On Some Non-Linear Integral Inequalities for Two-Variable Functions with Delay Involving Integration over Infinite Intervals

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Abstract: In this paper, we present some non-linear integral inequalities with a term of delay for functions of two independent variables that can be used in the theory of differential and integral equations with time delay. Also, we generalize these inequalities by integration over infinite intervals. An application is given as an illustration.

Keywords: Non-linear integral inequalities with delay; Two independent variables; Non-decreasing functions; Differential and integral equations with delay; Integration over infinite intervals

1 Introduction

The integral inequalities with a term of delay are used in the study of some partial differential equations with time delay. Many researches have established their basic properties, namely generalizations to the uni-dimensional case, applications to some partial differential equations, and the existence and uniqueness of solutions $[6, 9, 4]$.

Recently, a number of research papers have been written on retarded integral inequalities. For instance, Mi [5] investigated some generalized Gronwall-Bellman type impulsive integral inequalities containing integrations over infinite intervals. Likewise, Xu and Ma [8] established some non-linear retarded Volterra-Fredholm type integral inequalities in two independent variables. Also, Boudeliou and Khellaf [1] obtained some useful generalizations of two-variable integral inequalities with delay.

In 2013, Qingling and Zhonghua [7] gave the following inequality:

Lemma 1 *([7]) Let $a \in C(R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+, $f, g, h \in C(R_0^+ R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies*

$$u(x) \le a(x) + \int_0^x h(s)u(s)ds + \int_0^{\alpha(x)} f(s)\Big(\int_0^s g(\tau)u(\tau)d\tau\Big)ds,$$

for $x \in R_0^+$, then

$$u(x) \le a(x)\exp\left[\int_0^x h(s)ds + \int_0^{\alpha(x)} f(s)\Big(\int_0^s g(\tau)d\tau\Big)ds\right],$$

for all $x \in R_0^+$.

In 2017, Ghrissi and Hammami [2] obtained some generalized retarded integral inequalities of the Gronwall type for one-variable functions; their results were based on the results proved in [7]. For example:

(i) If $u \in C(R_0^+, R_0^+)$ satisfies

$$\begin{aligned} u^p(x) &\le a(x) + \int_0^x h(s)u^p(s)ds \\ &+ \int_0^{\alpha(x)} f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds, \end{aligned}$$

for $x \in R_0^+$, then

$$\begin{aligned} u(x) &\le \left[a(x) + \int_0^{\alpha(x)} m_2 f(s)\left(\int_0^s g(\tau)d\tau\right)ds\right]^{\frac{1}{p}} \exp\frac{1}{p} \\ &\left[\int_0^x h(s)ds + \int_0^{\alpha(x)} m_1 f(s)\left(\int_0^s g(\tau)d\tau\right)ds\right], \end{aligned}$$

for all $x \in R_0^+$.

(ii) If $u \in C(R_0^+, R_0^+)$ satisfies

$$u^p(x) \leq 1 + \int_0^x h(s)u^p(s)ds + \int_0^{\alpha(x)} f(s)\Big(\int_0^s g(\tau)u^q(\tau)d\tau\Big)ds,$$

for $x \in R_0^+$, then

$$u(x) \leq \exp\frac{1}{p}\Big[\int_0^x h(s)ds + \int_0^{\alpha(x)} f(s)\Big(\int_0^s g(\tau)d\tau\Big)ds\Big],$$

for all $x \in R_0^+$.

(iii) If $u \in C(R_0^+, R_0^+)$ satisfies

$$u(x) \leq a(x) + b(x)\int_0^x h(s)u(s)ds + \int_0^{\alpha(x)} f(s)\Big(\int_0^s g(\tau)u(\tau)d\tau\Big)ds,$$

for $x \in R_0^+$, then

$$u(x) \leq a(x)\exp\Big[b(x)\int_0^x h(s)ds + \int_0^{\alpha(x)} m_1 f(s)\Big(\int_0^s g(\tau)d\tau\Big)ds\Big],$$

for all $x \in R_0^+$.

The objective of this paper is to establish some non-linear retarded integral inequalities in two independent variables, and also to generalize those inequalities by integration over infinite intervals. These new inequalities present generalizations of the results proved by Ghrissi and Hammami [2]. An application is given to illustrate the efficiency of the obtained results in the study of the boundedness of solutions of certain hyperbolic partial differential equations.

2 Main Results

We denote $R_0^+ = [0, \infty]$ and $R^+ = (0, \infty)$. The first-order partial derivatives of the function $u(x, y)$ for $x, y \in R$ with respect to x and y are denoted by $\frac{\partial}{\partial x}u(x, y)$ and $\frac{\partial}{\partial y}u(x, y)$ respectively.

2.1 Some Retarded Non-linear Integral Inequalities in Two Independent Variables

In this section, we present some generalizations of the results proved in [2] and [7].

Theorem 2 *Let $a \in C(R_0^+ \times R_0^+, R^+)$ and α, $\beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \leq x$ on R_0^+ and $\beta(y) \leq y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$u(x, y) \leq a(x, y) + \int_0^x\int_0^y h(s, t)u(s, t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s, t)\Big(\int_0^s\int_0^t g(\tau, \sigma)u(\tau, \sigma)d\tau d\sigma\Big)dsdt, \quad (1)$$

for $(x, y) \in R_0^+ \times R_0^+$, then

$$u(x, y) \leq a(x, y)\exp\Big[\int_0^x\int_0^y h(s, t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s, t)\Big(\int_0^s\int_0^t g(\tau, \sigma)d\tau d\sigma\Big)dsdt\Big], \quad (2)$$

for all $(x, y) \in R_0^+ \times R_0^+$.

