The applications of the universal morphisms of LF-TOP the category of all fuzzy topological spaces

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Abstract: Finding the universal morphisms for a given category is considered as comprehensive study of the principal properties that this category can achieved. In this work, we build a category of fuzzy topological spaces with respect to Lowen's definition of Fuzzy TOPological space [3], that we denoted LF-TOP. Firstly, we collected universal morphisms of TOP category, listed by Sander Mac Lane [7]. Second, we studied universal morphisms of LF-TOP. We found through this study that the properties of this category are a generalization to the TOP category properties and TOP's universal morphisms are projections of LF-TOP's ones. This shows the power of Lowen's fuzziness effect on the ordinary topological space.

Key–Words: category, functor, fuzzy topological space, TOP-category, universal morphism.

1 Introduction

In the early 1940's, Samuel Eilenberg and Saunders Mac Lane invented Category theory [10], with the aim of bridging what may appears to be two quite different fields: Topology and Algebra. Later, it was propagated by Alexander Grothendieck in the 1960's. From another side, L.A. Zadeh [1] introduced the fuzzy set in 1965, since then many researchers used this tool to generalize different concepts of Mathematics. Fuzzy theory has many applications in mathematics and also in other fields such as information [4] and control [14].
Chang [4] was the first one who introduced the notion of fuzzy topology as an application of fuzzy sets. Later in 1976, Lowen [3] changed the definition of Chang because some intuitive and well known results in ordinary topology are not satisfied in the case of Chang's definition. For example, some constant functions fail to be continuous from one Chang topological space to another. After that, many researchers have given new other definitions of fuzzy topological space, as in [11, 12].
Regarding the importance of fuzzy applications and category theory, it seems more interesting to join both together. This leads us to speak about the applications of the universal morphisms of the fuzzy category.
The present work is organized as follows: in the next section, we recall some of the basic definitions (fuzzy set and operations on it, fuzzy topological space, fuzzy continuous application, universal morphisms,...). Then, we collect the universal morphisms of TOPological spaces category (TOP).
In the 3rd section, we study the universal morphisms of fuzzy topological spaces category (LF-TOP). And finally, we give a summary of the research and its results.

2 Preliminary Notions

Let X be a non empty set. A fuzzy set on X is a function A from X to $[0,1]$. The image $A(x)$ of the element $x \in X$ is called a degree of membership of x in A.

Definition 1 *[4] Let A and B be fuzzy sets on X. Then $\forall x \in X$*

1. $A = B \Longleftrightarrow A(x) = B(x)$.
2. $A \subset B \Longleftrightarrow A(x) \le B(x)$.
3. $C = A \bigvee B \Longleftrightarrow C(x) = max\{A(x), B(x)\}$.
4. $D = A \bigwedge B \Longleftrightarrow D(x) = min\{A(x), B(x)\}$.
5. $A = \overline{B} \Longleftrightarrow A(x) = 1 - B(x)$.

More generally, for a family of fuzzy sets, $A = \{A_i,\ i \in I\}$, the union, $C = \bigvee_I A_i$, and the intersection $D = \bigwedge_I A_i$, are defined by:

$$C(x) = sup_I\{A_i(x)\} \qquad for\, all\ x \in X.$$

$D(x) = inf_I\{A_i(x)\}$ $\qquad$ $for\ all\ x \in X.$

The symbol $\varnothing$ will be used to denote an empty fuzzy set ($\varnothing(x) = 0$ for all $x \in X$).
For X, we have by definition $X(x) = 1$, for all $x \in X$.

Definition 2 *[3] A Lowen fuzzy topology, or simply (F-TOP) is a family τ of fuzzy sets on X which satisfies the following conditions:*

1. *τ contains all constant fuzzy sets on X.*
2. *If $A,\ B \in \tau$, then $A \wedge B \in \tau$.*
3. *If $A_i \in \tau$ four all $i \in I$, then $\bigvee_I A_i \in \tau$.*

Definition 3 *[4,8] Let f be a function from X to Y, B is a fuzzy set on Y. Then the inverse image of B, written as $f^{-1}(B)$, is a fuzzy set on X defined by:*

$$f^{-1}(B)(x) = B(f(x)) \quad for\ all \quad x \in X.$$

Definition 4 *[5,16] Let (X, τ) be a F-TOP. We define the topological space $(X, \iota(\tau))$ such that: For $A \in \tau$:*

$$\iota(A) = \{\alpha(A),\ \alpha \in [0,1)\},\ \alpha(A) = \{x \in X : A(x) > \alpha\}$$

And $\iota(\tau) = \{\iota(A),\ A \in \tau\}$.

