\log x). \tag{18}$$

Let $-1 \in I'_{p,\tau,\lambda}$. We have,

$$c_{-1}(p,\tau,\lambda) = \frac{Z'_S(-1+\rho-\lambda,\tau)}{Z_S(-1+\rho-\lambda,\tau)}\frac{x^{2n-1}}{\prod_{\substack{l=0\\l\neq1}}^{2n}(-1+l)}.$$

Hence,

$$\begin{aligned}
&h^{-(2n-1)}\Delta^{+}_{2n-1}c_{-1}(p,\tau,\lambda)\\
&= h^{-(2n-1)}\frac{Z'_S(-1+\rho-\lambda,\tau)}{Z_S(-1+\rho-\lambda,\tau)}\frac{1}{-(2n-1)!}\times\\
&\times\Delta^{+}_{2n-1}\left(x^{2n-1}\right)
\end{aligned}$$

$$= \frac{Z_S^{'}\left(-1+\rho-\lambda,\tau\right)}{Z_S\left(-1+\rho-\lambda,\tau\right)} \frac{1}{-\left(2n-1\right)!} \left(\tilde{x}_5^{2n-1}\right)^{(2n-1)}$$
$$= -\frac{Z_S^{'}\left(-1+\rho-\lambda,\tau\right)}{Z_S\left(-1+\rho-\lambda,\tau\right)},$$

for $\tilde{x}_5 \in [x, x+(2n-1)h]$. We conclude,

$$h^{-(2n-1)}\Delta_{2n-1}^{+}c_{-1}\left(p,\tau,\lambda\right) = -\frac{Z_S^{'}\left(-1+\rho-\lambda,\tau\right)}{Z_S\left(-1+\rho-\lambda,\tau\right)}. \tag{19}$$

Finally, let $z \in B_{p,\tau,\lambda}^{'}$, $z=-j \leq -2$. Assume that $-j \in I_{p,\tau,\lambda}$. We deduce,

$$h^{-(2n-1)}\Delta_{2n-1}^{+}c_{-j}\left(p,\tau,\lambda\right)$$
$$= h^{-(2n-1)} \frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \Delta_{2n-1}^{+}\left(x^{-j+2n}\log x\right) +$$
$$h^{-(2n-1)} \frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \times$$
$$\times \left(-\sum_{\substack{l=0\\l\neq j}}^{2n}\frac{1}{-j+l} + a_{1,-j}^{p,\tau,\lambda}\right) \Delta_{2n-1}^{+}\left(x^{-j+2n}\right)$$
$$= \frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \left(\tilde{x}_6^{-j+2n}\log \tilde{x}_6\right)^{(2n-1)} +$$
$$\frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \left(-\sum_{\substack{l=0\\l\neq j}}^{2n}\frac{1}{-j+l} + a_{1,-j}^{p,\tau,\lambda}\right) \times$$
$$\left(\tilde{x}_7^{-j+2n}\right)^{(2n-1)}$$
$$= \frac{o_{-j}^{p,\tau,\lambda}}{(2n-j)!\,(-1)^j\, j!}\,(2n-j)!\,(-1)^{j-2}\,\frac{(j-2)!}{\tilde{x}_6^{j-1}} +$$
$$\frac{o_{-j}^{p,\tau,\lambda}}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \left(-\sum_{\substack{l=0\\l\neq j}}^{2n}\frac{1}{-j+l} + a_{1,-j}^{p,\tau,\lambda}\right) \cdot 0$$
$$= \frac{o_{-j}^{p,\tau,\lambda}}{j\,(j-1)} \frac{1}{\tilde{x}_6^{j-1}}$$

for some $\tilde{x}_6$, $\tilde{x}_7 \in [x, x+(2n-1)h]$. Since $\tilde{x}_6 \geq x$ and $j \geq 2$, we conclude that

$$h^{-(2n-1)}\Delta_{2n-1}^{+}c_{-j}\left(p,\tau,\lambda\right) = O\left(x^{-1}\right). \tag{20}$$