Proof: Since $a(x, y)$ is positive and non-decreasing, from (1) we have

$$\frac{u(x, y)}{a(x, y)} \leq 1 + \int_0^x\int_0^y h(s, t)\frac{u(s, t)}{a(s, t)}dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s, t) \Big(\int_0^s\int_0^t g(\tau, \sigma)\frac{u(\tau, \sigma)}{a(\tau, \sigma)}d\tau d\sigma\Big)dsdt.$$

Define a function $z(x, y)$ on $R_0^+ \times R_0^+$ by

$$z(x, y) = 1 + \int_0^x\int_0^y h(s, t)\frac{u(s, t)}{a(s, t)}dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s, t) \Big(\int_0^s\int_0^t g(\tau, \sigma)\frac{u(\tau, \sigma)}{a(\tau, \sigma)}d\tau d\sigma\Big)dsdt,$$

then $z(x, y)$ is positive and non-decreasing, $z(0, y) = z(x, 0) = 1$, $\frac{u(x,y)}{a(x,y)} \leq z(x, y)$, and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) = h(x,y)\frac{u(x,y)}{a(x,y)} + f(\alpha(x),\beta(y))$$

$$\alpha'(x)\beta'(y)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\frac{u(s,t)}{a(s,t)}dsdt.$$

i.e.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) \le h(x,y)z(x,y) + f(x,y)\alpha'(x)\beta'(y)$$

$$z(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt,$$

hence

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y)}{z(x,y)} \le h(x,y) + f(x,y)\alpha'(x)\beta'(y)$$

$$\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt.$$

Therefore

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z(x,y)}\right] \le h(x,y) + f(x,y)\alpha'(x)\beta'(y)$$

$$\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt. \tag{3}$$

By keeping y fixed, setting $x = s$, and integrating from 0 to x in (3), and again by keeping x fixed, setting $y = t$, and integrating from 0 to y in the resulting inequality, we obtain

$$z(x,y) \le \exp\left[\int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right].$$

Since $\frac{u(x,y)}{a(x,y)} \le z(x,y)$, we have

$$u(x,y) \le a(x,y)\exp\left[\int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right].$$

So the relation (2) is true.

Remark 3 *If we replace the functions in Theorem* 2 *with one-variable functions, we obtain Proposition 2 in* $[2]$.

Now we give two lemmas that would be used in the proofs of the coming theorems.

Lemma 4 *([2]) Assume that $a \ge 0$, $p \ge 1$. Then, for any $k > 0$, we have*

$$a^{\frac{1}{p}} \le \frac{1}{p}k^{\frac{1-p}{p}}\,a + \frac{p-1}{p}\,k^{\frac{1}{p}}. \tag{4}$$

Or equivalently $a^{\frac{1}{p}} \le m_1\,a + m_2$, *where* $m_1 = \frac{1}{p}k^{\frac{1-p}{p}}$ *and* $m_2 = \frac{p-1}{p}\,k^{\frac{1}{p}}$.

Lemma 5 *([3]) Assume that $a \ge 0$, $p \ge q \ge 0$, and $p \ne 0$. Then, for any $k > 0$, we have*

$$a^{\frac{q}{p}} \le \frac{q}{p}k^{\frac{q-p}{p}}\,a + \frac{p-q}{p}\,k^{\frac{q}{p}}. \tag{5}$$

Or equivalently $a^{\frac{q}{p}} \le m_3\,a + m_4$, *where* $m_3 = \frac{q}{p}k^{\frac{q-p}{p}}$ *and* $m_4 = \frac{p-q}{p}\,k^{\frac{q}{p}}$.

Theorem 6 *Let $a \in C(R_0^+ \times R_0^+, R^+)$ and $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+ and $\beta(y) \le y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge 1$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$u^p(x,y) \le a(x,y) + \int_0^x\int_0^y h(s,t)u^p(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{6}$$

for $(x,y) \in R_0^+ \times R_0^+$, then

$$u(x,y) \le A(x,y)\exp\frac{1}{p}\left[\int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} m_1 f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\right], \tag{7}$$

for all $(x,y) \in R_0^+ \times R_0^+$, where

$$A(x,y) = \left[a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} m_2 f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\right]^{\frac{1}{p}}. \tag{8}$$

Proof: Let $u^p(x,y)=z(x,y)$, then $z(x,y)$ satisfies

$$z(x,y) \le a(x,y)+\int_0^x\int_0^y h(s,t)z(s,t)dsdt+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z^{\frac{1}{p}}(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (4) we get

$$z(x,y) \le a(x,y)+\int_0^x\int_0^y h(s,t)z(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma) [m_1 z(\tau,\sigma)+m_2]d\tau d\sigma\Big)dsdt.$$

i.e.

$$z(x,y)\le a(x,y)+\int_0^x\int_0^y h(s,t)z(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} m_2 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} m_1 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 2 we get

$$z(x,y)\le \left[a(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_2 f(s,t) \Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right]\exp\left[\int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} m_1 f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\right].$$

Since $u(x,y)=z^{\frac{1}{p}}(x,y)$, we have

$$u(x,y) \le A(x,y)\exp\frac{1}{p}\left[\int_0^x\int_0^y h(s,t)dsdt+ \int_0^{\alpha(x)}\int_0^{\beta(y)} m_1 f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\right],$$

where $A(x,y)$ is defined in (8).