Definition 5 *[4,13,15] (definition of continuity) A function f from a F-TOP (X,T) to a F-TOP (Y,U) is fuzzy continuous (F-continuous) iff the inverse of each U-open fuzzy set is T-open.*

Definition 6 *[3](Lowen's definition of continuity) Let f be a function from an F-TOP (X,T) to an F-TOP (Y,S). Then, f is Lowen fuzzy continuous, (in short LF-continuous) iff $f : (X, \iota(T)) \longrightarrow (Y, \iota(S))$ is continuous.* [1]

Theorem 7 *[3]*
Let (X, δ) be an ordinary topological space. The following collection is an F-TOP:

$$\omega(A) = \{A \in I^X :\ \alpha(A) \in \delta,\ \forall \alpha \in [0,1[\}$$

Definition 8 *[2,8]*

(a) *Let τ be a F-TOP. A subfamily T of τ is a base for τ iff each member of τ can be expressed as the union of some members of T.*

(b) *A subfamily S of T is a subbase for τ iff the family of finite intersections of members of S forms a base for τ.*

(c) *A subbase for the product fuzzy topology on $(X,T) = (\prod_{i \in I} X_i, \prod_{i \in I} \tau_i)$ is given by $S = \{\pi_i^{-1}\theta_i;\ \theta_i \in \tau_i,\ i \in I\}$ (π_i the projection from X onto X_i) so that a base can be taken to be $B = \{\bigwedge_{j=1}^{n} \pi_{i_j}^{-1}\theta_{i_j};\ \theta_{i_j} \in \tau_{i_j},\ i_j \in I,\ j = 1...n,\ n \in \mathbb{N}\}$.*

Proposition 9 *[7] (THE UNIVERSAL MORPHISMS OF TOP)*
TOP is the category of all topological spaces and continuous maps.

(a) *The element of Co-product of (X, τ_X) and (Y, τ_Y) in TOP is their disjoint union.*

(b) *The element of Co-equalizer of $f, g : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ in TOP is the topological space $(Y/\sim, \tau_{Y/\sim})$, where $\sim$ is the least equivalence relation which contains all pairs $< f(x), g(x) >$, for $x \in X$.*

(c) *The element of Push-out of $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $g : (X, \tau_X) \longrightarrow (Z, \tau_Z)$ in TOP is the disjoint union $(Y \uplus Z, \tau_{Y \uplus Z})$ with the elements $f(x)$ and $g(x)$ identified for each $x \in X$.*

(d) *The element of Product of (X, τ_X), (Y, τ_Y) in TOP is their cartesian product.*

(e) *The element of Co-equalizer of $f, g : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ in TOP is the topological space (D, τ_D), where $D = \{x \in X, f(x) = g(x)\}$.*

(f) *The element of Pull-back of $f : (X, \tau_X) \longrightarrow (Z, \tau_Z)$, $g : (Y, \tau_Y) \longrightarrow (Z, \tau_Z)$ in TOP is the topological space (C, τ_C), where $C = \{(x,y) \in X \times Y, f(x) = g(y)\}$.*

3 Main results

The fuzzy topological spaces F-TOP and fuzzy continuous mappings (F-continuous) form a category which we denote by LF-TOP. Now, we investigate the morphism of this category.

3.1 Co-product

Definition 10 *Let $(X_1, \tau_1), (X_2, \tau_2)$ be two fuzzy topological spaces. The disjoint union of $(X_1, \tau_1), (X_2, \tau_2)$ is defined as:*

$$(X_1, \tau_1) \uplus (X_2, \tau_2) = (X_1 \uplus X_2, \tau_{X_1 \uplus X_2})$$

where

$$X_1 \uplus X_2 = \{X_1 \times \{1\}\} \cup \{X_2 \times \{2\}\};$$

[1] we use definition(5) instead this definition [see 9].

