Watersheds for Solutions of Nonlinear Parabolic Equations

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Abstract: In this paper we describe a technique that we have used in a number of publications to find the "watershed" under which the initial condition of a positive solution of a nonlinear reaction-diffusion equation must lie, so that this solution does not develop into a traveling wave, but decays into a trivial solution. The watershed consists of the positive solution of the steady-state problem together with positive pieces of nodal solutions (with identical boundary conditions). We prove in this paper that our method for finding watersheds works in R^k, $k \geq 1$, for increasing functions $f(z)/z$. In addition, we weaken the condition that $f(z)/z$ be increasing, and show that the method also works in R^1 when $f(z)/z$ is bounded. The decay rate is exponential.

Key–Words: Nonlinear parabolic equations, positive solutions, nodal solutions.

1 Introduction

One of the interesting situations in thermodynamics is to study the behavior of a heat equation when the initial condition to the problem begins in a neighborhood of an unstable solution of the associated steady-state problem. For simplicity, in this introduction, we will illustrate what we mean in one space dimension, leaving the details about R^k to Section 2. Consider the reaction-diffusion equation

$$u_t = \delta u_{xx} + f(u), \quad x \in \Omega, \quad t > 0, \tag{1}$$

where δ is a nonnegative diffusion coefficient and $\Omega = [a, b]$ is a closed bounded interval on the reals. Suppose that we want the solution of (1) to satisfy Neumann boundary conditions

$$u_x(a, t) = 0 = u_x(b, t), \quad t > 0, \tag{2}$$

and that $f(u) = 0$ has several roots, $u = u_j$, $j = 0, 1, ..., n$. Then each of these roots is a solution to the problem (1)-(2), so that the problem has multiple constant stationary solutions. When $\delta = 0$, the problem is a first-order differential equation (in t for each x) with multiple constant solutions. The solutions are stable when $f'(u_j) < 0$, unstable when $f'(u_j) > 0$, and their stability can be determined by analyzing the higher order terms of the Taylor series of $f(u)$ at u_j when $f'(u_j) = 0$. Assume we are given the initial condition

$$u(x, 0) = \phi(x), \quad a \leq x \leq b. \tag{3}$$

If $\phi(x)$ lies entirely in the trough between a stable and an unstable constant solution, it will tend (for each x) to the stable solution, while if part of $\phi(x)$ lies in one trough while the rest belongs to an adjacent trough, with the unstable solution between, the parts in each trough will tend to the stable solutions forming stationary fronts.

When $\delta > 0$, and $\phi(x)$ lies in a single trough, the solution u again approaches the stable constant solution, but if different parts of $\phi(x)$ lie in two adjacent troughs as above, then $u(x, t)$ will be influenced by both the diffusion term and the different stable constant solutions. Parts of $u(x, t)$ will move trying to adjust and produce, in some cases, nonstationary fronts, called traveling waves, first investigated in the celebrated papers of Fisher [5] and Kolmogoroff et al. [6]. Although traveling waves are common, and have a substantial literature (see Fife [4], even when $\phi(x)$

lies in two adjacent troughs, the solution $u(x,t)$ need not form a traveling wave. Instead, the solution $u(x,t)$ may collapse directly to one of the stable constant solutions, or blow up if not bounded by a stable solution above. Chen and Derrick [1] studied this situation for Dirichlet boundary conditions in R^k, and in Derrick et al. [2] and [3] we studied it for Neumann boundary conditions in R^1. In Section 2 of this paper we prove that the technique works for bounded domains in R^k with boundary conditions $\alpha u_n = \beta u$, with α, β not both 0. We showed in Derrick et al. [3] that the positive and existing nodal solutions (for sufficiently small $\delta > 0$) of the steady-state problem

$$\delta v_{xx} + f(v) = 0, \quad x \in \Omega, \tag{4}$$

with identical boundary conditions, form a watershed $v^*(x)$ (see Watt et al. [7]) when patched together appropriately. If $\phi(x)$ lies between the watershed $v^*(x)$ and a stable solution $u_j(x)$, then the solution will collapse to $u_j(x)$, even though $\phi(x)$ may intersect an unstable solution $u_{j\pm1}(x)$ repeatedly. If we divide (4) by δ, we obtain a nonlinear eigenvalue problem with $1/\delta$ as the eigenvalue; higher eigenvalues lead to increasing nodes, so the smaller δ is, the more nodal solutions may exist.