Corollary 7 *Let $a\in C(R_0^+\times R_0^+,R^+)$ and $\alpha,\ \beta\in C^1(R_0^+,R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x)\le x$ on R_0^+ and $\beta(y)\le y$ on R_0^+, $f,g,h\in C(R_0^+\times R_0^+,R_0^+)$, $p\ge 1$. If $u\in C(R_0^+\times R_0^+,R_0^+)$ satisfies*

$$u^p(x,y) \le a(x,y)+\int_0^x\int_0^y h(s,t)u(s,t)dsdt+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)u^p(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{9}$$

for $(x,y)\in R_0^+\times R_0^+$, then

$$u(x,y) \le B(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y m_1 h(s,t)dsdt+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \tag{10}$$

for all $(x,y)\in R_0^+\times R_0^+$, where

$$B(x,y)=\left[a(x,y)+\int_0^x\int_0^y m_2 h(s,t)dsdt\right]^{\frac{1}{p}}. \tag{11}$$

Proof: Let $u^p(x,y)=z(x,y)$, then $z(x,y)$ satisfies

$$z(x,y) \le a(x,y)+\int_0^x\int_0^y h(s,t)z^{\frac{1}{p}}(s,t)dsdt+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (4) we get

$$z(x,y) \le a(x,y)+\int_0^x\int_0^y h(s,t)[m_1 z(s,t)+m_2]dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

i.e.

$$z(x,y) \le a(x,y)+\int_0^x\int_0^y m_2 h(s,t)dsdt + \int_0^x\int_0^y m_1 h(s,t)z(s,t)dsdt$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 2 we get

$$z(x,y)\leq\left[a(x,y)+\int_0^x\int_0^y m_2h(s,t)dsdt\right]\exp$$
$$\Big[\int_0^x\int_0^y m_1h(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)$$
$$\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].$$

Since $u(x,y)=z^{\frac{1}{p}}(x,y)$, we have

$$u(x,y)\leq B(x,y)\exp\frac{1}{p}\Big[m_1\int_0^x\int_0^y h(s,t)dsdt$$
$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],$$

where $B(x,y)$ is defined in (11).

Remark 8 *(i) If we replace the functions in Theorem* 6 *and* 7 *with one-variable functions, we obtain Propositions 9 and 11 in* [2].
(ii) If we take $p=1$, *then Theorem* 6 *and Corollary* 7 *reduce to Theorem* 2.

Theorem 9 *Let* $\alpha,\ \beta\ \in\ C^1(R_0^+,R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x)\leq x$ *on* R_0^+ *and* $\beta(y)\leq y$ *on* R_0^+, $f,g,h\in C(R_0^+\times R_0^+,R_0^+)$, $p\geq q\geq 1$. *If* $u\in C(R_0^+\times R_0^+,R_0^+)$ *satisfies*

$$u^p(x,y)\leq 1+\int_0^x\int_0^y h(s,t)u^p(s,t)dsdt+$$
$$\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)u^q(\tau,\sigma)d\tau d\sigma\right)dsdt, \tag{12}$$

for $(x,y)\in R_0^+\times R_0^+$, *then*

$$u(x,y)\leq C(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y h(s,t)dsdt+$$
$$\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\Big], \tag{13}$$

for all $(x,y)\in R_0^+\times R_0^+$, *where*

$$C(x,y)=\Big[1+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_4f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt\Big]^{\frac{1}{p}}. \tag{14}$$

Proof: Define a function $z(x,y)$ by the right side of (12), then $z(x,y)>0$, $u^p(x,y)\leq z(x,y)$, $u(x,y)\leq z^{\frac{1}{p}}(x,y)$, $u^q(x,y)\leq z^{\frac{q}{p}}(x,y)$, and

$$z(x,y)\leq 1+\int_0^x\int_0^y h(s,t)z(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)z^{\frac{q}{p}}(\tau,\sigma)d\tau d\sigma\right)dsdt.$$

Using (5) we get

$$z(x,y)\leq 1+\int_0^x\int_0^y h(s,t)z(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)[m_3z(\tau,\sigma)+m_4]d\tau d\sigma\right)dsdt.$$

i.e.

$$z(x,y)\leq 1+\int_0^x\int_0^y h(s,t)z(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_4f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\right)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\right)dsdt.$$

Using Theorem 2 we get

$$z(x,y)\leq\Big[1+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_4f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big]\exp\Big[\int_0^x\int_0^y h(s,t)dsdt$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].$$

Finally,

$$u(x,y)\le C(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y h(s,t)dsdt+$$

$$\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],$$

where $C(x,y)$ is defined in (14).

Remark 10 *If we take $p=q=1$ and $a(x,y)=1$, then Theorem 9 reduces to Theorem 2.*

Corollary 11 *Let $a\in C(R_0^+\times R_0^+,R^+)$ and $\alpha,\ \beta\in C^1(R_0^+,R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x)\le x$ on R_0^+ and $\beta(y)\le y$ on R_0^+, $f,g,h\in C(R_0^+\times R_0^+,R_0^+)$, $p\ge q\ge 1$. If $u\in C(R_0^+\times R_0^+,R_0^+)$ satisfies*

$$\begin{aligned}u^p(x,y) &\le a(x,y)+\int_0^x\int_0^y h(s,t)u^p(s,t)dsdt\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\\ &\qquad u^q(\tau,\sigma)d\tau d\sigma\Big)dsdt,\end{aligned}\tag{15}$$

for $(x,y)\in R_0^+\times R_0^+$, then

$$\begin{aligned}u(x,y) &\le a^{\frac{1}{p}}(x,y)C(x,y)\exp\frac{1}{p}\Big[\\ &\int_0^x\int_0^y h(s,t)dsdtr+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3 f(s,t)\\ &\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],\end{aligned}\tag{16}$$

for all $(x,y)\in R_0^+\times R_0^+$, where $C(x,y)$ is defined in (14).