$\tau_{X_1 \uplus X_2} = \{\theta,\ \theta$ *is a fuzzy set on* $X_1 \uplus X_2,\ \varphi_1^{-1}(\theta) \in \tau_1$ *and* $\varphi_2^{-1}(\theta) \in \tau_2\}$

$$\begin{aligned} \varphi_1 : (X_1, \tau_1) &\longrightarrow (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \\ x &\longmapsto \varphi_1(x) = (x, 1) \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : (X_2, \tau_2) &\longrightarrow (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \\ x &\longmapsto \varphi_2(x) = (x, 2) \end{aligned}$$

Proposition 11 *The disjoint union* $(X_1 \uplus X_2, \tau_{X_1 \uplus X_2})$ *is a fuzzy topological space.*

Proof: Let $C \in [0,1]^{X_1 \uplus X_2}$ be any constant fuzzy set, by definition(3), $\varphi_1^{-1}(C)$ is a fuzzy set on X_1 and $\varphi_1^{-1}(C)(x) = C(\varphi_1(x)) = C((x,1)) = C$, but (X_1, τ_1) is a fuzzy topological space so $\varphi_1^{-1}(C) \in \tau_1$. And using the same method we find $\varphi_2^{-1}(C) \in \tau_2$, so $C \in \tau_{X_1 \uplus X_2}$. Now, let $\theta_1, \theta_2 \in \tau_{X_1 \uplus X_2}$, then $\theta_1 \wedge \theta_2$ is a fuzzy set on $X_1 \uplus X_2$ and $\varphi_1^{-1}(\theta_1 \wedge \theta_2) = \varphi_1^{-1}(\theta_1) \wedge \varphi_1^{-1}(\theta_2) \in \tau_1$
$(\varphi_2^{-1}(\theta_1 \wedge \theta_2) = \varphi_2^{-1}(\theta_1) \wedge \varphi_1^{-1}(\theta_2) \in \tau_2)$. Hence $\theta_1 \wedge \theta_2 \in \tau_{X_1 \uplus X_2}$.
Finally, if $\theta_i \in \tau_{X_1 \uplus X_2}, \forall i \in I$, then $\vee_{i \in I} \theta_i$ is a fuzzy set on $X_1 \uplus X_2$, and $\varphi_1^{-1}(\vee_{i \in I} \theta_i)$, and $\varphi_1^{-1}(\vee_{i \in I} \theta_i) = \vee_{i \in I} \varphi_1^{-1}(\theta_i) \in \tau_1$
$(\varphi_2^{-1}(\vee_{i \in I} \theta_i) = \vee_{i \in I} \varphi_2^{-1}(\theta_i) \in \tau_2)$, so $\vee_{i \in I} \theta_i \in \tau_{X_1 \uplus X_2}$.
Hence $\tau_{X_1 \uplus X_2}$ is a topological space on $X_1 \uplus X_2$.

Proposition 12 *The applications* φ_1, φ_2 *are F-continuous.*

Proof: Clear (by definition of $\tau_{X_1 \uplus X_2}$).

Theorem 13 *Let* h *be a function from a fuzzy topological space* $(X_1 \uplus X_2, \tau_{X_1 \uplus X_2})$ *into a* (C, τ_C). *Then:* $(h \circ \varphi_1)$ *and* $(h \circ \varphi_2)$ *are F-continuous* $\Longrightarrow$ h *is F-continuous*

Proof: Let $\theta \in \tau_C$. As $h \circ \varphi_1$ and $h \circ \varphi_2$ are F-continuous, then $(h \circ \varphi_1)^{-1}(\theta) \in \tau_1$ and $(h \circ \varphi_2)^{-1}(\theta) \in \tau_2$, then $\varphi_1^{-1}(h^{-1}(\theta)) \in \tau_1$ and $\varphi_2^{-1}(h^{-1}(\theta))) \in \tau_2$.
So $h^{-1}(\theta) \in \tau_{X_1 \uplus X_2}$. Hence h is F-continuous.

Corollary 14 *The element of Co-product of* $(X_1, \tau_1), (X_2, \tau_2) \in$ <u>*LF-TOP*</u> *is a topological space* $(X_1, \tau_1) \uplus (X_2, \tau_2)$ *(defined above).*

Proof: By proposition (11), $(X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \in$ <u>LF-TOP</u>. Also by proposition (12) φ_1, φ_2 are F-continuous.
Let $f : (X_1, \tau_1) \longrightarrow (C, \tau_C)$, $g : (X_2, \tau_2) \longrightarrow (C, \tau_C)$ be F-continuous applications, then there exists a unique F-continuous application h defined by:

$$\begin{aligned} h : (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) &\longrightarrow (C, \tau_C) \\ (x, k) &\longmapsto h(x,k) = \begin{cases} f(x) & if\, k = 1. \\ g(x) & if\, k = 2. \end{cases} \end{aligned}$$

It is clear that: $f = h \circ \varphi_1$, $g = h \circ \varphi_2$.
By theorem (13), h is F-continuous .
Proof of the unicity of h:
Let $h' : (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \longrightarrow (C, \tau_C)$ be another F-continuous application, where $f = h' \circ \varphi_1$ and $g = h' \circ \varphi_2$. We have:

$$(h' \circ \varphi_1)(x) = h'(\varphi_1(x)) = h'(x, 1) = f(x).$$

and

$$(h' \circ \varphi_2)(x) = h'(\varphi_2(x)) = h'(x, 2) = g(x).$$

therefore h is unique.

