Prime geodesic theorem for compact even-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature

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Abstract: Applying a suitably derived, Titchmarsh-Landau style approximate formula for the logarithmic derivative of the Ruelle zeta function, we obtain an another proof of the recently improved variant of the prime geodesic theorem for compact, even-dimensional, locally symmetric spaces of real rank one.

Key–Words: Prime geodesic theorem, Selberg zeta function, Ruelle zeta function

1 Introduction

In [3], the authors have derived a prime geodesic theorem with error terms.

As it is pointed out in the concluding section of that paper, the obtained asymptotic form refines the corresponding one (best known until that time) established by DeGeorge [7] for compact, even- dimensional, locally symmetric spaces of real rank one.

According to the arguments and the techniques that have been made use of, [3] is completely based on Randol's approach in the compact Riemann surfaces case [21]. In particular, this assumes the use of the zeta functions of Selberg and Ruelle, the use of the complex integration over a square contour, and the use of a higher degree counting function of appropriate order. The required properties of the zeta functions are deduced from the Bunke–Olbrich's investigation [4] (see also, [2]).

Earlier, in [1], the authors improved the error term in Park's prime geodesic theorem for real hyperbolic manifolds with cusps [19].

Park's argumentation included a modification of Hejhal's techniques [11, 12] by means of the Ruelle zeta function described by Gon–Park [10] and the complex integration over a circular contour suggested by Fried [8].

Keeping in place these ingredients and increasing the order of the counting function to agree with the corresponding Randol's one, the authors adjusted Park's result to the form [1].

In this paper, motivated by the result [1], we apply the formula for the logarithmic derivative of the Ruelle zeta function (based on [24], and analogous to the one used in [19]), to obtain yet another proof of the prime geodesic theorem [3].

2 Preliminaries

Our notation is based on [4].

Let $Y = \Gamma \backslash G/K = \Gamma \backslash X$ be a compact, $n-$ dimensional (n even), locally symmetric Riemannian manifold of strictly negative sectional curvature, where G is a connected semi-simple Lie group of real rank one, K is a maximal compact subgroup of G and Γ is a discrete, co-compact, torsion-free subgroup of G.

We assume that the Riemannian metric over Y, induced from the Killing form is normalized such that the sectional curvature of Y varies between -4 and -1.

Assume that G is a linear group.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and M the centralizer of $\mathfrak{a}$ in K.

Let $\Phi(\mathfrak{g}, \mathfrak{a})$ be the root system and $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$ a system of positive roots. Let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{n}_\alpha$$

be the sum of the root spaces. Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^{+}(\mathfrak{g},\mathfrak{a})} \dim(\mathfrak{n}_\alpha)\,\alpha.$$

Put $\mathfrak{a}^{+}$ to be the half line in $\mathfrak{a}$ on which the positive roots take positive values and $A^{+} = \exp(\mathfrak{a}^{+})$.

In [4], the authors introduced the concept of $\sigma-$admissibility for $\sigma \in \hat{M}$ and even n.

By [4, p. 49, Lemma 1.18], there exists a $\sigma-$admissible element $\gamma_\sigma \in R(K)$ for every $\sigma \in \hat{M}$, where $R(K)$ denotes the representation ring over $\mathbb{Z}$ of K. Therefore, we shall assume that $\sigma-$admissible $\gamma_\sigma \in R(K)$ is chosen and fixed for every $\sigma \in \hat{M}$ that appears below.

Since $\Gamma \subset G$ is co-compact and torsion-free, there are only two types of conjugacy classes: the class of the identity $e \in \Gamma$ and classes of hyperbolic elements.

Let Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ denote the set of the $\Gamma-$ conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

As it is known, every hyperbolic element $g \in G$ is conjugated to some element $a_g m_g \in A^{+}M$ (see, e.g., [9]). Following [4, p. 59], we define $l(g) = |\log(a_g)|$. Note that for $g \in \Gamma$, the number $l(g)$ is the length of the closed geodesic on Y defined by g.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \rho$ resp. $\mathrm{Re}(s) > 2\rho$, the Selberg zeta function $Z_{S,\chi}(s,\sigma)$ resp. the Ruelle zeta function $Z_{R,\chi}(s,\sigma)$ is defined as the infinite product given by Definition 3.2 resp. Definition 3.1 in [4, pp. 96–97], where σ and χ are finite-dimensional unitary representations of M and Γ, respectively.

The singularities of meromorphically continued $Z_{S,\chi}(s,\sigma)$ are determined in [4, p. 113, Th. 3.15].