Proof: Since $a(x,y)$ is positive and non-decreasing, from (15) we have

$$\frac{u^p(x,y)}{a(x,y)} \le 1+\int_0^x\int_0^y h(s,t)\frac{u^p(s,t)}{a(s,t)}dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u^q(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

i.e.

$$\frac{u^p(x,y)}{a(x,y)} \le 1+\int_0^x\int_0^y h(s,t)\frac{u^p(s,t)}{a(s,t)}dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u^q(\tau,\sigma)}{a^{\frac{q}{p}}(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Or equivalently

$$\begin{aligned}\left(\frac{u(x,y)}{a^{\frac{1}{p}}(x,y)}\right)^p &\le 1+\int_0^x\int_0^y h(s,t)\left(\frac{u(s,t)}{a^{\frac{1}{p}}(s,t)}\right)^p dsdt\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\\ &\qquad\left(\frac{u(\tau,\sigma)}{a^{\frac{1}{p}}(\tau,\sigma)}\right)^q d\tau d\sigma\Big)dsdt.\end{aligned}$$

Using Theorem 9 we get

$$\begin{aligned}\frac{u(x,y)}{a^{\frac{1}{p}}(x,y)} &\le C(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y h(s,t)dsdt\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].\end{aligned}$$

Finally,

$$\begin{aligned}u(x,y) &\le a^{\frac{1}{p}}(x,y)C(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y h(s,t)dsdt\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_3 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],\end{aligned}$$

where $C(x,y)$ is defined in (14).

Remark 12 *If we take $p=q=1$, then Corollary 11 reduces to Theorem 2.*

Corollary 13 *Let $\alpha,\ \beta\in C^1(R_0^+,R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x)\le x$ on R_0^+ and $\beta(y)\le y$ on R_0^+, $f,g,h\in C(R_0^+\times R_0^+,R_0^+)$, $p\ge q\ge 1$. If $u\in C(R_0^+\times R_0^+,R_0^+)$ satisfies*

$$u^p(x,y)\le 1+\int_0^x\int_0^y h(s,t)u^q(s,t)dsdt+$$

$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \qquad (17)$$

for $(x,y)\in R_0^+\times R_0^+$, *then*

$$u(x,y)\le K(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y m_3h(s,t)dsdt$$
$$\int_0^{\alpha(x)}\int_0^{\beta(y)} m_1f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],$$

for all $(x,y)\in R_0^+\times R_0^+$, *where*

$$K(x,y) = \Big[1+\int_0^x\int_0^y m_4h(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_2f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big]^{\frac{1}{p}}. \qquad (18)$$

Proof: Define a function $z(x,y)$ by the right side of (17), then $z(x,y)>0$, $u^p(x,y)\le z(x,y)$, $u(x,y)\le z^{\frac{1}{p}}(x,y)$, $u^q(x,y)\le z^{\frac{q}{p}}(x,y)$, and

$$z(x,y)\le 1+\int_0^x\int_0^y h(s,t)z^{\frac{q}{p}}(s,t)dsdt +$$
$$\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z^{\frac{1}{p}}(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (4) and (5), we get

$$z(x,y) \le 1+\int_0^x\int_0^y h(s,t)[m_3z(s,t)+m_4]dsdt+$$
$$\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)(\int_0^s\int_0^t g(\tau,\sigma)[m_1z(\tau,\sigma)+m_2]d\tau d\sigma)dsdt.$$

i.e.

$$z(x,y) \le 1+\int_0^x\int_0^y m_4h(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_2 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt+$$
$$\int_0^x\int_0^y m_3h(s,t)z(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_1 f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 2, we get

$$z(x,y)\le\Big[1+\int_0^x\int_0^y m_4h(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_2f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big]\exp\Big[\int_0^x\int_0^y m_3h(s,t)dsdt$$
$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_1f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].$$

Since $u(x,y)\le z^{\frac{1}{p}}(x,y)$, we have

$$u(x,y) \le K(x,y)\exp\frac{1}{p}\Big[\int_0^x\int_0^y m_3h(s,t)dsdt$$
$$+\int_0^{\alpha(x)}\int_0^{\beta(y)} m_1f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],$$

where $K(x,y)$ is defined in (18).

Corollary 14 *Let* $\alpha,\ \beta\in C^1(R_0^+,R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x)\le x$ *on* R_0^+ *and* $\beta(y)\le y$ *on* R_0^+, $f,g,h\in C(R_0^+\times R_0^+,R_0^+)$, $p\ge q\ge 1$. *If* $u\in C(R_0^+\times R_0^+,R_0^+)$ *satisfies*

$$u(x,y) \le 1+\int_0^x\int_0^y h(s,t)u^p(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s\int_0^t g(\tau,\sigma)u^q(\tau,\sigma)d\tau d\sigma\right)dsdt, \qquad (19)$$

for $(x,y)\in R_0^+\times R_0^+$, *then*

$$u(x,y)\le\Big[1+(1-p)\Big(\int_0^x\int_0^y h(s,t)dsdt+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big)\Big]^{\frac{1}{1-p}}, \qquad (20)$$

for all $(x,y)\in R_0^+\times R_0^+$.

Proof: Define a function $z(x,y)$ by the right side of (19), then

$$u(x,y) \le z(x,y), \;\; z(0,y) = z(x,0) = 1, \quad \frac{\partial}{\partial x} z(x,0) = \frac{\partial}{\partial y} z(0,y) = 0, \tag{21}$$

and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) \le h(x,y) z^p(x,y) + \alpha'(x)\beta'(y) f(x,y)\Big(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t) z^q(s,t) dsdt\Big).$$

i.e.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) \le z^p(x,y)\Big[h(x,y) + \alpha'(x)\beta'(y) f(x,y)\Big(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t) dsdt\Big)\Big],$$

hence

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) z(x,y)}{z^{p+1}(x,y)} - \frac{\frac{\partial}{\partial x} z(x,y)\frac{\partial}{\partial y}\, z(x,y)}{z^{p+1}(x,y)} \le h(x,y) + \alpha'(x)\beta'(y) f(x,y)\Big(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t) dsdt\Big).$$

Therefore

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x} z(x,y)}{z^p(x,y)}\right] \le h(x,y) + \alpha'(x)\beta'(y) f(x,y) \Big(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t) dsdt\Big).$$

By keeping y fixed, setting $x = s$, and integrating from 0 to x in the above inequality and using (21), and again by keeping x fixed, setting $y = t$, and integrating from 0 to y in the resulting inequality and using (21), we obtain the inequality (20).