3.2 Co-Equalizer

Definition 15 *Let* (A, τ_A) *be a fuzzy topological space,* $\sim$ *is the equivalence relation on* A *and* $P : A \longrightarrow A/\sim$ *is the natural projection map, we define* $\tau_{A/\sim}$ *by:*

$$\tau_{A/\sim} = \{\theta\,,\ \theta \textit{ is a fuzzy set on } A/\sim,\ \textit{where } P^{-1}(\theta) \in \tau_A\}.$$

Proposition 16 $(A/\sim, \tau_{A/\sim})$ *is a fuzzy topological space.*

Proof: Let $C \in [0,1]^{A/\sim}$ be any constant fuzzy set, by definition (3), $P^{-1}(C)$ is a fuzzy set on A and $P^{-1}(C)(x) = C(P(x)) = C(\overline{x}) = C$, but (A, τ_A) is a fuzzy topological space, so $P^{-1}(C) \in \tau_A$, then $C \in \tau_{A/\sim}$.
Now, let $\theta_1, \theta_2 \in \tau_{A/\sim}$, then $\theta_1 \wedge \theta_2$ is a fuzzy set on $A/\sim$, and
$P^{-1}(\theta_1 \wedge \theta_2) = P^{-1}(\theta_1) \wedge P^{-1}(\theta_2) \in \tau_A$. Hence, $\theta_1 \wedge \theta_2 \in \tau_{A/\sim}$.
Finally, if $\theta_i \in \tau_{A/\sim}, \forall i \in I$, then $\vee_{i \in I} \theta_i$ is a fuzzy set on $A/\sim$, and
$P^{-1}(\vee_{i \in I} \theta_i) = \vee_{i \in I} P^{-1}(\theta_i) \in \tau_A$, so $\vee_{i \in I} \theta_i \in \tau_{A/\sim}$, and hence $\tau_{A/\sim}$ is a topological space on $A/\sim$.

Proposition 17 P *is F-continuous.*

Proof: evident (by definition of $\tau_{A/\sim}$).

Theorem 18 *Let* $(A, \tau_A), (B, \tau_B) \in F-TOP$*,* $\sim$ *is the equivalence relation of* A *and* $P : A \longrightarrow A/\sim$ *is the associated projection. If* $h : (A, \tau_A) \longrightarrow (B, \tau_B)$ *is the F-continuous application compatible with* $\sim$*, then there exists a unique F-continuous application* h'*, where* $h = h' \circ P$*. In addition:*

$$h \textit{ is F-continuous} \Longrightarrow h' \textit{ is F-continuous}.$$

Proof: Let's define h' by :

$$h' : (A/\sim, \tau_{A/\sim}) \longrightarrow (B, \tau_B)$$
$$\overline{x} \longmapsto h'(\overline{x}) = h(x)$$

It is clear that h' is unique and $h = h' \circ P$.
Let $\theta \in \tau_B$, $(h' \circ P)$ is F-continuous, then $(h' \circ P)^{-1}(\theta) \in \tau_A$, so $P^{-1}(h'^{-1}(\theta)) \in \tau_A$. Therefore $h'^{-1}(\theta) \in \tau_{A/\sim}$ and h' is F-continuous.

Corollary 19 *The element of Co-equalizer of $f, g : (B, \tau_B) \longrightarrow (A, \tau_A)$ in LF-TOP is the topological space $(A/\sim, \tau_{A/\sim})$, where $\sim$ is the least equivalence relation which contains all pairs $< f(x), g(x) >$, such that $x \in B$.*

Proof: Let $h : (A, \tau_A) \longrightarrow (C, \tau_C)$ be an F-continuous application where $h \circ f = h \circ g$. For the existence of a unique h', by theorem (18) it is sufficient to prove that h is compatible with $\sim$:
Let $x_1, x_2 \in A$, $x_1 \sim x_2 \iff \exists b \in B, x_1 = f(b) \quad \wedge \quad x_2 = g(b)$. And
$h(x_1) = h(f(b)) = (h \circ f)(b) = (h \circ g)(b) = h(g(b)) = h(x_2)$, so h is compatible with $\sim$.