Let $-j \in I_{p,\tau,\lambda}^{'}$. Then,

$$h^{-(2n-1)}\Delta_{2n-1}^{+}c_{-j}\left(p,\tau,\lambda\right)$$
$$= h^{-(2n-1)} \frac{Z_S^{'}\left(-j+\rho-\lambda,\tau\right)}{Z_S\left(-j+\rho-\lambda,\tau\right)} \frac{1}{\prod\limits_{\substack{l=0\\l\neq j}}^{2n}(-j+l)} \times$$
$$\times \Delta_{2n-1}^{+}\left(x^{-j+2n}\right)$$
$$= \frac{Z_S^{'}\left(-j+\rho-\lambda,\tau\right)}{Z_S\left(-j+\rho-\lambda,\tau\right)} \frac{1}{(2n-j)!\,(-1)^j\, j!} \times$$
$$\times \left(\tilde{x}_8^{-j+2n}\right)^{(2n-1)}$$
$$= \frac{Z_S^{'}\left(-j+\rho-\lambda,\tau\right)}{Z_S\left(-j+\rho-\lambda,\tau\right)} \frac{1}{(2n-j)!\,(-1)^j\, j!} \cdot 0 = 0$$

for some $\tilde{x}_8 \in [x, x+(2n-1)h]$. Hence,

$$h^{-(2n-1)}\Delta_{2n-1}^{+}c_{-j}\left(p,\tau,\lambda\right) = 0. \tag{21}$$

Taking into account (16)-(21) as well as the fact that the set $B_{p,\tau,\lambda}^{'}$ is finite one, we obtain

$$\sum_{z\in B_{p,\tau,\lambda}^{'}} h^{-(2n-1)}\Delta_{2n-1}^{+}c_z\left(p,\tau,\lambda\right) = O\left(x\log x\right). \tag{22}$$

Finally, consider the sum over $C_{p,\tau,\lambda}^{3}$ on the right hand side of (15). Let $z \in C_{p,\tau,\lambda}^{3}$. By the definition of $C_{p,\tau,\lambda}^{3}$,

$$\left|h^{-(2n-1)}\Delta_{2n-1}^{+}c_z\left(p,\tau,\lambda\right)\right|$$
$$= \left|h^{-(2n-1)} o_z^{p,\tau,\lambda} \frac{1}{z\,(z+1)\ldots(z+2n)} \times \right.$$
$$\left. \times \Delta_{2n-1}^{+}\left(x^{z+2n}\right)\right|$$
$$= \left|o_z^{p,\tau,\lambda} \frac{1}{z\,(z+1)\ldots(z+2n)} \left(\tilde{x}_9^{z+2n}\right)^{(2n-1)}\right|$$
$$= \left|o_z^{p,\tau,\lambda} \frac{1}{z\,(z+1)} \tilde{x}_9^{z+1}\right| = \left|o_z^{p,\tau,\lambda}\right| \frac{1}{|z|\,|z+1|} \tilde{x}_9^{z+1}$$
$$\leq \left|o_z^{p,\tau,\lambda}\right| \frac{1}{|z|\,|z+1|} \tilde{x}_9^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}},$$

where $\tilde{x}_9 \in [x, x + (2n-1)h]$. Hence, $h \leq \frac{x}{2}$ yields

$$h^{-(2n-1)} \Delta^{+}_{2n-1} c_z (p, \tau, \lambda) = O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

We have,

$$\begin{aligned} &\sum_{z \in C^3_{p,\tau,\lambda}} h^{-(2n-1)} \Delta^{+}_{2n-1} c_z (p, \tau, \lambda) \\ &= O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{23}$$

Combining (15), (22) and (23), we obtain

$$\begin{aligned} &h^{-(2n-1)} \Delta^{+}_{2n-1} \psi_{2n} (x) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} 1 \times \\ &\times \sum_{s^{p,\tau,\lambda} \in \left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1} - 1, 2\rho\right]} 1 \times \\ &\times \frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} + O\left(h^{2\rho+1}\right) + \\ &O\left(x^{\rho+1} M^{n-2}\right) + \\ &O\left(h^{-(2n-1)} x^{\rho+2n} M^{-n-1}\right) + \\ &O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{24}$$

Substituting $h = x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M = x^{\frac{\rho}{n+2\rho-1}}$ into (24) and taking into account that $\psi_1 (x) \leq h^{-(2n-1)} \Delta^{+}_{2n-1} \psi_{2n} (x)$, we conclude

$$\begin{aligned} &\psi_1 (x) \\ &\leq \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} 1 \times \\ &\times \sum_{s^{p,\tau,\lambda} \in \left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1} - 1, 2\rho\right]} 1 \times \\ &\times \frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} + \\ &O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{25}$$