Remark 15 *(i) If we replace the functions in Theorem* 9 *and Corollaries* 11, 13, *and* 14 *with one-variable functions, we obtain Propositions 13, 15, 17, and 19 in* [2] *respectively.*
(ii) If we take $p = q = 1$ *and* $a(x,y) = 1$, *then Corollaries* 13 *and* 14 *reduce to Theorem* 2.

Theorem 16 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \le x$ *on* R_0^+ *and* $\beta(y) \le y$ *on* R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $b \in C(R_0^+ \times R_0^+, R_0^+)$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$\begin{aligned} u(x,y) &\le a(x,y) + b(x,y)\int_0^x\int_0^y h(s,t) u(s,t) dsdt \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma) u(\tau,\sigma) d\tau d\sigma\Big) dsdt, \end{aligned} \tag{22}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$\begin{aligned} u(x,y) &\le a(x,y)\exp\Big[b(x,y)\int_0^x\int_0^y h(s,t) dsdt \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma) d\tau d\sigma\Big) dsdt\Big], \end{aligned} \tag{23}$$

for all $(x,y) \in R_0^+ \times R_0^+$.

Proof: Since $a(x,y)$ is positive and non-decreasing, from (22) we have

$$\begin{aligned} \frac{u(x,y)}{a(x,y)} &\le 1 + b(x,y)\int_0^x\int_0^y h(s,t)\frac{u(s,t)}{a(s,t)} dsdt \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)} d\tau d\sigma\Big) dsdt. \end{aligned}$$

Fix any $(X,Y) \in R_0^+ \times R_0^+$. Then, for $0 \le x \le X$, $0 \le y \le Y$, we have

$$\begin{aligned} \frac{u(x,y)}{a(x,y)} &\le 1 + b(X,Y)\int_0^x\int_0^y h(s,t)\frac{u(s,t)}{a(s,t)} dsdt \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)} d\tau d\sigma\Big) dsdt. \end{aligned}$$

Define a function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= 1 + b(X,Y)\int_0^x\int_0^y h(s,t)\frac{u(s,t)}{a(s,t)} dsdt \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)} d\tau d\sigma\Big) dsdt, \end{aligned}$$

then $z(x,y)$ is positive and non-decreasing, $z(0,y) = z(x,0) = 1$, $\frac{u(x,y)}{a(x,y)} \le z(x,y)$, and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) = b(X,Y)h(x,y)\frac{u(x,y)}{a(x,y)} + \alpha'(x)\beta'(y)f(\alpha(x),\beta(y))\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\frac{u(s,t)}{a(s,t)}dsdt.$$

i.e.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) \le z(x,y)\Big[b(X,Y)h(x,y) + \alpha'(x)\beta'(y)f(x,y)\Big(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt\Big)\Big],$$

hence

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y)z(x,y)}{z^2(x,y)} - \frac{\frac{\partial}{\partial x}z(x,y)\frac{\partial}{\partial y}z(x,y)}{z^2(x,y)} \le b(X,Y)h(x,y) + \alpha'(x)\beta'(y)f(x,y)\Bigg(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt\Bigg).$$

Therefore

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z(x,y)}\right] \le b(X,Y)h(x,y) + \alpha'(x)\beta'(y)f(x,y)\Bigg(\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)dsdt\Bigg).$$

By keeping y fixed, setting $x = s$, and integrating from 0 to x in the above inequality, and again by keeping x fixed, setting $y = t$, and integrating from 0 to y in the resulting inequality, we have

$$z(X,Y) \le \exp\Big[b(X,Y)\int_0^X\int_0^Y h(s,t)dsdt + \int_0^{\alpha(X)}\int_0^{\beta(Y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].$$

Since (X,Y) is arbitrary and $\frac{u(x,y)}{a(x,y)} \le z(x,y)$, by replacing (X,Y) with (x,y) in the above inequality we obtain the inequality (23).

Remark 17 *(i) If we replace the functions in Theorem* 16 *with one-variable functions, we obtain Proposition 20 in* [2].
(ii) If we take $b(x,y) = 1$, *then Theorem* 16 *reduces to Theorem* 2.

Theorem 18 *Let* $a \in C^1(R_0^+ \times R_0^+, R^+)$ *and* $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \le x$ *on* R_0^+ *and* $\beta(y) \le y$ *on* R_0^+, $f,g,h \in C(R_0^+ \times R_0^+, R_0^+)$. *Let* ω *be a non-decreasing continuous function such that* $\omega \ge 1$, $\omega(x) \ge x$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$u(x,y) \le a(x,y) + \int_0^x\int_0^y h(s,t)\omega(u(s,t))dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{24}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \le a(x,y)W^{-1}\Big[W(1) + \int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \tag{25}$$

for all $(x,y) \in R_0^+ \times R_0^+$, *where* W^{-1} *is the inverse function of*

$$W(r) = \int_0^r \frac{ds}{\omega(s)}, \qquad r > 0, \tag{26}$$

and $(x_1,y_1) \in R_0^+ \times R_0^+$ *is chosen so that* $W(1) + \int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt$ *in* $Dom(W^{-1})$ *for all* (x,y) *lying in* $[0,x_1]\times[0,y_1]$.