3.3 Push-out

Definition 20 *Let $(A, \tau_A), (B, \tau_B) \in F-TOP$, $\sim$ equivalence relation on $A \uplus B$, note $X_0 = (A \uplus B)/\sim$, we define τ_{X_0} by: $\tau_{X_0} = \{\theta$, θ is a fuzzy set on X_0, $\varphi_1^{-1}(P^{-1}(\theta)) \in \tau_A$ and $\varphi_2^{-1}(P^{-1}(\theta)) \in \tau_B\}$. where*

$$P : A \uplus B \longrightarrow X_0$$
$$(x, k) \longmapsto P(x, k) = \overline{(x, k)}$$

$$\varphi_1 : (A, \tau_A) \longrightarrow (A \uplus B, \tau_{A \uplus B})$$
$$x \longmapsto \varphi_1(x) = (x, 1)$$

and

$$\varphi_2 : (B, \tau_B) \longrightarrow (A \uplus B, \tau_{A \uplus B})$$
$$x \longmapsto \varphi_2(x) = (x, 2)$$

Proposition 21 *The space (X_0, τ_{X_0}) is a fuzzy topological space.*

Proof: The proof is based on the proofs of proposition (11) and proposition (16).

Proposition 22 *The following applications:*

$$N : (A, \tau_A) \longrightarrow (X_0, \tau_{X_0})$$
$$x \longmapsto N(x) = \overline{(x, 1)}$$

$$M : (B, \tau_B) \longrightarrow (X_0, \tau_{X_0})$$
$$x \longmapsto M(x) = \overline{(x, 2)}$$

are F-continuous.

Proof: First, let's prove that N is F-continuous.
Let $\theta \in \tau_{X_0}$, then $\varphi_1^{-1}(P^{-1}(\theta)) \in \tau_A$. So, $(P \circ \varphi_1)^{-1}(\theta) \in \tau_A$. But:
$(P \circ \varphi_1)(x) = P(\varphi_1(x)) = P((x, 1)) = \overline{(x, 1)} = N(x), \quad \forall x \in A$. Therefore $N^{-1}(\theta) \in \tau_A$.
Using the same method, we prove that M is F-continuous.

Theorem 23 *Let (A, τ_A), (B, τ_B) and (C, τ_C) be F-TOP and $f : (C, \tau_C) \longrightarrow (A, \tau_A)$, $g : (C, \tau_C) \longrightarrow (B, \tau_B)$ be two F-continuous applications. The element of Push-out of $< f, g >$ is (X_0, τ_{X_0}), where $X_0 = (A \uplus B)/\sim$ and $\sim$ is the least equivalence relation which contains all pairs $< (\varphi_1 \circ f)(c), (\varphi_2 \circ g)(c) >$, such that $c \in C$.*

Proof: By proposition (21), $(X_0, \tau_{X_0}) \in$ F-TOP. Also, by proposition (22), $\{N, M\}$ are F-continuous.
Let $(D, \tau_D) \in$ F-TOP, and $U : (A, \tau_A) \longrightarrow (D, \tau_D)$, $V : (B, \tau_B) \longrightarrow (D, \tau_D)$ are two F-continuous applications, where $V \circ g = U \circ f$.
The proof of the existence of a unique F-continuous application $h : (X_0, \tau_{X_0}) \longrightarrow (D, \tau_D)$ where $U = h \circ N$, $V = h \circ M$ requires the following steps:

Step 1: We know that the Co-product of (A, τ_A), (B, τ_B) is a disjoint union $(A \uplus B, \tau_{A \uplus B})$, then for $\{N, M\}$ there exists an F-continuous application $\pi : (A \uplus B, \tau_{A \uplus B}) \longrightarrow (X_0, \tau_{X_0})$, where $N = \pi \circ \varphi_1$ *and* $M = \pi \circ \varphi_2$.