It is known that the Ruelle zeta function can be expressed in terms of Selberg zeta functions (see, e.g., [8]). In our case (see, [4, pp. 99–100]), there exist sets $I_p = \left\{(\tau,\lambda) \mid \tau \in \hat{M}, \lambda \in \mathbb{R}\right\}$ such that

$$\begin{aligned} &Z_{R,\chi}(s,\sigma) \\ &= \prod_{p=0}^{n-1} \prod_{(\tau,\lambda)\in I_p} Z_{S,\chi}(s+\rho-\lambda, \tau\otimes\sigma)^{(-1)^p}. \end{aligned} \tag{1}$$

Let $\mathfrak{T}$ be the set of all elements $\tau \in \hat{M}$ that appear in (1).

The following theorems hold true.

Theorem A [2, p. 530, Cor. 4.2.] *A meromorphic extension over $\mathbb{C}$ of the Ruelle zeta function $Z_{R,\chi}(s,\sigma)$ can be expressed as*

$$Z_{R,\chi}(s,\sigma) = \frac{Z_R^1(s)}{Z_R^2(s)},$$

where $Z_R^1(s)$, $Z_R^2(s)$ are entire functions of order at most n over $\mathbb{C}$.

Theorem B *Let $\varepsilon > 0$, $2\rho \geq \eta > 0$. Suppose $t \gg 0$ is chosen so that $\mathrm{i}t$ is not a singularity of $Z_{S,\chi}(s,\tau\otimes\sigma)$, $\tau \in \mathfrak{T}$. Then,*

(i)

$$\begin{aligned} &\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)} \\ &= O\left(t^{n-1+\varepsilon}\right) + \\ &\sum_{p=0}^{n-1} (-1)^p \sum_{\substack{(\tau,\lambda)\in I_p \\ \lambda = 2\rho}} \sum_{\left|t-\gamma_{S,p}^{\tau,\lambda}\right| \leq 1} \frac{1}{s-\rho_{S,p}^{\tau,\lambda}} \end{aligned}$$

for $s = \sigma_1 + \mathrm{i}t$, $\rho \leq \sigma_1 < \frac{1}{4}t - \rho$, where $\rho_{S,p}^{\tau,\lambda} = -\rho + \lambda + \mathrm{i}\gamma_{S,p}^{\tau,\lambda}$ is a singularity of $Z_{S,\chi}(s+\rho-\lambda, \tau\otimes\sigma)$ on the line $\mathrm{Re}(s) = -\rho + \lambda$.

(ii)

$$\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)} = O\left(\frac{1}{\eta} t^{n-1+\varepsilon}\right)$$

for $s = \sigma_1 + \mathrm{i}t$, $\rho + \eta \leq \sigma_1 < \frac{1}{4}t - \rho$.

Note that the proof of this theorem follows from an analogous reasoning as in the proof of the corresponding result in [24] (see also, [19]).

The following lemma will be applied in the sequel (see, [8, p. 509, Prop. 7.])

Lemma C *Suppose $Z(s)$ is the ratio of two nonzero entire functions of order at most n. Then, there is a $D > 0$ such that for arbitrarily large choices of r*

$$\int_r \left| \frac{Z'(s)}{Z(s)} \right| |ds| \leq D r^n \log r.$$

3 Main result

As it is known, a prime geodesic C_γ over Y corresponds to a conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$.

Let $\pi_\Gamma(x)$ be the number of prime geodesics C_γ over Y of length $l(\gamma)$, whose norm $N(\gamma) = e^{l(\gamma)}$ is not larger than x.

We fix $\chi \in \hat{\Gamma}$, $\sigma \in \hat{M}$ and omit them in the notation.

For $g \in \Gamma$, let $n_\Gamma(g) = \#(\Gamma_g / \langle g \rangle)$, where Γ_g is the centralizer of g in Γ and $\langle g \rangle$ is the group generated by g.

If $\gamma \in \Gamma_{\mathrm{h}}$ then $\gamma = \gamma_0^{n_\Gamma(\gamma)}$ for some $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$.

For $\gamma \in \Gamma_{\mathrm{h}}$ we introduce $\Lambda(\gamma) = \Lambda\left(\gamma_0^{n_\Gamma(\gamma)}\right) = \log N(\gamma_0)$.

Finally, we define $\psi_j(x) = \int\limits_0^x \psi_{j-1}(t)\,dt$, $j \in \mathbb{N}$, where $\psi_0(x) = \sum\limits_{\gamma \in \Gamma_{\mathrm{h}}, N(\gamma) \le x} \Lambda(\gamma)$.