In the sections that follow, we will assume without loss of generality that the trivial solution $u \equiv 0$ is stable and some positive solution v of (4) bounds the initial condition ϕ. This is not a restriction since it is frequently possible to recast the problem this way by substitution. We will assume that the function f is piecewise continuous, and that $f(0) = 0$. In the papers Chen and Derrick [1], and Derrick et al. [2], [3], we required that the function $f(u)/u$ be increasing in u, over the range of v. Typical functions $f(u)$ that apply are u^p, $p > 1$, $u(1 + u^p)$, and $e^u - 1$, for $u \geq 0$, $u(0.5 - u)(1 - u)$ over $0 \leq u \leq 0.75$, and certain zeros of the Kamenetskii combustion equation $e^{-1/u} - \delta(u - a)$. Many other functions will also work. The requirement that $f(u)/u$ be increasing is not necessary: one of the main results in this paper (Sections 3 and 4 for R^1) will be to remove that condition, and to replace it by boundedness.

2 Preliminaries

Let Ω be the closure of a bounded domain in R^k, $k \geq 1$, and let its boundary $\partial\Omega$ be C^1 (for $k > 1$), so that normal derivatives will exist. Consider the reaction-diffusion problem

$$\begin{gathered} u_t = \delta\Delta u + f(u), \quad x \in \Omega, \quad t > 0, \\ \alpha u(x,t) = \beta u_n(x,t), \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \phi(x), \quad x \in \Omega, \end{gathered} \tag{5}$$

where δ is a positive diffusion coefficient, α and β are not both zero, and n is the unit outward normal to Ω. Let $v(x)$ be a steady-state solution to problem (5); hence v solves

$$\begin{gathered} \delta\Delta v + f(v) = 0, \quad x \in \Omega, \\ \alpha v(x) = \beta v_n(x), \quad x \in \partial\Omega. \end{gathered} \tag{6}$$

Suppose that positive solutions $u(x,t)$ to problem (5) and $v(x)$ to problem (6) both exist, and that $0 \leq u(x,t) < \lambda v(x)$, $0 < \lambda < 1$, for every $x \in \Omega$ and $0 \leq t \leq T$. Then the function

$$g_n(t) = \int_\Omega \frac{u^{n+2}(x,t)}{v^n(x)}dx = \int_\Omega \left(\frac{u}{v}\right)^{n+2} v^2 dx, \quad n \geq 1, \tag{7}$$

is properly defined for all $0 \leq t \leq T$, since $(u/v)^{n+2} \leq \lambda^{n+2} < 1$. Indeed, $(g_n(t))^{\frac{1}{n+2}}$ is the L_{n+2}-norm of $(u(.,t)/v)$ over Ω with respect to the measure v^2dx:

$$g_n(t) = \left(\| \frac{u}{v} \|_{n+2}\right)^{n+2}. \tag{8}$$

Note that with this strong condition imposed on u and v, $g_n(t)$ decreases monotonically as n increases, for each t in $0 \leq t \leq T$.

Lemma 1 *Let u and v be positive solutions of problems (5) and (6), respectively. Then*

$$g_n'(t) = (n+2)\left\{\int_\Omega \frac{u^{n+2}}{v^n}\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)dx - \delta(n+1)\int_\Omega \frac{u^n}{v^{n+2}}|v\nabla u - u\nabla v|^2 dx\right\}. \tag{9}$$

Proof: Differentiating (7) with respect to t inside the integral sign and replacing u_t by the right side of the first equation of (5) we have

$$g_n'(t) = (n+2)\int_\Omega \frac{u^{n+1}}{v^n}(\delta\Delta u + f(u))dx.$$

By the first equation in (6)

$$0 = (n+2)\int_\Omega \frac{u^{n+2}}{v^{n+1}}(\delta\Delta v + f(v))dx,$$

so, if we subtract,

$$g_n'(t) = (n+2)\left\{\int_\Omega \frac{u^{n+2}}{v^n}\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)dx\right.$$

$$+\delta \int_\Omega \left(\frac{u}{v}\right)^{n+1} (v\Delta u - u\Delta v)dx \Bigg\}.$$