Step 2: Let's define the new application $U \uplus V$ by:

$$U \uplus V : (A \uplus B, \tau_{A \uplus B}) \longrightarrow (D, \tau_D)$$
$$(x, k) \longmapsto (U \uplus V)(x, k) = \begin{cases} U(x) & if k = 1. \\ V(x) & if k = 2. \end{cases}$$

If $U \uplus V$ is compatible with $\sim$, then there exists a unique F-continuous application $h : (X_0, \tau_{X_0}) \longrightarrow (D, \tau_D)$, where $U \uplus V = h \circ \pi$ (theorem (18)).
Let (x, k), $(x', k') \in A \uplus B$, then:
$(x, k) \sim (x', k') \Longrightarrow \exists c \in C$,
$(x, k) = (\varphi_1 \circ g)(c)$ *and* $(x', k') = (\varphi_2 \circ f)(c)$.
$(U \uplus V)(x, k) = (U \uplus V)(\varphi_1 \circ f)(a) = (U \uplus V)(f(c), 1) = U(f(c)) = (U \circ f)(c)$.
$(U \uplus V)(x', k') = (U \uplus V)(\varphi_2 \circ g)(c) = (U \uplus V)(g(c), 2) = V(g(c)) = (V \circ g)(c)$.
But $V \circ g = U \circ f$, then $U \uplus V$ is compatible with $\sim$.

Step 3: Prove that $U = h \circ N, V = h \circ M$.
$(h \circ N)(x) = (h \circ (\pi \circ \varphi_1))(x) = (h \circ \pi)(x, 1) = (U \uplus V)(x, 1) = U(x), \forall x \in A$.
$(h \circ M)(x) = (h \circ (\pi \circ \varphi_2))(x) = (h \circ \pi)(x, 2) = (U \uplus V)(x, 2) = V(x), \forall x \in B$.

3.4 Product

Definition 24 *Let* $(A, \tau_A), (B, \tau_B)$ *be two F-TOP. We define* $\tau_{A\times B}$ *by:*
$\tau_{A\times B} = \{\theta,\ \theta$ *is a fuzzy set on* $A \times B$ *where* $\theta = \bigvee_{i\in I} A_i \times B_i\ ,\ A_i \in \tau_A,\ B_i \in \tau_B,\ \forall i \in I\}$

Proposition 25 *It is clear that* $(A \times B, \tau_{A\times B})$ *is a fuzzy topological space.*

Proposition 26 *The projections* P_1, P_2 *are F-continuous, where :*

$$\begin{aligned} P_1 : (A \times B, \tau_{A\times B}) &\longrightarrow (A, \tau_A) \\ (x, y) &\longmapsto P_1(x, y) = x \end{aligned}$$

$$\begin{aligned} P_2 : (A \times B, \tau_{A\times B}) &\longrightarrow (B, \tau_B) \\ (x, y) &\longmapsto P_2(x, y) = y \end{aligned}$$

Proof: Let's prove that P_1 is F-continuous.
If $\theta \in \delta_1$, for $(x, y) \in A \times B$, then

$$\begin{aligned} P_1^{-1}(\theta)(x, y) &= \theta(P_1(x, y)) = \theta(x) \\ &= min\{\theta(x),\ 1(y)\} \\ &= (\theta \times 1)(y) \end{aligned}$$

This implies that $P_1^{-1}(\theta) \in \tau_{A\times B}$, so P_1 is F-continuous.
The same method, we prove that P_2 is F-continuous.

Theorem 27 *Let* (C, τ_C) *be a F-TOP and* $f_1 : (C, \tau_C) \longrightarrow (A, \tau_A)$, $f_2 : (B, \tau_B) \longrightarrow (C, \tau_C)$ *be two F-continuous applications. If* f *is a function from* C *to* $A \times B$ *defined by* $f(x) = (f_1(x), f_2(x))$, *then:*

$$f_1,\ f_2 \text{ are F-continuous} \Longrightarrow f \text{ is F-continuous}$$

Proof: *Let* $\theta \in \tau_{A\times B}$, *then* $\theta = \bigvee_{i\in I} A_i \times B_i$, *where* $A_i \in \tau_A$ *and* $B_i \in \tau_B$,
as $f^{-1}(\theta) = f^{-1}(\bigvee_{i\in I} A_i \times B_i) = \bigvee_{i\in I} f^{-1}(A_i \times B_i)$.
But: $f^{-1}(A_i \times B_i)$ *is a fuzzy set on* C, *let* $x \in C$, *then*

$$\begin{aligned} f^{-1}(A_i \times B_i)(x) &= (A_i \times B_i)(f(x)) \\ &= (A_i \times B_i)(f_1(x), f_2(x) \\ &= min\{A_i(f_1(x),\ B_i(f_2(x))\} \\ &= (f_1^{-1}(A_i) \wedge f_2^{-1}(B_i))(x) \in \tau_C. \end{aligned}$$