Theorem 1. *(Prime Geodesic Theorem) Let Y be as above. Then,*

$$\begin{aligned} &\pi_\Gamma(x) \\ &= \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda) \in I_p} \sum_{s^{p,\tau,\lambda} \in \left(2\rho \frac{n+\rho-1}{n+2\rho-1}, 2\rho\right]} 1 \times \\ &\quad \times \mathrm{li}\left(x^{s^{p,\tau,\lambda}}\right) + O\left(x^{2\rho \frac{n+\rho-1}{n+2\rho-1}} (\log x)^{-1}\right) \end{aligned}$$

as $x \to +\infty$, where $s^{p,\tau,\lambda}$ denotes a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda, \tau)$.

Proof: Let $k \ge 2n$ be an integer and $x > 1$, $c > 2\rho$.

As in [3, pp. 311-312], we obtain

$$\begin{aligned} \psi_k(x) &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} 1 \times \\ &\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds. \end{aligned} \tag{2}$$

Choose any $T_1 \gg 0$ and consider the interval $\mathrm{i}\,t$, $T_1 - 1 < t \le T_1 + 1$.

Reasoning as in Lemma 8 [3, pp. 309-310] (see also, [21], [20]), we conclude that there exists a point in the interval, denote it by $\mathrm{i}\,\tilde{T}$, such that

$$\left|\mathrm{i}\,\tilde{T} - \alpha\right| > \frac{C}{\tilde{T}^n} \tag{3}$$

for some fixed $C > 0$, where $\alpha \in S_R$ and S_R denotes the set of all singularities of all zeta functions $Z_S(s,\tau)$, $\tau \in \mathfrak{T}$.

We define

$$\begin{aligned} &R(T) \\ &= \{s \in \mathbb{C} \mid |s| \le T, \mathrm{Re}(s) \le \rho\} \cup \\ &\quad \left\{s \in \mathbb{C} \mid \rho \le \mathrm{Re}(s) \le c, -\tilde{T} \le \mathrm{Im}(s) \le \tilde{T}\right\}, \end{aligned}$$

where $T = \sqrt{\tilde{T}^2 + \rho^2}$.

By our choice of the point $\tilde{T}$, we know that no pole of $\frac{Z_R'(s)}{Z_R(s)}$ occurs on the square part of the boundary of $R(T)$. Moreover, following [19, p. 98], we may also assume that no pole of the integrand of $\psi_k(x)$ occurs on the circular part of the boundary of $R(T)$.

Applying the Cauchy residue theorem to the integrand of $\psi_k(x)$ over $R(T)$, we obtain

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}} \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds \\ &= \sum_{z \in R(T)} 1 \times \\ &\quad \times \mathrm{Res}_{s=z} \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) \end{aligned} \tag{4}$$

$$\begin{aligned} &-\frac{1}{2\pi \mathrm{i}} \int_{C_T} + \frac{1}{2\pi \mathrm{i}} \int_{\rho+\mathrm{i}\tilde{T}}^{\rho+\delta+\mathrm{i}\tilde{T}} + \frac{1}{2\pi \mathrm{i}} \int_{\rho+\delta-\mathrm{i}\tilde{T}}^{\rho-\mathrm{i}\tilde{T}} + \\ &\frac{1}{2\pi \mathrm{i}} \int_{\rho+\delta+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}} + \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\tilde{T}}^{\rho+\delta-\mathrm{i}\tilde{T}}, \end{aligned}$$

where $0 < \delta < c - \rho$ is fixed and C_T denotes the circular part of the boundary of $R(T)$ with the anticlockwise orientation.

By [4, pp. 96–97, (3.4)],

$$\frac{Z_R'(s)}{Z_R(s)} = -\sum_{\gamma \in \Gamma_{\mathrm{h}}} \frac{\Lambda(\gamma)}{N(\gamma)^s}$$

for $\mathrm{Re}(s) > 2\rho$. Therefore, if $a > 0$,

$$\left|\frac{Z_R'(s+a)}{Z_R(s+a)}\right| \leq \sum_{\gamma\in\Gamma_{\mathrm{h}}} \frac{\Lambda(\gamma)}{N(\gamma)^{2\rho+a}} = -\frac{Z_R'(2\rho+a)}{Z_R(2\rho+a)}$$

for $\mathrm{Re}(s) \geq 2\rho$, i.e., $\frac{Z_R'(s)}{Z_R(s)}$ is bounded for $\mathrm{Re}(s) \geq 2\rho + a$.