Using Green's theorem on the second integral above yields

$$\int_\Omega \left(\frac{u}{v}\right)^{n+1} (v\Delta u - u\Delta v)dx$$

$$= -(n+1)\int_\Omega \left(\frac{u}{v}\right)^{n} |\nabla u|^2 dx$$

$$+n\int_\Omega \left(\frac{u}{v}\right)^{n+1} \nabla u \nabla v dx$$

$$+(n+2)\int_\Omega \left(\frac{u}{v}\right)^{n+1} \nabla u \nabla v dx$$

$$-(n+1)\int_\Omega \left(\frac{u}{v}\right)^{n+2} |\nabla v|^2 dx$$

$$+\int_{\partial\Omega} \left(\frac{u}{v}\right)^{n+1} (vu_n - uv_n)dx,$$

or

$$\begin{aligned}\int_\Omega \left(\frac{u}{v}\right)^{n+1} (v\Delta u - u\Delta v)dx \\ = -(n+1)\int_\Omega \left(\frac{u^n}{v^{n+2}}\right) |v\nabla u - u\nabla v|^2 dx\end{aligned} \quad (10)$$

$$+\int_{\partial\Omega} \left(\frac{u}{v}\right)^{n+1} (vu_n - uv_n)dx.$$

From the boundary conditions in (5) and (6), either $u = v = 0$ (when $\beta = 0$), or $u_n = v_n = 0$ (when α=0), or $u_n = (\alpha/\beta)u$ and $v_n = (\alpha/\beta)v$ on $\partial\Omega$, so that $vu_n - uv_n = 0$, when u_n and v_n are the normal derivatives of u and v for $x \in \partial\Omega$. Since $0 < (u/v) < \lambda$, this shows that the integral on $\partial\Omega$ is zero. □

Remark 2 *If $k = 1$, then $\Omega = [a, b]$ and the boundary term in (10) becomes*

$$\left(\frac{u}{v}\right)^{n+1} (vu_x - uv_x)\Bigg|_b^a,$$

which is zero for the boundary conditions in (5)-(6), which include the following common conditions:

$$u(a,t) = 0 = u(b,t), \quad t \geq 0,$$

$$\textit{and} \quad v(a) = 0 = v(b) \quad (Dirichlet),$$

$$u(a,t) = 0 = u_x(b,t), \quad t \geq 0,$$

$$\textit{and} \quad v(a) = 0 = v_x(b) \quad (Mixed),$$

$$u_x(a,t) = 0 = u(b,t), \quad t \geq 0,$$

$$\textit{and} \quad v_x(a) = 0 = v(b) \quad (Mixed),$$

$$u_x(a,t) = 0 = u_x(b,t), \quad t \geq 0,$$

$$\textit{and} \quad v_x(a) = 0 = v_x(b) \quad (Neumann).$$

Furthermore, the boundary term is also zero for periodic conditions

$$u(a,t) = u(b,t), u_x(a,t) = u_x(b,t),$$

$$t \geq 0 \quad \textit{and} \quad v(a) = v(b), v_x(a) = v_x(b).$$

When $k = 1$, equation (9) becomes

$$\begin{aligned} g_n'(t) = (n+2)\Bigg\{\int_a^b \frac{u^{n+2}}{v^n}\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)dx \\ -\delta(n+1)\int_a^b \frac{u^n}{v^{n+2}}|vu_x - uv_x|^2 dx\Bigg\}. \end{aligned} \quad (11)$$

We used the strong condition $0 \leq u(x,t) \leq \lambda v(x)$, $0 < \lambda < 1$, for all $x \in \Omega$ and $0 \leq t \leq T$ to properly define $g_n(t)$ above, but a much weaker assumption will do the job for certain functions f.

Lemma 3 *Let $f(z)/z$ be an increasing function for $z \geq 0$. If $0 \leq \phi(x) < \lambda v(x)$, $\lambda < 1$, where ϕ is defined in problem (5) and v is the positive solution of the steady state problem (6). Then the positive solution u to problem (6) satisfies $0 \leq u(x,t) < \lambda v(x)$, for all $t > 0$ and $x \in \Omega$.*

Proof: Since $u(x, 0) = \phi(x) < \lambda v(x)$ with $\lambda < 1$, $u(x,t) < v(x)$ for all $x \in \Omega$ with t near 0. Assume there are $x_j \in \Omega$, $t_j > 0$ for which $u(x_j,t_j) \geq v(x_j)$, and let $t_0 > 0$ be the least of these numbers t_j. Since $v > u \geq 0$ in $\Omega \times [0, t_0)$, the monotonicity of $f(z)/z$ implies that in (9) the first integral is negative, so that $g_n'(t) < 0$. Hence $g_n(t) < g_n(0)$ for $0 < t < t_0$.