So $f^{-1}(A_i \times B_i) \in \tau_C$ *and* $f^{-1}(\theta) \in \tau_C$. *Hence* f *is F-continuous.*

Corollary 28 *Let* $(A, \tau_A), (B, \tau_B) \in$ *<u>LF-TOP</u>. The element of product of* (A, τ_A), (B, τ_B) *is the topological space* $(A \times B, \tau_{A\times B})$ *(defined above).*

Proof:

If $f_1 : (C, \tau_C) \longrightarrow (A, \tau_A), f_2 : (C, \tau_C) \longrightarrow (B, \tau_B)$ be two F-continuous applications, then there exists a unique F-continuous application defined by:

$$\begin{aligned} f : (C, \tau_C) &\longrightarrow (A \times B, \tau_{A\times B}) \\ x &\longmapsto h(x) = (f_1(x), f_2(x)) \end{aligned}$$

Clearly, $f_1 = P_1 \circ f$ and $f_2 = P_2 \circ f$.
By theorem (27), f is F-continuous.
Proof of the unity of f. Let f' be another F-continuous application where
$f' : (C, \delta) \longrightarrow (A \times B, \tau_{A\times B})$ and $f_1 = P_1 \circ f'$, $f_2 = P_2 \circ f'$.
We suppose that: $f'(x) = (a, b)$.
$a = P_1(a, b) = (P_1 \circ f')(x) = f_1(x),\ b = P_2(a, b) = (P_2 \circ f')(x) = f_2(x)$.
Then $f'(x) = (f_1(x), f_2(x)) = h(x)$, so f is unique.

3.5 Equalizer

Definition 29 *Let* $f, g : (B, \tau_B) \longrightarrow (A, \tau_A)$ *be two F-continuous applications.* D *is a subset of* B *defined by* $D = \{x \in B,\ f(x) = g(x)\}$, *we define* τ_D *by* $\tau_D = \{\theta\ ,\ \theta = F(D) \bigwedge B_i$ *is a fuzzy set on* D, $B_i \in \tau_B$ *and* $F(D)\, is\ a\ fuzzy\ set\ on\, B, where$
$F(D)(x) = \chi_D(x)\}$

Proposition 30 *It is clear that* (D, τ_D) *is a fuzzy topological space.*

Proposition 31 e *is F-continuous, where:*

$$\begin{aligned} e : (D, \tau_D) &\longrightarrow (B, \tau_B) \\ x &\longmapsto e(x) = x \end{aligned}$$

Proof: Let $\theta \in \tau_B$, and let $x \in D$,
$e^{-1}(\theta)(x) = \theta(e(x)) = \theta(x)$, we put $\theta = \theta \wedge F(D)$.
Hence $e^{-1}(\theta) \in \tau_D$, then e is F-continuous.

Corollary 32 *The element of Equalizer of* $f, g : (B, \tau_B) \longrightarrow (A, \tau_A)$ *in <u>LF-TOP</u> is the topological space* (D, τ_D) *(defined above).*

Proof: Let $(C, \tau_C) \in$ F-TOP, for $h : (C, \tau_C) \longrightarrow (B, \tau_B)$ the F-continuous application, where $f \circ h = g \circ h$, then there exists a unique F-continuous application h' defined by:

$$\begin{aligned} h' : (C, \tau_C) &\longrightarrow (D, \tau_D) \\ x &\longmapsto h'(x) = h(x) \end{aligned}$$

h' is F-continuous since h is F-continuous.
Let $x \in C$: $(e \circ h')(x) = e(h'(x)) = e(h(x)) = h(x)$ then $e \circ h' = h$.
Proof of the unity of h', let $h'' : (C, \tau_C) \longrightarrow (D, \tau_D)$ be another F-continuous application, where $e \circ h'' = h$, then
$(e \circ h'')(x) = (e \circ h')(x) \Longrightarrow e(h''(x)) = e(h'(x)) \Longrightarrow h''(x) = h'(x),\ \forall x \in C$.

3.6 Pull-back

Definition 33 *Let* $f : (B, \tau_B) \longrightarrow (A, \tau_A)$, $g : (D, \tau_D) \longrightarrow (A, \tau_A)$ *be two F-continuous applications.* C *is a subset of* $B \times D$ *defined by:* $C = \{(x, y) \in B \times D,\ f(x) = g(y)\} \subseteq B \times D$. *We define* τ_C *by:*
$\tau_C = \{\theta, \theta = F(C) \cap \theta'$*is a fuzzy set on* C, $\theta' \in \tau_{B\times D}$ *and* $F(C)$ *is a fuzzy set on* $B \times D$, $where\ F(C)(x, y) = \chi_C(x, y)\}$.