Analogously (see, e.g., [12]),

$$\begin{aligned} &\psi_1 (x) \geq \\ &\sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} 1 \times \\ &\times \sum_{s^{p,\tau,\lambda} \in \left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1} - 1, 2\rho\right]} 1 \times \\ &\times \frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} + \\ &O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{26}$$

Now, combining (25) and (26), we end up with

$$\begin{aligned} &\psi_1 (x) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} 1 \times \\ &\times \sum_{s^{p,\tau,\lambda} \in \left((2\rho+1)\frac{n+\rho-1}{n+2\rho-1} - 1, 2\rho\right]} 1 \times \\ &\times \frac{x^{s^{p,\tau,\lambda}+1}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)} + \\ &O\left(x^{(2\rho+1)\frac{n+\rho-1}{n+2\rho-1}}\right). \end{aligned} \tag{27}$$

The assertion of the theorem follows from (27) and the fact that

$$\frac{\psi_1 (x)}{x} = \sum_{\gamma \in \Gamma_{\mathrm{h}}, N(\gamma) \leq x} \Lambda_0 (\gamma) \left(1 - \frac{N(\gamma)}{x}\right).$$

This completes the proof. □

4 Final remarks

We deal with weighted generalized prime geodesic theorem for compact, even-dimensional, locally symmetric Riemannian manifolds of strictly negative sectional curvature, which is given in terms of order one counting function $\frac{\psi_1(x)}{x}$ (see, Theorem 1).

However, the following theorem holds also true.

Theorem 2. *Let Y be as above. Then,*

$$\frac{1}{2}\sum_{\gamma\in\Gamma_h,N(\gamma)\leq x}\Lambda_0(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^2$$
$$=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\times$$
$$\times\sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]}1\times$$
$$\times\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}$$
$$+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}-2\frac{\rho}{n+2\rho-1}}\right)$$

as $x\to+\infty$, where $s^{p,\tau,\lambda}$ is a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$.

Proof. It is enough to consider the counting function $\frac{\psi_2(x)}{x^2}$ instead of $\frac{\psi_1(x)}{x}$, and to follow the lines of the proof of Theorem 1.

In this case, (4) and (5) read as:

$$\Delta^+_{2n-2}f(x)$$
$$=\int_x^{x+h}\int_{t_{2n-2}}^{t_{2n-2}+h}\ldots\int_{t_2}^{t_2+h}f^{(2n-2)}(t_1)\,dt_1\ldots dt_{2n-2}$$

and

$$\Delta^+_{2n-2}f(x)=h^{2n-2}f^{(2n-2)}(\tilde{x}),$$

where $\tilde{x}\in[x,x+(2n-2)h]$, and f is at least $2n-2$ times differentiable function.

Moreover,

$$\Delta^+_{2n-2}f(x)$$
$$=\sum_{i=0}^{2n-2}(-1)^i\binom{2n-2}{i}f(x+(2n-2-i)h).$$

Now,

$$\sum_{s^{p,\tau,\lambda}\in O^{\varepsilon}_{2\rho}}h^{-(2n-2)}\Delta^+_{2n-2}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)$$
$$=\sum_{s^{p,\tau,\lambda}\in O^{\varepsilon}_{2\rho}}\frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}\tag{28}$$
$$+O\left(h^{2\rho+2}\right).$$

If $z\in S^{p,\tau,\lambda}_{-\rho+\lambda}$, then

$$h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=O\left(h^{-(2n-2)}|z|^{-2n-1}x^{\rho+2n}\right),$$
$$h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=O\left(|z|^{-3}x^{\rho+2}\right).$$

Hence, for $M>2\rho$,

$$\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|-\rho+\lambda|<|z|\leq M}}h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)+$$
$$\sum_{\substack{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}\\|z|>M}}h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=O\left(x^{\rho+2}\int_{|-\rho+\lambda|}^{M}t^{-3}dN_{p,\tau,\lambda}(t)\right)+\tag{29}$$
$$O\left(h^{-(2n-2)}x^{\rho+2n}\times\right.$$
$$\left.\times\int_M^{+\infty}t^{-2n-1}dN_{p,\tau,\lambda}(t)\right)$$
$$=O\left(x^{\rho+2}M^{n-3}\right)+$$
$$O\left(h^{-(2n-2)}x^{\rho+2n}M^{-n-1}\right).$$