Taking $(n + 2)$-nd roots in this last inequality, we get the L_{n+2}-norms (with measure v^2dx)

$$||\frac{u}{v}||_{n+2} = \Big(\int_\Omega \left(\frac{u}{v}\right)^{n+2} v^2 dx\Big)^{\frac{1}{n+2}}$$

$$\leq \Big(\int_\Omega \left(\frac{\phi}{v}\right)^{n+2} v^2 dx\Big)^{\frac{1}{n+2}} = ||\frac{\phi}{v}||_{n+2},$$

for $t \in [0, t_0)$. Since the L_{n+2}-norm tends to the L_∞-norm as $n \to \infty$ (see the Appendix), we have by continuity

$$\frac{u(x_0,t)}{v(x_0)} \le \sup_\Omega \frac{u}{v} \le \sup_\Omega \frac{\phi}{v} < \lambda < 1 \qquad (12)$$

$$\text{for} \quad t \quad \text{in} \quad [0, t_0].$$

This contradicts the assumption that $u(x_0, t_0) \ge v(x_0)$. Hence, $u(x,t) < v(x)$ for all $x \in \Omega$, $t \ge 0$. Thus, the first integral in (9) is always negative, so the second inequality in (12) holds for all $t \ge 0$. Hence, $0 \le u(x,t) < \lambda v(x)$, for all $x \in \Omega$ and $t > 0$. □

It is not necessary to assume $f(z)/z$ is increasing for all $z \ge 0$, but simply over the watershed.

We have shown that the solutions $u(x,t)$ stay below $v(x)$ for all t. We now show that $u(x,t)$ decays exponentially to zero, implying that the steady state solution $v(x)$ is repelling.

Theorem 4 *Let $f(z)/z$ be an increasing function for $0 \le z \le \max_\Omega v$. If $0 \le \phi < \lambda v$, $\lambda < 1$, where v is the solution of the steady state problem (6), then there exists a constant $c_0 > 0$, so that the solution u to (5) satisfies*

$$u(x,t) < \lambda v(x) \exp(-c_0 t) \qquad (13)$$
$$\textit{for all} \quad t > 0 \quad \textit{and} \quad x \in \Omega.$$

Proof: We showed at the conclusion of Lemma 3 that $u(x,t) \le \lambda v(x) < v(x)$ for all $t > 0$, so there is a constant $c_0(x)$ such that

$$\Big(\frac{f(v(x))}{v(x)} - \frac{f(u(x,t))}{u(x,t)}\Big) \ge c_0(x) > 0, \qquad (14)$$

for each x in Ω. But Ω is compact, so $c_0 = \min_\Omega c_0(x)$ exists and $c_0 > 0$. By (9)

$$g_n'(t) = (n+2)\Bigg\{\int_\Omega \frac{u^{n+2}}{v^n}\Bigg(\frac{f(u)}{u} - \frac{f(v)}{v}\Bigg)dx$$
$$-\delta(n+1)\int_\Omega \frac{u^n}{v^{n+2}}|vu_x - uv_x|^2 dx\Bigg\}$$

$$\le -(n+2)\int_\Omega \frac{u^{n+2}}{v^n}\Bigg(\frac{f(v)}{v} - \frac{f(u)}{u}\Bigg)dx$$

$$\le -c_0(n+2)g_n(t).$$

Integrating this inequality we obtain

$$g_n(t) \le g_n(0)\exp(-c_0(n+2)t),$$

and again, taking the $(n+2)$-nd root of both sides, and letting $n \to \infty$, we have in the $L_\infty(v^2 dx)$-norm:

$$\frac{u(x,t)}{v(x)} \le \sup_\Omega \frac{u}{v} \le \sup_\Omega \frac{\phi}{v}\exp(-c_0 t)$$
$$< \lambda \exp(-c_0 t).$$

□

Remark 5 *Note that in Theorem 4 the monotonicity of $f(z)/z$ insures that $g_n'(t) < 0$, which guarantees that $g_n(t)$ is decreasing for all t. However, it may not be necessary to require that $f(z)/z$ be increasing, because the second integral in (11), if nonzero, is multiplied by $(n+2)$ which, for sufficiently large n, will dominate the first integral, if it is bounded. Thus, it is critical to determine under what conditions the second integral is zero. This will be the topic (in $k = 1$ dimension) that we study in the next section.*

3 Conditions in 1-dimension under which the second integral in (11) is zero

In order for the second integral in the right side of equation (11) to be zero it is necessary that $|vu_x - uv_x|^2 = 0$, or equivalently that

$$vu_x = uv_x \quad \text{or} \quad \frac{u_x}{u} = \frac{v_x}{v}. \qquad (15)$$

We integrate to obtain $\ln u = \ln v + k$, with $k = k(t)$. Hence, only when $u = K(t)v$ is it possible for the second integral to be zero. But can such a $u(x,t)$ be a solution to problem (5)?