Proposition 34 *It is clear that* (C, τ_C) *is a fuzzy topological space.*

Proposition 35 *The projections* p, q *are F-continuous, where*

$$p : (C, \tau_C) \longrightarrow (B, \tau_B)$$
$$(x, y) \longmapsto p(x, y) = x$$

$$q : (C, \tau_C) \longrightarrow (D, \tau_D)$$
$$(x, y) \longmapsto q(x, y) = y$$

Proof: We prove that p is F-continuous, let $B_i \in \tau_B$, and as $p^{-1}(B_i) = B_i$.
We put $p^{-1}(B_i) = (B_i \times D) \bigwedge F(C)$, then p is F-continuous.
Using the same method, we prove that q is F-continuous.

Theorem 36 *Let* $f : (B, \tau_B) \longrightarrow (A, \tau_A)$, $g : (D, \tau_D) \longrightarrow (A, \tau_A)$ *are two F-continuous applications and* (E, τ_E) *a fuzzy topological space.*
If $h : (E, \tau_E) \longrightarrow (B, \tau_B)$, $k : (E, \tau_E) \longrightarrow (D, \tau_D)$ *be two F-continuous applications where* $f \circ h = g \circ k$ *and* r *defined by:*

$$\begin{aligned} r : (E, \tau_E) &\longrightarrow (C, \tau_C) \\ x &\longmapsto r(x) = (h(x), k(x)). \end{aligned} \tag{1}$$

then: h, k *are F-continuous* $\Longrightarrow$ r *is F-continuous.*

Proof: Let $\theta \in \tau_E$, but:

$$\begin{aligned} r^{-1}(\theta) &= r^{-1}(F(C) \bigwedge (\bigvee_{i\in I} B_i \times D_i) \\ &= r^{-1}(F(C)) \bigwedge r^{-1}(\bigvee_{i\in I} B_i \times D_i) \\ &= r^{-1}(F(C)) \bigwedge \bigvee_{i\in I}(h^{-1}(B_i) \bigwedge k^{-1}(D_i)) \in \tau_E. \end{aligned}$$

As (h, k are F-continuous and $r^{-1}(F(C)) = E$) .

Corollary 37 *Let* $f : (B, \tau_B) \longrightarrow (A, \tau_A), g : (D, \tau_D) \longrightarrow (A, \tau_A)$ *in* *LF-TOP*.
The element of Pull-Back of $\langle f, g\rangle$ *is a topological space* (C, τ_C) *(defined above).*

Proof: First, by definition of C, it is clear that $f \circ p = g \circ q$.
Second, by theorem (36), if $h : (E, \tau_E) \longrightarrow (B, \tau_B)$, $k : (E, \tau_E) \longrightarrow (D, \tau_D)$ are two F-continuous applications where $f \circ h = g \circ k$, then there exists a F-continuous application r defined by (1).
It is clear that $k = q \circ r$ and $h = p \circ r$.
Proof of the unicity of r: let r' be another F-continuous application, where
$r' : (E, \tau_E) \longrightarrow (C, \tau_C)$ and $k = q \circ r'$, $h = p \circ r'$.
We suppose that $r'(x) = (a, b)$, therefore
$a = p(a, b) = (p \circ r')(x) = h(x)\ and$
$b = q(a, b) = (q \circ r')(x) = k(x)$. So $r = r'$.

4 Conclusion

The main aim of the new Lowen's definition of fuzzy topological space, proposed in 1976, is the adding of the properties that can not be verified by Chang's definition [4]. This research showed that:

1. All the theorems and propositions used in this work, are a generalization of that used in finding the universal morphisms in the TOP's category, as fundamental property of the topology product and fundamental property of the topology quotient .

2. TOP's universal morphisms are projections of LF-TOP's ones.

3. Lowen's fuzziness of the ordinary topological space is strong because it generalized the theorems and the propositions of fuzzy topological space. In addition, it is possible to extract from each ordinary space a fuzzy topological space [theorem 7].

4. If we consider the Lowen's definition of continuity (definition (6)), the most fundamental theorems become wrong (ex. theorem (13)), because each a function f from a F-TOP (X, T) to a F-TOP (Y, U) is F-continuous implied f is LF-continuous and the inverse is wrong.

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