We estimate

$$\frac{1}{2\pi\,\mathrm{i}} \int_{c+\mathrm{i}\,\tilde{T}}^{c+\mathrm{i}\,\infty} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds = O\left(x^{c+k} \int_{\tilde{T}}^{+\infty} \frac{dt}{t^{k+1}}\right) = O\left(x^{c+k}\tilde{T}^{-k}\right) \tag{5}$$

and

$$\frac{1}{2\pi\,\mathrm{i}} \int_{c-\mathrm{i}\,\infty}^{c-\mathrm{i}\,\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds = O\left(x^{c+k}\tilde{T}^{-k}\right). \tag{6}$$

Now, we estimate the integrals on the right hand side of (4).

By Theorem A and Lemma C,

$$\begin{aligned} &\frac{1}{2\pi\,\mathrm{i}} \int_{C_T} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds \\ &= O\left(x^{\rho+k}T^{-k-1} \int_{C_T} \left|\frac{Z_R'(s)}{Z_R(s)}\right| |ds|\right) \\ &= O\left(x^{\rho+k}T^{-k-1} \int_{|s|=T} \left|\frac{Z_R'(s)}{Z_R(s)}\right| |ds|\right) \\ &= O\left(x^{\rho+k}T^{-k-1+n}\log T\right). \end{aligned} \tag{7}$$

In order to estimate the remaining integrals, we apply (3) and Theorem B. Fix some $\varepsilon > 0$.

We obtain

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\rho+\mathrm{i}\,\tilde{T}}^{\rho+\delta+\mathrm{i}\,\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds =$$

$$O\left(x^{\rho+\delta+k}T^{-k-1} \int_{\rho+\mathrm{i}\,\tilde{T}}^{\rho+\delta+\mathrm{i}\,\tilde{T}} \left|\frac{Z_R'(s)}{Z_R(s)}\right| |ds|\right).$$

By Theorem B (*i*) and (3),

$$\begin{aligned} &\frac{Z_R'(s)}{Z_R(s)} \\ &= O\left(\tilde{T}^{n-1+\varepsilon}\right) + \sum_{p=0}^{n-1} (-1)^p \sum_{\substack{(\tau,\lambda)\in I_p \\ \lambda=2\rho}} \sum_{\left|\tilde{T}-\gamma_{S,p}^{\tau,\lambda}\right|\leq 1} \frac{1}{s-\rho_{S,p}^{\tau,\lambda}} \\ &= O\left(\tilde{T}^{n-1+\varepsilon}\right) + O\left(\tilde{T}^{n} \sum_{p=0}^{n-1} (-1)^p \sum_{\substack{(\tau,\lambda)\in I_p \\ \lambda=2\rho}} \sum_{\left|\tilde{T}-\gamma_{S,p}^{\tau,\lambda}\right|\leq 1} 1\right) \\ &= O\left(\tilde{T}^{n-1+\varepsilon}\right) + O\left(\tilde{T}^{2n} \sum_{p=0}^{n-1} (-1)^p \sum_{\substack{(\tau,\lambda)\in I_p \\ \lambda=2\rho}} 1\right) \\ &= O\left(\tilde{T}^{n-1+\varepsilon}\right) + O\left(\tilde{T}^{2n}\right) = O\left(\tilde{T}^{2n}\right) = O\left(T^{2n}\right) \end{aligned}$$

for $s = \sigma_1 + \mathrm{i}\,\tilde{T}$, $\rho \leq \sigma_1 \leq \rho + \delta$. Hence,

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\rho+\mathrm{i}\,\tilde{T}}^{\rho+\delta+\mathrm{i}\,\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) ds = O\left(x^{\rho+\delta+k}T^{-k-1+2n}\right). \tag{8}$$

Similarly,

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\rho+\delta-\mathrm{i}\tilde{T}}^{\rho-\mathrm{i}\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\dots(s+k)}\right)ds = O\left(x^{\rho+\delta+k}T^{-k-1+2n}\right). \tag{9}$$

By Theorem B (*ii*),

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\rho+\delta+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\dots(s+k)}\right)ds = O\left(x^{c+k}T^{-k-1}\int\limits_{\rho+\delta+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right) = O\left(\delta^{-1}x^{c+k}T^{-k-2+n+\varepsilon}\right). \tag{10}$$

Similarly,

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\tilde{T}}^{\rho+\delta-\mathrm{i}\tilde{T}} 1\times \left(-\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\dots(s+k)}\right)ds = O\left(\delta^{-1}x^{c+k}T^{-k-2+n+\varepsilon}\right). \tag{11}$$

Combining (1), (2), (4)-(11) and letting $T \to +\infty$, we get

$$\psi_k(x) = \sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A_k^{p,\tau,\lambda}} c_z(p,\tau,\lambda,k),$$

where $A_k^{p,\tau,\lambda}$ denotes the set of poles of $\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{s(s+1)\dots(s+k)}$ and $c_z(p,\tau,\lambda,k) = \mathrm{Res}_{s=z}\left(\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{s(s+1)\dots(s+k)}\right)$.