We now show an example of a situation where $u(x,t) = K(t)v(x)$, with $K(t)$ a function only of the variable t (so that the second integral in (11) is equal to zero), for a more general set of problems than (5)-(6), with Dirichlet or mixed boundary conditions and a constant multiple of v as the initial function ϕ.

Example 1. (Dirichlet case) Consider the problem (in $[0,\pi] \times (0,\infty)$)

$$u_t = u_{xx} + u(u^2 + u_x^2), \quad u(0,t) = u(\pi,t) = 0,$$

with initial value $\phi(x) = \sin x/\sqrt{1+c}$. It is trivial to check that

$$u(x,t) = \frac{\sin x}{\sqrt{1+ce^{2t}}}$$

is a solution of this problem, while $v(x) = \sin x$ is a solution of the steady-state problem

$$v_{xx} + v(v^2 + v_x^2) = 0, \quad v(0) = v(\pi) = 0.$$

If $c > 0$, then $\sqrt{1+c} = \lambda^{-1} > 1$, implying that $\phi(x) = u(x,0) < \lambda v(x)$, and we observe that $u(x,t) \to 0$ as $t \to \infty$. The decay is exponential. When $-1 < c < 0$, then $\sqrt{1+c} = \lambda^{-1} < 1$, so $\phi(x) > \lambda v(x)$. In this case $u(x,t)$ blows up when $ce^{2t} = -1$. Note that $u(x,t) = K(t)v(x)$, indicating that the second integral in (11) is zero.

Observe that this example also holds on $[0,\pi/2] \times (0,\infty)$ for the same functions for the first mixed boundary conditions, and replacing all the sines by cosines will make it work for the other mixed one, or the Neumann conditions.

Although Example 1 shows that it is possible for a solution to a more general set of problems than (5)-(6) to satisfy $u(x,t) = K(t)v(x)$, this cannot happen for nonlinear problems (5)-(6).

Theorem 6 *Let $f(u)$ be an analytic function, and let $u(x,t)$ and $v(x)$ be positive solutions, respectively, of the equations*

$$u_t = \delta u_{xx} + f(u) \quad \textit{and} \quad \delta v_{xx} + f(v) = 0.$$

Then $u(x,t) = K(t)v(x)$, with K not identically equal to 1, can only happen when $f(u)$ is homogeneous of degree 1 and K is a constant.

Proof: Substituting $u(x,t) = K(t)v(x)$ into the parabolic equation, we obtain by (6)

$$K'v = KDv_{xx} + f(Kv) = f(Kv) - Kf(v).$$

Assume that $f(v) = \sum_{n=0}^{\infty} \beta_n v^n$. Then we can rewrite the two ends of the above equation as

$$K'v = \beta_0(1-K) + \sum_{n=2}^{\infty} \beta_n (K^n - K)v^n.$$

Matching powers of v, we get $K' = 0$, implying that K is constant. If $|K| \neq 1$, then all the $\beta_n = 0$ except β_1, yielding $f(u) = \beta_1 u$, homogeneous of degree 1. Clearly $K \neq -1$, since the solution u is positive. Finally, if $K \equiv 1$, then $u \equiv v$.

Ruling these linear homogeneous of degree 1 problems out is no disadvantage to our technique, since linear problems can be easily solved by such classical methods as Fourier expansions. Thus, for any other analytic function, the second integral in (11) will not be constantly zero. Hence, there is some hope that the second integral in (11) may be used to dominate the first integral. This will be what we consider in the next section, where we show (in one dimension) that the results in Section 2 also hold for bounded $f(z)/z$. □

4 Theorem 4 holds in 1-dimension for bounded $f(z)/z$

It is still possible that even though $u(x,t) \neq K(t)v(x)$, some individual values of t may exist where u does equal a constant times v. We will show now that this only happens at discrete values of t.