The error terms $O\left(h^{2\rho+2}\right)$, $O\left(x^{\rho+2}M^{n-3}\right)$ and $O\left(h^{-(2n-2)}x^{\rho+2n}M^{-n-1}\right)$ that appear in (28) and (29) all become $O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right)$ for the choice $h=x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M=x^{\frac{\rho}{n+2\rho-1}}$.

Consequently, we define the following sets:

$$B_{p,\tau,\lambda}$$
$$=\Big\{-j\in I_{-2n}\mid c_{-j}(p,\tau,\lambda)=$$
$$O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right)\Big\},$$
$$B'_{p,\tau,\lambda}=I_{-2n}\backslash B_{p,\tau,\lambda},$$

$$C^2_{p,\tau,\lambda}$$
$$=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\mid -2n-1<s^{p,\tau,\lambda}\leq -2n+(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}\right\},$$
$$C^3_{p,\tau,\lambda}$$
$$=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\mid -2n+(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq -2+(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}\right\},$$
$$C^4_{p,\tau,\lambda}$$
$$=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\mid -2+(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq 2\rho\right\}.$$

We leave the set $C^1_{p,\tau,\lambda}$ in the same form as before. Hence, (10) remains valid.

The following estimates hold obviously true:

$$\sum_{z\in B_{p,\tau,\lambda}} c_z(p,\tau,\lambda)=O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right),$$
$$\sum_{z\in C^2_{p,\tau,\lambda}} c_z(p,\tau,\lambda)=O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

The definition of the set $C^4_{p,\tau,\lambda}$ and the relation (28) give us

$$\sum_{z\in C^4_{p,\tau,\lambda}} h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=\sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]} 1\times$$
$$\times\frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}+O\left(h^{2\rho+2}\right).$$

Hence, the relation that corresponds to (13) comes with $O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right)$ now.

Furthermore, (15) reads as

$$\begin{aligned}&h^{-(2n-2)}\Delta^+_{2n-2}\psi_{2n}(x)\\&=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}}1\times\\&\times h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)+\\&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^3_{p,\tau,\lambda}}1\times\\&\times h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)+\\&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times\\&\times\sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]}1\times\\&\times\frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}+\\&O\left(h^{2\rho+2}\right)+\\&O\left(x^{\rho+2}M^{n-3}\right)+\\&O\left(h^{-(2n-2)}x^{\rho+2n}M^{-n-1}\right)+\\&O\left(h^{-(2n-2)}x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned}\tag{30}$$

Reasoning as earlier, we obtain that:

$$h^{-(2n-2)}\Delta^+_{2n-2}c_0(p,\tau,\lambda)=O\left(x^2\log x\right),$$
$$h^{-(2n-2)}\Delta^+_{2n-2}c_{-1}(p,\tau,\lambda)=O\left(x\log x\right),$$
$$h^{-(2n-2)}\Delta^+_{2n-2}c_{-2}(p,\tau,\lambda)=O\left(\log x\right),$$
$$h^{-(2n-2)}\Delta^+_{2n-2}c_{-j}(p,\tau,\lambda)=O\left(x^{-1}\right)$$

for $3\leq j\leq 2n$.

Therefore,

$$\sum_{z\in B'_{p,\tau,\lambda}} h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)=O\left(x^2\log x\right).$$

Finally, we find that

$$\sum_{z\in C^3_{p,\tau,\lambda}} h^{-(2n-2)}\Delta^+_{2n-2}c_z(p,\tau,\lambda)$$
$$=O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

Consequently, (30) become

$$h^{-(2n-2)}\Delta^+_{2n-2}\psi_{2n}(x)$$
$$=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times$$
$$\times\sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]}1\times$$

$$\times \frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}+$$
$$O\left(h^{2\rho+2}\right)+$$
$$O\left(x^{\rho+2}M^{n-3}\right)+O\left(h^{-(2n-2)}x^{\rho+2n}M^{-n-1}\right)+$$
$$O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