Now, proceeding exactly in the same way as in [3, pp. 313f], we obtain the claim. This completes the proof. □

4 Consequences and concluding remarks

By (1), Theorem 1 can be written as

$$\pi_\Gamma(x) = -\sum_{s_R\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\mathrm{li}\left(x^{s_R}\right)+ O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right) \tag{12}$$

as $x \to +\infty$, where s_R denotes a singularity of the Ruelle zeta function $Z_R(s)$.

Being Riemannian symmetric space of rank one, X is either a real, a complex or a quaternionic hyperbolic space or the hyperbolic Cayley plane, i.e., X is one of the following spaces:

$$H\mathbb{R}^k\,(k \text{ even}, k\geq 2),\ H\mathbb{C}^m\,(m\geq 1),$$
$$H\mathbb{H}^l\,(l\geq 1),\ H\mathbb{C}a^2.$$

Hence, $n = k$, $2m$, $4l$, 16 and $\rho = \frac{1}{2}(k-1)$, m, $2l+1$, 11, respectively.

Since $H\mathbb{C}^1 \cong H\mathbb{R}^2$ and $H\mathbb{H}^1 \cong H\mathbb{R}^4$ (see, e.g., [13]), we may assume $m \geq 2$ and $l \geq 2$.

Let $X = H\mathbb{R}^k$, k even, $k \geq 2$.

Now, by (12) and [5, p. 45],

$$\pi_\Gamma(x) = \sum_{p=0}^{\frac{k}{2}-1}(-1)^p\sum_{s(p)\in\left(\frac{3}{4}(k-1),k-1\right]}\mathrm{li}\left(x^{s(p)}\right) + O\left(x^{\frac{3}{4}(k-1)}(\log x)^{-1}\right) \tag{13}$$

as $x \to +\infty$, where $s(p)$ denotes a singularity of the Selberg zeta function $Z_S\left(s-\frac{k-1}{2}+p,\sigma_p\right)$ and σ_p is the $p-$th exterior power of the standard representation of $SO(k-1)$ (see, e.g., [4, p. 23]). In particular, $Z_S(s,\sigma_p)$ is the Selberg zeta function whose singularity pattern is given by Proposition 5.5 [4, p. 150].

Note that (13) agrees with the corresponding result in the case of real hyperbolic manifolds with cusps [1].

Let $X = H\mathbb{C}^m$, $m \geq 2$.

Now, by (12) and [4, p. 138, Lemma 4.10],

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{p=0}^{m}\sum_{q=m-p+1}^{m}(-1)^{p+q}\times\\&\times\sum_{s^{p,q}\in\left(2m\frac{3m-1}{4m-1},2m\right]}\operatorname{li}\left(x^{s^{p,q}}\right)+\\&O\left(x^{2m\frac{3m-1}{4m-1}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$, where $s^{p,q}$ denotes a singularity of the Selberg zeta function $Z_S(s+m-p-q,\sigma_{p-1,q-1})$ and $\sigma_{p,q}$ is the irreducible representation introduced in [4, p. 138] (see also, [4, p. 24]). $Z_S(s,\sigma_{p,q})$ is the Selberg zeta function whose singularities are described by Proposition 5.7 [4, p. 152].

If $X=H\mathbb{H}^l$, $l\geq 2$ or $X=H\mathbb{C}a^2$, then, according to [4, p. 154], a satisfactory understanding of the sets I_p, p large, in terms of finding an explicit expression for the Ruelle zeta function via Selberg zeta functions, does not seem to be available for now.

Let $X=H\mathbb{H}^l$, $l\geq 2$.

By Theorem 1 and (12)

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{p=0}^{4l-1}(-1)^{p+1}\times\\&\times\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(12l\frac{2l+1}{8l+1},4l+2\right]}\operatorname{li}\left(x^{s^{p,\tau,\lambda}}\right)+\\&O\left(x^{12l\frac{2l+1}{8l+1}}(\log x)^{-1}\right)\\&=-\sum_{s_R\in\left(12l\frac{2l+1}{8l+1},4l+2\right]}\operatorname{li}\left(x^{s_R}\right)+\\&O\left(x^{12l\frac{2l+1}{8l+1}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$.