Assume $W(x,t) = uv_x - vu_x$ is real analytic on $[a,b] \times [0,\infty) \equiv \Gamma$. Let $Z(W) = \{(x,t) \in \Gamma : W(x,t) = 0\} \neq \Gamma$. In particular, W analytic implies that $Z(W)$ has no interior (since then $Z(W) \equiv \Gamma$). Let $E = \{t : W(x,t) \equiv 0$ for all $x \in [a,b]\}$; that is, E consists of the values t at which the second integral in (11) is zero.

Lemma 7 *E has no finite limit points, that is $E = \{0 \leq t_0 < t_1 < t_2 < ...\}$.*

Proof: If the statement in the Lemma is not true, there is a sequence $\{t_n\} \in E, t_n \to t^* \in E, t^* < \infty$, and since W is continuous, there is some value $x_0 \in (a,b)$ for which

$$W(x,t) = \sum_{i\geq 0}\sum_{j\geq 0} A_{ij}(x-x_0)^i(t-t^*)^j,$$
$$\text{with} \quad A_{00} = W(x_0,t^*) = 0.$$

Assume that $W(x_0,t)$ is not identically zero in t for $0 < |t-t^*| < \eta$. Then

$$W(x_0,t) = \sum_{j\geq 1} A_{0j}(t-t^*)^j.$$

By a generalization of the Weierstrass preparation theorem (see [8]) we may express W as follows for some fixed j:

$$W(x,t) = [(t-t^*)^{j_0} + A_{j_0-1}(x)(t-t^*)^{j_0-1} + \\ ... + A_1(x)(t-t^*) + A_0(x)]G(x,t),\ j_0 \geq 1, \tag{16}$$

where the coefficients $A_j(x)$ are real analytic and G is nonzero in a neighborhood N of the point $P_0 = (x_0,t^*)$.

For each fixed x' near x_0, the term in square brackets in (16) is a polynomial in t of degree j_0, so $W(x',t) = 0$ can have at most j_0 roots in N. But, since $W(x',t_n) = 0$ for $t_n \to t^*$, $W(x',t)$ has an infinite number of zeros in N, which is a contradiction. Hence

$$W(x_0,t) = \sum A_{0j}(t-t^*)^j = 0, \quad \text{for} \quad t \quad \text{near} \quad t^*.$$

Now choose a direction line L through P_0, with $W(x,t)$ not identically 0 on L near P_0. Repeating the previous process at P_0 with L replacing the t-axis (and rotated coordinates) we get the same contradiction. Hence E has no finite accumulation points. □

Lemma 8 *Let t_1 be the first nonzero t in E, let $0 < t^* < t_1$, and let $0 \le \phi < \lambda v$, $0 < \lambda < 1$, for all $x \in [a,b]$, where v is the positive solution to problem (6). Then, if $f(z)/z$ is bounded, the positive solution $u(x,t)$ to problem (5) satisfies $0 \le u(x,t) < \lambda v(x)$, for all x in $[a,b]$ and $0 \le t \le t^*$.*

Proof: Let $0 \le t^* < t_1$ and note that on each line segment (x,t'), $a \le x \le b$, for $0 \le t' \le t^*$, we have points (x_j, t'_j) where $W^2(x_j,t'_j) > 0$. By continuity, there is an $\epsilon_j = \epsilon_j(x_j,t'_j) > 0$ such that in the box

$$B((x_j,t'_j),\epsilon_j) = \{(x,t) :$$

$$(|x-x_j| \le \epsilon_j) \times (|t-t'_j| \le \epsilon_j)\}$$

we have $W^2(x,t) \ge \frac{1}{2}W^2(x_j,t'_j)$ for all $(x,t) \in B((x_j,t'_j),\epsilon_j)$.