Proof: Since $a(x,y)$ is positive and non-decreasing, from (24) we have

$$\frac{u(x,y)}{a(x,y)} \le 1 + \int_0^x\int_0^y h(s,t)\omega\Big(\frac{u(s,t)}{a(s,t)}\Big)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Define a function $z(x,y)$ by the right side of the above inequality, then $z(x,y) > 0$, $z(0,y) = z(x,0) = 1$,

$\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, and

$$\frac{\partial}{\partial x} z(x,y) \leq \omega(z(x,y)) \int_0^y h(x,t)dt + \alpha'(x)z(x,y) \int_0^{\beta(y)} f(x,t)\Big(\int_0^{\alpha(x)}\int_0^t g(s,\sigma)dsd\sigma\Big)dt.$$

i.e.

$$\frac{\frac{\partial}{\partial x} z(x,y)}{\omega(z(x,y))} \leq \int_0^y h(x,t)dt + \alpha'(x)\int_0^{\beta(y)} f(x,t) \Big(\int_0^{\alpha(x)}\int_0^t g(s,\sigma)dsd\sigma\Big)dt.$$

By integration and using (26), we have

$$z(x,y) \leq W^{-1}\Big[W(1) + \int_0^x\int_0^y h(s,t)dsdt + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big(\int_0^s\int_0^t g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big],$$

where $W(1) = W(z(0,y))$. Since $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, it is easy to obtain the inequality (25).

Remark 19 *(i) If we replace the functions in Theorem* 18 *with one-variable functions, we obtain Proposition 22 in* [2].
(ii) If we take $\omega(u(x,y)) = u(x,y)$, *then Theorem* 18 *reduces to Theorem* 2.

2.2 Some Generalized Inequalities by Integration over Infinite Intervals

In this section, we give some generalizations of the results obtained in section 2.1.

Theorem 20 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \leq x$ *on* R_0^+ *and* $\beta(y) \leq y$ *on* R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$\begin{aligned} u(x,y) &\leq a(x,y) + \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty h(s,t)u(s,t)dsdt \\ &+ \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty f(s,t)\Big(\int_s^\infty\int_t^\infty g(\tau,\sigma) u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \end{aligned} \quad (27)$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$\begin{aligned} u(x,y) &\leq a(x,y)\exp\Big[\int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty h(s,t)dsdt \\ &+ \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty f(s,t)\Big(\int_s^\infty\int_t^\infty g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \end{aligned} \quad (28)$$

for all $(x,y) \in R_0^+ \times R_0^+$.

Proof: Since $a(x,y)$ is positive and non-decreasing, from (27) we have

$$\frac{u(x,y)}{a(x,y)} \leq 1 + \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty h(s,t)\frac{u(s,t)}{a(s,t)}dsdt + \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty f(s,t)\Big(\int_s^\infty\int_t^\infty g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Define a function $z(x,y)$ on $R_0^+ \times R_0^+$ by

$$z(x,y) = 1 + \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty h(s,t)\frac{u(s,t)}{a(s,t)}dsdt + \int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty f(s,t)\Big(\int_s^\infty\int_t^\infty g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt,$$

then $z(x,y)$ is positive and non-decreasing, $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, and

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\partial}{\partial y} z(x,y) &\leq h(x,y)z(x,y)\alpha'(x)\beta'(y) + f(x,y) \alpha'(x)\beta'(y)\int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty g(s,t)z(s,t)dsdt \\ &\leq h(x,y)z(x,y)\alpha'(x)\beta'(y) + f(x,y) \alpha'(x)\beta'(y)z(x,y)\int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty g(s,t)dsdt, \end{aligned}$$

hence

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y} z(x,y)z(x,y)}{z^2(x,y)} - \frac{\frac{\partial}{\partial x}z(x,y)\frac{\partial}{\partial y}z(x,y)}{z^2(x,y)} \leq h(x,y) \alpha'(x)\beta'(y) + f(x,y)\alpha'(x)\beta'(y)\int_{\alpha(x)}^\infty\int_{\beta(y)}^\infty g(s,t)dsdt.$$

Therefore

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z(x,y)}\right] \leq h(x,y)\alpha'(x)\beta'(y) + f(x,y)\alpha'(x)\beta'(y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt. \tag{29}$$

By keeping x fixed, setting $y = t$, and integrating (29) we have

$$-\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z(x,y)}\right] \leq \alpha'(x)\int_{\beta(y)}^{\infty} h(x,t) + \alpha'(x)\int_{\beta(y)}^{\infty} f(x,t)\Big(\int_{\alpha(x)}^{\infty}\int_{t}^{\infty} g(s,\sigma)dsd\sigma\Big)dt.$$

Again by keeping y fixed, setting $x = s$, and integrating the resulting inequality, we obtain

$$z(x,y) \leq \exp\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right].$$

Since $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, we obtain the inequality (28).

Remark 21 *If we replace* $\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}$ *with* $\int_{0}^{\alpha(x)}$ *and the functions with one-variable functions in Theorem* 20, *we obtain Proposition 2 in* [2].

Theorem 22 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* $\alpha, \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \leq x$ *on* R_0^+ *and* $\beta(y) \leq y$ *on* R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \geq 1$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$u^p(x,y) \leq a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u^p(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{30}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \leq A_1(x,y)\exp\frac{1}{p}\Big[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \tag{31}$$

for all $(x,y) \in R_0^+ \times R_0^+$, *where*

$$A_1(x,y) = \left[a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right]^{\frac{1}{p}}. \tag{32}$$

Proof: Define a function $z(x,y)$ by the right side of (30), then $u^p(x,y) \leq z(x,y)$, $u(x,y) \leq z^{\frac{1}{p}}(x,y)$, and

$$z(x,y) \leq a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)z^{\frac{1}{p}}(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (4) we get

$$z(x,y) \leq a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 20 we have

$$z(x,y) \leq \left[a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\right]\exp\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \right.$$

$$\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg].$$

Finally, since $u(x,y) \le z^{\frac{1}{p}}(x,y)$, it is easy to obtain the inequality (31) with the condition (32).