Finally, let $X=H\mathbb{C}a^2$.

Now, by Theorem 1 and (12)

$$\begin{aligned}&\pi_\Gamma(x)\\&=\sum_{p=0}^{15}(-1)^{p+1}\times\\&\times\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(\frac{572}{37},22\right]}\operatorname{li}\left(x^{s^{p,\tau,\lambda}}\right)\\&+O\left(x^{\frac{572}{37}}(\log x)^{-1}\right)\\&=-\sum_{s_R\in\left(\frac{572}{37},22\right]}\operatorname{li}\left(x^{s_R}\right)+\\&O\left(x^{\frac{572}{37}}(\log x)^{-1}\right)\end{aligned}$$

as $x\to+\infty$.

As it has been already proved in [3], the error term in Theorem 1 is sharper than the optimal error term $O\left(x^{\left(1-\frac{1}{2n}\right)2\rho}\right)$ given in [7].

Our error term is of the form $O\left(x^{\alpha}(\log x)^{-1}\right)$, where $\alpha<\frac{3}{2}\rho$ if $X=H\mathbb{C}^m$, $(m\geq 2)$, $H\mathbb{H}^l$, $(l\geq 2)$, $H\mathbb{C}a^2$ and $\alpha=\frac{3}{2}\rho$ if $X=H\mathbb{R}^k$, k even, $k\geq 2$.

As we already noted, (13) agrees with the corresponding result [1]. Moreover, for $k=2$, (13) becomes [21], i.e.,

$$\begin{aligned}\pi_\Gamma(x)&=\sum_{s(0)\in\left(\frac{3}{4},1\right]}\operatorname{li}\left(x^{s(0)}\right)+\\&O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)\\&=\sum_{s(0)\in\left(\frac{3}{4},1\right]}\operatorname{li}\left(x^{s(0)}\right)+E(x)\end{aligned}\tag{14}$$

as $x\to+\infty$, where $\lambda(0)=s(0)(1-s(0))$ is a small eigenvalue in $\left[0,\frac{3}{16}\right]$ of the Laplacian Δ_0 on $L^2(Y)$, Y is a compact hyperbolic Riemann surface, and Γ is a discontinuous subgroup of $G=\mathrm{PSL}(2,\mathbb{R})$.

If $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$, (14) becomes

$$\pi_\Gamma(x)=\operatorname{li}(x)+E(x),$$

as $x\to+\infty$ (see, [15], [18], [6], [23]), where $E(x)\ll x^{\frac{3}{4}+\varepsilon}$, $\varepsilon>0$. Here, the error term $E(x)$ is related to zeros of the corresponding Selberg zeta function in an analogous way as the error term in the prime number theorem is related to zeros of the Riemann zeta function. There are differences, however. Namely, the Selberg zeta function is a meromorphic function of order two. On the other side, the Riemann zeta function is a meromorphic function of order one. Moreover, the Selberg zeta function is known to satisfy an analog of the Riemann hypothesis. Bearing this in mind, one should expect that the estimate $E(x)\ll x^{\frac{1}{2}+\varepsilon}$, $\varepsilon>0$ holds true. This has yet to be proven, though. The main difficulty in achieving this estimate stems from the fact that the Selberg zeta function (as a function of order two), has much more zeros than the Riemann zeta function. Thus, $E(x) \ll x^{\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$ remains an open problem up to these days.

The following breaks are known (see, e.g., [23]).

Iwaniec [15], proved that $E(x) \ll x^{\frac{35}{48}+\varepsilon}$, $\varepsilon > 0$. Luo-Sarnak [18], improved Iwaniec's result to $E(x) \ll x^{\frac{7}{10}+\varepsilon}$, $\varepsilon > 0$. Cai [6], improved the last result to get $E(x) \ll x^{\frac{71}{102}+\varepsilon}$, $\varepsilon > 0$. Finally, Soundararajan-Young [23], established the best known estimate $E(x) \ll x^{\frac{25}{36}+\varepsilon}$, $\varepsilon > 0$.

In future research, the author is going to pay attention to weighted form of the prime geodesic theorem derived in this paper and in [3]. The motivation for it stems from the fact that the error term in (14) is far from the expected $E(x) \ll x^{\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$, and the fact that an analog of the Riemann hypothesis is known to be true in this case. The goal will be to obtain a ψ_k-level analogue, $k \geq 0$, of the result derived here.