Project the interior of the boxes $B((x_j,t'_j),\epsilon_j)$ onto the t-axis to form the open intervals $C(t'_j,\epsilon_j) = \{|t-t'_j| < \epsilon_j\}$ and note they form an open cover of $[0,t^*]$:

$$\bigcup_{t'\in[0,t^*]} C(t',\epsilon) \supseteq [0,t^*].$$

By compactness, there is a finite subcover, say $C(t'_1,\epsilon_1), ..., C(t'_k,\epsilon_k)$. Let

$$0 < \eta \equiv \min_{1\le i\le k}\left\{\frac{1}{2}W^2(x_i,t'_i)\right\}.$$

Any $0 \le t \le t^*$ belongs to some $C(t'_i,\epsilon_i)$, so $W^2(x,t) \ge \eta > 0$, for all x in $|x-x_i| < \epsilon_i$. We can rewrite (11) in the form

$$g'_n(t) = (n+2)\left\{\int_a^b \frac{u^{n+2}}{v^n}\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)dx \right.$$
$$\left. -\delta(n+1)\int_a^b \frac{u^n}{v^{n+2}}|vu_x - uv_x|^2 dx\right\} \quad (17)$$

$$= (n+2)\left\{\int_a^b \frac{u^n}{v^{n+2}}\left(u^2v^2\Big(\frac{f(u)}{u} - \frac{f(v)}{v}\Big)\right.\right.$$
$$\left.\left. -\delta n W^2(x,t) - \delta W^2(x,t)\right)dx\right\}.$$

Since $f(z)/z$ is bounded (say by M), there exists an $N > 2 \parallel v \parallel_\infty^4 M/\delta\eta$ such that $g'_n(t) < 0$, for all $n \ge N$ and $0 \le t \le t^*$. Hence, $g_n(t) \le g_n(0)$ for $0 \le t \le t^*$. As before, taking $(n+2)$-nd roots, with $n \ge N$, we have

$$||\frac{u}{v}||_{n+2} \le ||\frac{\phi}{v}||_{n+2}.$$

Letting $n \to \infty$, we get $\sup_{(a,b)}(u/v) \le \sup_{(a,b)}(\phi/v)$, and $\sup_{(a,b)}(\phi/v) < \lambda$, so $u(x,t) < \lambda v(x)$, for all $x \in [a,b]$ and $0 \le t \le t^*$. □

So, now that we have a start on the inequality ($0 \le u(x,t) < \lambda v(x)$ on $0 \le t \le t^*$, $a \le x \le b$) we show that we can extend the proof all the way to t_1 and beyond.

Theorem 9 *If $f(z)/z$ is bounded, there exists a number λ^*, with $0 < \lambda < \lambda^* < 1$, so that if $0 \le \phi(x) < \lambda v(x)$, where v is the positive solution to problem (6), then the positive solution $u(x,t)$ to problem (5) satisfies*

$$0 \le u(x,t) < \lambda^* v(x), \quad \textit{for all} \quad a \le x \le b,$$

and $u(x,t)$ decays exponentially as t increases.

Proof: Let $\sigma = 1 - \lambda$, then by continuity, we can select t^* sufficiently close to t_1, so that $u(x,t) < \lambda_1 v(x)$ for all $x \in [a,b]$, $0 \le t \le t_1$, with $\lambda_1 < \lambda + (\sigma/4)$. Repeat the constructions in Lemma 8 for the interval $[t_1,t_2]$ of the set E. An identical compactness argument shows that $g'_n(t) < 0$ for all $t \in [t_1,t_1^*]$ with $t_1 < t_1^* < t_2$. Again select t_1^* sufficiently close to t_2 so that $u(x,t) < \lambda_2 v(x)$ for all $x \in [a,b]$, $t_1 \le t \le t_2$, with $\lambda_2 < \lambda_1 + (\sigma/8)$. Continue in this fashion for either the finite, or the countably infinite number of intervals in E.

If the number of intervals is infinite, the procedure above guarantees that $u(x,t) < [\lambda + (\sigma/2)]v(x) < v(x)$, for all $t \ge 0$ and $x \in [a,b]$, and we can select $\lambda^* = \lambda + (\sigma/2)$. If the number of intervals in E is finite, let t_k be the largest value in E. Then, $u < \lambda_k v$, with $\lambda_k < \lambda^*$, for $x \in [a,b]$ and $0 \le t \le t_k$. Now select $t^*_{k+1} > t_k$ as large as we want, and repeat the argument in Lemma 8 on the interval $[t_k, t^*_{k+1}]$ getting $u < \lambda_k v$ on $0 \le t \le t^*_{k+1}$ for all x in $[a,b]$.