Corollary 23 *Let $a \in C(R_0^+ \times R_0^+, R^+)$ and $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+ and $\beta(y) \le y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge 1$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u^p(x,y) &\le a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u(s,t)dsdt \\ &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma) \\ &\qquad u^p(\tau,\sigma)d\tau d\sigma\Big)dsdt, \end{aligned} \tag{33}$$

for $(x,y) \in R_0^+ \times R_0^+$, then

$$\begin{aligned} u(x,y) &\le B_1(x,y)\exp\frac{1}{p}\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 h(s,t)dsdt \\ &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg], \end{aligned} \tag{34}$$

for all $(x,y) \in R_0^+ \times R_0^+$, where

$$B_1(x,y) = \Bigg[a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 h(s,t)dsdt\Bigg]^{\frac{1}{p}}. \tag{35}$$

Proof: Define a function $z(x,y)$ by the right side of (33), then $u^p(x,y) \le z(x,y)$, $u(x,y) \le z^{\frac{1}{p}}(x,y)$, and

$$z(x,y) \le a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z^{\frac{1}{p}}(s,t)dsdt +$$

$$\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (4) we get

$$z(x,y) \le a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 h(s,t)dsdt$$

$$+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)$$

$$\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 20 we have

$$\begin{aligned} z(x,y) &\le \Bigg[a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_2 h(s,t)dsdt\Bigg]\exp \\ &\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_1 h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t) \\ &\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg]. \end{aligned}$$

Finally, since $u(x,y) \le z^{\frac{1}{p}}(x,y)$, we obtain the inequality (34) with the condition (35).

Remark 24 *(i) If we replace $\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}$ with $\int_0^{\alpha(x)}$ and the functions with one-variable functions in Theorem 22 and Corollary 23, we obtain Propositions 9 and 11 in [2].*
(ii) If we take $p = 1$, then Theorem 22 and Corollary 23 reduce to Theorem 20.

Theorem 25 *Let $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+ and $\beta(y) \le y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge q \ge 1$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u^p(x,y) &\le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u^p(s,t)dsdt + \\ &\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)u^q(\tau,\sigma)d\tau d\sigma)dsdt, \end{aligned} \tag{36}$$

for $(x,y) \in R_0^+ \times R_0^+$, then

$$\begin{aligned} u(x,y) &\le C_1(x,y)\exp\frac{1}{p}\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \\ &\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_3 f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg], \end{aligned} \tag{37}$$

for all $(x,y) \in R_0^+ \times R_0^+$, *where*

$$C_1(x,y) = \Big[1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_4 f(s,t) \Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big]^{\frac{1}{p}}. \tag{38}$$

Proof: Define a function $z(x,y)$ by the right side of (36), then $u^p(x,y) \le z(x,y)$, $u(x,y) \le z^{\frac{1}{p}}(x,y)$, $u^q(x,y) \le z^{\frac{q}{p}}(x,y)$, and

$$z(x,y) \le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)z^{\frac{q}{p}}(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using (5) we get

$$z(x,y) \le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)[m_3 z(\tau,\sigma)+m_4]d\tau d\sigma\Big)dsdt.$$

i.e.

$$z(x,y) \le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)z(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_4 f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_3 f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)z(\tau,\sigma)d\tau d\sigma\Big)dsdt.$$

Using Theorem 20 we have

$$z(x,y) \le \Big[1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_4 f(s,t) \Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big] \exp \Big[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_3 f(s,t) \Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big].$$

Finally, since $u(x,y) \le z^{\frac{1}{p}}(x,y)$, it is easy to obtain the inequality (37) with the condition (38).

Corollary 26 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* α, $\beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \le x$ *on* R_0^+ *and* $\beta(y) \le y$ *on* R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge q \ge 1$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$u^p(x,y) \le a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u^p(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)u^q(\tau,\sigma)d\tau d\sigma)dsdt, \tag{39}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \le a^{\frac{1}{p}}(x,y)C_1(x,y)\exp\frac{1}{p}\Big[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} m_3 f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \tag{40}$$

for all $(x,y) \in R_0^+ \times R_0^+$, *where* $C_1(x,y)$ *is defined in* (38).

Proof: Since $a(x,y)$ is a positive and non-decreasing function, from (39) we have

$$\frac{u^p(x,y)}{a(x,y)} \le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\frac{u^p(s,t)}{a(s,t)}dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)\frac{u^q(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

i.e.

$$\left[\frac{u(x,y)}{a^{\frac{1}{p}}(x,y)}\right]^p \le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\left[\frac{u(s,t)}{a^{\frac{1}{p}}(s,t)}\right]^p dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)\left[\frac{u(\tau,\sigma)}{a^{\frac{1}{p}}(\tau,\sigma)}\right]^q d\tau d\sigma\Big)dsdt.$$

Using Theorem 25, we obtain the inequality (40) with the condition (38).

Corollary 27 *Let $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+ and $\beta(y) \le y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge q \ge 1$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u^p(x,y) &\le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u^q(s,t)dsdt \\ &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma) \\ &\qquad u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \end{aligned} \tag{41}$$

for $(x,y) \in R_0^+ \times R_0^+$, then

$$\begin{aligned} u(x,y) &\le \exp\frac{1}{p}\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \\ &\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg], \end{aligned} \tag{42}$$

for all $(x,y) \in R_0^+ \times R_0^+$.

Proof: Define a function $z^p(x,y)$ by the right side of (41), then $u^p(x,y) \le z^p(x,y)$, $u(x,y) \le z(x,y)$, and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} z^p(x,y) \le h(x,y)z^q(x,y)\alpha'(x)\beta'(y)$$

$$+\alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)z(s,t)dsdt\Big),$$

hence

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} z^p(x,y) \le h(x,y)z^p(x,y)\alpha'(x)\beta'(y)$$

$$\alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)z^p(s,t)dsdt\Big).$$

i.e.

$$\begin{aligned} \frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y} z^p(x,y)}{z^p(x,y)} &\le h(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y) \\ &\quad f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big). \end{aligned}$$

Therefore

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y} z^p(x,y)z^p(x,y)}{(z^p(x,y))^2} - \frac{\frac{\partial}{\partial x} z^p(x,y)\frac{\partial}{\partial y}z^p(x,y)}{(z^p(x,y))^2} \le h(x,y)$$

$$\alpha'(x)\beta'(y)+\alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big).$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x} z^p(x,y)}{z^p(x,y)}\right] &\le h(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y) \\ &\quad f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big). \end{aligned}$$

By keeping y fixed, setting $x = s$, and integrating the above inequality, and again by keeping x fixed, setting $y = t$, and integrating the resulting inequality, we obtain

$$\begin{aligned} z^p(x,y) &\le \exp\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt \\ &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg]. \end{aligned}$$

Since $u(x,y) \le z(x,y)$, we obtain the inequality (42).