5 Applications of prime geordesic theorem

Prime geodesic theorem gives the number $\pi_\Gamma(x)$ of prime geodesics C_γ over Y whose length is not larger than $\log x$. As it is known, a prime geodesic C_γ over Y corresponds to a conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$. In other words,

$$\pi_\Gamma(x) = \#\{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}} \,:\, l(\gamma_0) \leq \log x\}.$$

Actually, there is a canonical dynamical system on SY, i.e., the geodesic flow φ determined by the metric of Y, given by (see, [4, p. 95])

$$\varphi : \mathbb{R} \times SY \ni (t, \Gamma g M) \to \Gamma g \exp(-tH) M \in SY,$$

where H is the unit vector in $\mathfrak{a}^+$, $SY = \Gamma \backslash G/M$ is the unit sphere bundle of Y, and M is the centralizer of $\mathfrak{a}$ in K with Lie algebra $\mathfrak{m}$. Our interest is in number of closed geodesics on Y, i.e., in the number of closed orbits of φ. The most common invariant measures for a dynamical system are those carried by periodic orbits. Counting periodic orbits is thus a natural task from the point of view of ergodic theory. The most effective tool to do the counting are zeta functions (see, [22]).

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1},$$

$\mathrm{Re}(s) > 1$ generates prime numbers. Hence, it represents an appropriate tool to count prime numbers. Prime number theorem (in its best form up to now), states that

$$\begin{aligned}\pi(x) &= \mathrm{li}(x) + E(x) \\ &= \mathrm{li}(x) + O\left(xe^{-c\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}}\right),\end{aligned}$$

as $x \to +\infty$, where $\pi(x)$ denotes the number of prime numbers not larger than x. Recall that the Riemann hypothesis, or the RH (which states that the non-trivial zeros of $\zeta(s)$ are on the line $\mathrm{Re}(s) = \frac{1}{2}$) is equivalent to the fact that $E(x) = O\left(x^{\frac{1}{2}} \log x\right)$. Moreover, it has been noted, (assuming the RH), that the spacings of the imaginary parts of the non-trivial zeros of $\zeta(s)$ behave like eigenvalues of random Hermitian matrices. This fact relates zeta and quantum chaos. This is so called Montgomery-Odlyzko law or Gaussian Unitary Ensemble (see, [16]).

There is the Dedekind zeta function of an algebraic number field K ($\mathbb{Q}\left(\sqrt{3}\right)$ for example). This zeta function is an infinite product over prime ideals $\mathfrak{p}$ in the ring O_K of algebraic integers of K. Hence, Riemann's work can be extended to this function to prove the prime ideal theorem (see, [17]).

The Weil zeta function

$$\zeta_W(z) = \exp \sum_{m=1}^{+\infty} \frac{z^m}{m} |\mathrm{Fix}\, f^m|$$

counts periodic orbits for the dynamical system (M, f), where M is a space obtained by the extension of an algebraic variety over finite filed $\mathbb{F}_q$ to the algebraic closure of $\mathbb{F}_q$, $f : M \to M$ is the Frobenius map (acting as $z \to z^q$ on coordinates), and $|\mathrm{Fix}\, f^m|$ is the number of fixed points of m-th iterate of f. A natural way to count periodic orbits for a map f is to weight them with topological index $L(x, f) = \mathrm{sgn}\det(1 - T_x f)$, where f is a diffeomorphism on the compact manifold M, $x \in \mathrm{Fix}\, f^m$, $1 - T_x f$ is invertible, and $T_x f$ denotes the tangent map to f at x (see, [22]).

In this paper, the Ruelle zeta function is introduced by following Definition 3.1 [4, p. 96]. In such cases, a discrete time dynamical system generated by $f : M \to M$ is replaced by a continuous dynamical system, i.e., a semiflow or flow on M. The most common type of flow is the geodesic flow on a Riemann manifold (the geodesic flow on a compact manifold of negative curvature is an Anosov flow). Therefore, a zeta function is defined as a product over prime periodic orbits γ_0, where $l(\gamma_0)$ is the period of γ_0. As we noted in the Preliminary section of the paper, the period $l(\gamma_0)$ of the periodic orbit γ_0 (for the geodesic flow) is the length of the closed geodesic determined by γ_0.