To show that the decay is exponential select $n \ge 1 + \max_{1\le j\le k} N_j$. The second term in parenthesis in (17) will dominate the first, so

$$g'_n(t) < -\delta(n+2)\left\{\int_a^b \frac{u^n}{v^{n+2}}W^2(x,t)dx\right\},$$

$$\text{on} \quad [0,t^*].$$

Hence, denoting by η the minimum of the η_j, we have

$$g'_n(t) < -\delta\eta\frac{n+2}{\lambda^{*2}||v||_\infty^4}\int_a^b \frac{u^{n+2}}{v^n}dx$$

$$< -\delta\eta \frac{n+2}{\lambda^{*2}||v||_{\infty}^{4}} g_n(t).$$

Integrating this inequality from on $[0, t^*]$, we get

$$g_n(t) < \exp(\frac{-\delta\eta(n+2)}{\lambda^{*2}||v||_{\infty}^{4}} t) g_n(0).$$

Taking the $(n+2)$-nd roots, we have

$$\| \frac{u}{v} \|_{n+2} < \| \frac{\phi}{v} \|_{n+2} \exp(\frac{-\delta\eta t}{\lambda^{*2}||v||_{\infty}^{4}}).$$

Letting $n \to \infty$ yields

$$\sup_{(a,b)} \frac{u}{v} \leq \sup_{(a,b)} \left(\frac{\phi}{v}\right) \exp(\frac{-\delta\eta t}{\lambda^{*2}||v||_{\infty}^{4}})$$

$$< \lambda^* \exp(\frac{-\delta\eta t}{\lambda^{*2}||v||_{\infty}^{4}})$$

for all t in $[0, t^*]$. Thus, $u(x,t)$ decreases exponentially for t as large as we want. $\square$

5 Conclusios

In this paper we have developed a technique for showing when traveling waves do not occur in a nonlinear parabolic system, even when the initial equation may exceed an unstable solution of the associated steady-state problem. We have shown that this method holds when the nonlinear term f satisfies the condition that $f(z)/z$ is monotone, or even bounded (in R^1).

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Appendix

In this appendix we prove that

(A):$\| f \|_{L_p}$ is increasing as $p \to \infty$, and that

(B): $\| f \|_{L_p} \to \|f\|_{L_\infty}$ = ess $\sup_\Omega$ |f(x)|.

We need **Holder's Inequality:** Let μ be a measure on the space Ω, $f \in L_p$, $g \in L_q$, with $p, q > 1$ and $(1/p) + (1/q) = 1$. Then $fg \in L_1$ and

$$\int_\Omega fg d\mu \leq \| f \|_{L_p} \| g \|_{L_q} .$$

(a proof of this theorem is in [9], pp. 119–120.)

In what we have been doing we shall consider the positive measure $d\nu = v^2(x)dx$ on the set Ω. Suppose that

$$\int_\Omega v^2 dx = \int_\Omega d\nu = K.$$

Then define $d\mu = (1/K)d\nu$, so that $\mu(\Omega) = 1$ in what follows.

(A) Suppose $1 < p < q$ and $f \in L_p$. Then $f^p \in L_1$, and $m = q/p > 1$. Let m' be such that $(1/m) + (1/m') = 1$. Then by Holder's inequality

$$\int_\Omega |f|^p d\mu \leq \Big(\int_\Omega (|f|^p)^m d\mu\Big)^{1/m} \Big(\int_\Omega d\mu\Big)^{1/m'}$$

$$= \Big(\int_\Omega |f|^q d\mu\Big)^{p/q},$$

from which it follows that $\| f \|_{L_p} \leq \| f \|_{L_q}$.

(B) Without loss of generality, by division, we may assume that $\| f \|_{L_\infty} = 1$ (if it is finite).

There is a set Q with $\mu(Q) > 0$ such that $|f(x)| > 1 - \epsilon$ for $x \in Q$, with $\epsilon > 0$ arbitrary. Then

$$\| f \|_{L_p} \geq \Big(\int_Q |f|^p d\mu\Big)^{1/p} > (1-\epsilon)(\mu(Q))^{1/p}.$$

Letting p increase to ∞, we see that $\| f \|_{L_p} \to 1 - \epsilon$, and since ϵ is arbitrary, that implies that $\| f \|_{L_p}$ increases to $1 = \| f \|_{L_\infty}$.

If $\| f \|_{L_\infty} = \infty$, then for arbitrarily large N there is a set Q so that $|f(x)| > N$ for all $x \in Q$. So,

$$\| f \|_{L_p} \geq \left(\int_Q |f|^p d\mu \right)^{1/p} > N\mu(Q)^{1/p},$$

and $\| f \|_{L_p} \to N$ as $p \to \infty$. Since N is arbitrary, the result holds.