Substituting $h=x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M=x^{\frac{\rho}{n+2\rho-1}}$, and bearing in mind that $\psi_2(x)\leq h^{-(2n-2)}\Delta^+_{2n-2}\psi_{2n}(x)$, we obtain

$$\psi_2(x)$$
$$\leq \sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times$$
$$\times \sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]}1\times$$
$$\times \frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}+$$
$$O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

Since the opposite inequality holds also true, we end up with

$$\psi_2(x)$$
$$=\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}1\times$$
$$\times \sum_{s^{p,\tau,\lambda}\in\left((2\rho+2)\frac{n+\rho-1}{n+2\rho-1}-2,2\rho\right]}1\times$$
$$\times \frac{x^{s^{p,\tau,\lambda}+2}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\left(s^{p,\tau,\lambda}+2\right)}+$$
$$O\left(x^{(2\rho+2)\frac{n+\rho-1}{n+2\rho-1}}\right).$$

Therefore, the claim of the theorem follows from the fact that

$$\frac{\psi_2(x)}{x^2}=\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x}\Lambda_0(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^2.$$

This completes the proof. □

According to Theorem 2, the error term in the prime geodesic theorem (3) can be even further improved (in a weighted sense).

It would be worth to follow the method described in this paper in order to obtain an analogous result for a higher order counting function

$$\frac{\psi_j(x)}{x^j}=\frac{1}{j!}\sum_{\gamma\in\Gamma_{\mathrm{h}},N(\gamma)\leq x}\Lambda_0(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^j,$$

$j>2$, and to possibly determine the optimal size of the error term.

We highlight the following contribution of the used literature to the results derived in this paper.

Our main result is a generalization of the corresponding, classical result [11, Th. 30.] to the case of compact, even-dimensional, locally symmetric Riemannian manifolds of strictly negative sectional curvature. In the case of the Riemann zeta function $\zeta(s)$, the corresponding prime number theorem states that

$$\psi_0(x)=x+O\left(x^{\theta}\log^2 x\right),$$

where $\frac{1}{2}\leq\theta\leq 1$ denotes the upper bound of the real parts of the zeros of $\zeta(s)$. At the same time, a weighted form of the prime number theorem yields a better result

$$\frac{\psi_1(x)}{x}=\frac{1}{2}x+O\left(x^{\theta}\right).$$

The references [2] and [8] represent a suitable starting point for our current research since they provide proofs of the prime geodesic theorem in the case at hand. [2], however, is founded on Randol's approach [13] in the case of compact Riemann surfaces. On the other side, [8] follows Park's approach [12] in the case of real hyperbolic manifolds with cusps.

Note that [1] improves Park's result [12].

In general, prime geodesic theorems [2], [8], [13], [12] and [1] stem from the use of properties of the corresponding Ruelle zeta function and the use of appropriately chosen higher order counting function.

In particular, [2] and [8] improve DeGeorge's result [4].

Some of the results derived in [6] are applied in [8], [12] and [1]. These results are related to the behaviour of the logarithmic derivative of a meromorphic function of order not larger than n along arbitrarily large circles in the complex plane.

Our preliminary material is almost completely based on the book of Bunke and Olbrich [3]. There, the authors offered a complete investigation of the theta functions and the Selberg zeta functions associated with locally homogeneous vector bundles over compact locally symmetric space of rank one. Hence, as a reference, [3] plays the key role in our work.

Some of the necessary results on the Weyl's asymptotic law (in our setting) are adopted from [5].

Similarly, some of the results on classification of locally symmetric spaces are adopted from [10].

The reference [7] is valuable one since it provides a more detailed insight into some aspects of the Lie algebras theory than [3]. Additionally, it gives a prime geodesic theorem (in our setting) based on an application of the corresponding Selberg zeta function.

Gangolli's result [7] is weaker than Degeorge's [4], however.

Hejhal [9], also offers a prime geodesic theorem based on an application of the corresponding Selberg zeta function. In particular, he extensively studies the Selberg zeta function over a hyperbolic Riemannian surface Y, i.e., when Γ is a co-finite discrete subgroup of $G = \mathrm{PSL}\,(2, \mathbb{R})$. The contribution of the reference [9] to our work lies in the fact that it offers a very efficient method for residues calculating.

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