A natural ally of the Ruelle zeta function is the Selberg zeta function (see, [4, p. 97, Def. 3.2]). If Y is a compact hyperbolic Riemann surface, i.e., if $Y = \Gamma \setminus \mathbb{H}$, where Γ is a torsion free Fuchsian group operating on the Lobatchevsky plane $\mathbb{H}$ with the Poincaré metric, then, the zeros of the Selberg zeta function are related to the eigenvalues of the Laplace-Beltrami operator Δ on $\Gamma \setminus \mathbb{H}$ (see, previous section). In this way, we obtain a connection between classical mechanics (the geodesic flow) and quantum mechanics (with the Hamiltonian Δ). This connection is also related to quantum chaos (see, [22]).

A dynamical zeta function is a zeta function

$$\zeta(z) = \exp \sum_{m=1}^{+\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} g\left(f^k x\right)$$

associated with the weighted dynamical system (M, f, g), where $\operatorname{Fix} f^m$ is finite set for all $m > 0$, and the dynamical system (M, f) is equipped with a weight $g = \exp A$, where A is a real function. The radius of convergence of $\zeta(s)$ is $\exp(-P(A))$, where

$$P(A) = \limsup_{m \to \infty} \frac{1}{m} \log \sum_{x \in \operatorname{Fix} f^m} \exp \sum_{k=0}^{m-1} A\left(f^k x\right).$$

The function $A \to P(A)$, called pressure, stems from a theory called thermodynamic formalism, which is based on techniques of statistical mechanics. The dynamical zeta function is introduced in such a way to count periodic orbits with general weights.

Summarizing what is said above, one can say that the counting of periodic orbits with weights is related to various areas of research: Selberg zeta functions, thermodynamic formalism, hyperbolic dynamics, as well as to the areas: Grothendieck-Fredholm determinants, kneading determinants, etc. (see, [22]).

The graph theory zetas first appeared in work of Ihara on p-adic groups. In particular, the Ihara (vertex) zeta function of X is defined at $u \in \mathbb{C}$, for $|u|$ sufficiently small, by (see, [14])

$$\zeta_V(u, X) = \prod_{[P]} \left(1 - u^{v(P)}\right)^{-1},$$

where $[P]$ runs over primes of X and V denotes the set of vertices of X. More precisely, X is a finite, connected, rank≥ 1 graph, with no danglers, i.e., with no degree 1 vertices. Rank means the rank of the fundamental group of the graph. If X is a finite connected undirected graph with vertex set V and undirected edge set E, one orients its edges arbitrarily and obtains $2|E|$ oriented edges indexed

$$e_1, e_2, ..., e_n, e_{n+1} = e_1^{-1}, ..., e_{2n} = e_n^{-1}.$$

Primes $[P]$ in X are equivalence classes of closed backtrackless tailless primitive paths P. For example, $C = a_1\, a_2\, ...\, a_s$ is a path, where a_j is an oriented edge of X. The length of C is $v(C) = s$. Backtrackless means that $a_{i+1} \neq a_i^{-1}$ for all i. Tailless means that $a_s \neq a_1^{-1}$. The equivalence class $[C]$ is the set

$$[C] = \{a_1 a_2 ... a_s, a_2 a_3 ... a_s a_1, ..., a_s a_1 ... a_{s-1}\}.$$

$[P]$ is primitive means that $P \neq D^m$ for any integer $m \geq 2$ and any path D in X. The rank of the fundamental group of X is denoted by r_X and is defined by $r_X - 1 = |E| - |V|$. In other words, r_X is the number of edges deleted from X to form a spanning tree. The following graph theory prime number theorem holds true (see, [14])

Suppose that R_X is the radius of the largest circle of convergence of the Ihara function. Then, for a connected graph X with $\Delta_X = 1$, we have

$$\pi(m) \sim \frac{R_X^{-m}}{m}$$

as $m \to +\infty$. If $\Delta_X > 1$, then $\pi(m) = 0$ unless Δ_X divides m. If Δ_X divides m, then

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m}$$

as $m \to +\infty$. Here, $\pi(m)$ is the prime counting function and is defined by

$$\pi(n) = \#\{[C] \ prime \ in \ X : v(C) = n\}.$$

Furthermore, Δ_X is the greatest common divisor of the prime path lengths in X, i.e.,

$$\Delta_X = \gcd\{v(P) : [P] \ prime \ in \ X\}.$$

Finally, note that Pavey [20], applied a prime geodesic theorem to prove an asymptotic formula for class numbers of orders in totally complex quartic fields with no real quadratic subfield. His prime geodesic theorem is derived for a compact symmetric space formed as a quotient of the Lie group $\mathrm{SL}_4(R)$.

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