Remark 28 *(i) If we take $p = q = 1$ and $a(x,y) = 1$, then Theorem 25 and Corollary 27 reduce to Theorem 20.*
(ii) If we take $p = q = 1$, then Corollary 26 reduces to Theorem 20.

Corollary 29 *Let $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ be non-decreasing functions in each variable, with $\alpha(x) \le x$ on R_0^+ and $\beta(y) \le y$ on R_0^+, $f, g, h \in C(R_0^+ \times R_0^+, R_0^+)$, $p \ge q \ge 1$. If $u \in C(R_0^+ \times R_0^+, R_0^+)$ satisfies*

$$\begin{aligned} u(x,y) &\le 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u^p(s,t)dsdt \\ &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma) \\ &\qquad u^q(\tau,\sigma)d\tau d\sigma\Big)dsdt, \end{aligned} \tag{43}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \leq \Big[1+(1-p)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big)\Big]^{\frac{1}{1-p}}, \quad (44)$$

for all $(x,y) \in R_0^+ \times R_0^+$.

Proof: Define a function $z(x,y)$ by right side of (43), then

$$u(x,y) \leq z(x,y), \quad (45)$$

and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} z(x,y) \leq h(x,y)z^p(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)z^q(s,t)dsdt\Big).$$

i.e.

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} z(x,y) \leq h(x,y)z^p(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y)f(x,y)z^p(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big).$$

It follows that

$$\frac{\frac{\partial}{\partial x}\frac{\partial}{\partial y} z(x,y)z(x,y)}{z^{p+1}(x,y)} - \frac{\frac{\partial}{\partial x}z(x,y)\frac{\partial}{\partial y} z(x,y)}{z^{p+1}(x,y)} \leq h(x,y) \alpha'(x)\beta'(y)+\alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big).$$

Therefore

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z^p(x,y)}\right] \leq h(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y) f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big).$$

By keeping y fixed, setting $x=s$, and integrating the above inequality and using (45), and again by keeping x fixed, setting $y=t$, and integrating the resulting inequality and using (45), we obtain the inequality (44).

Remark 30 *If we replace* $\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}$ *with* $\int_0^{\alpha(x)}$ *and the functions with one-variable functions in Theorem* 25 *and Corollaries* $26,\ 27,$ *and* 29 *we obtain Propositions 13, 15, 17, and 19 in* $[2]$ *respectively.*

Theorem 31 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \leq x$ *on* R_0^+ *and* $\beta(y) \leq y$ *on* R_0^+, $f,g,h \in C(R_0^+ \times R_0^+, R_0^+)$, $b \in C(R_0^+ \times R_0^+, R_0^+)$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$u(x,y) \leq a(x,y) + b(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)u(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \quad (46)$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \leq a(x,y)\exp\Big[b(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Big], \quad (47)$$

for all $(x,y) \in R_0^+ \times R_0^+$.

Proof: Since $a(x,y)$ is positive and non-decreasing, from (46) we have

$$\frac{u(x,y)}{a(x,y)} \leq 1 + b(x,y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\frac{u(s,t)}{a(s,t)}dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Fix any $(X,Y) \in R_0^+ \times R_0^+$. Then, for $0 \leq x \leq X$, $0 \leq y \leq Y$, we have

$$\frac{u(x,y)}{a(x,y)} \leq 1 + b(X,Y)\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\frac{u(s,t)}{a(s,t)}dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_s^{\infty}\int_t^{\infty} g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Define a function $z(x,y)$ by the right side of the above inequality, then $z(x,y)$ is positive and non-decreasing, $z(0,y) = z(x,0) = 1$, $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\, z(x,y) \leq z(x,y)\Bigg[b(X,Y)h(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big)\Bigg].$$

It follows that

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x}z(x,y)}{z(x,y)}\right] \leq b(X,Y)h(x,y)\alpha'(x)\beta'(y) + \alpha'(x)\beta'(y)f(x,y)\Big(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(s,t)dsdt\Big).$$

By keeping y fixed, setting $x = s$, and integrating the above inequality, and again by keeping x fixed, setting $y = t$, and integrating the resulting inequality, we get

$$z(X,Y) \leq \exp\Bigg[b(X,Y)\int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty} h(s,t)dsdt + \int_{\alpha(X)}^{\infty}\int_{\beta(Y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg].$$

Since (X,Y) is arbitrary and $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, by replacing (X,Y) with (x,y) in the above inequality we obtain the result (47).

Remark 32 *(i) If we replace* $\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}$ *with* $\int_{0}^{\alpha(x)}$ *and the functions with one-variable functions in Theorem* 31, *we obtain Proposition 20 in* [2].
(ii) If we take $b(x,y) = 1$, *then Theorem* 31 *reduces to Theorem* 20.

Theorem 33 *Let* $a \in C(R_0^+ \times R_0^+, R^+)$ *and* $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$ *be non-decreasing functions in each variable, with* $\alpha(x) \leq x$ *on* R_0^+ *and* $\beta(y) \leq y$ *on* R_0^+, $f,g,h \in C(R_0^+ \times R_0^+, R_0^+)$. *Let* ω *be a non-decreasing continuous function such that* $\omega \geq 1$, $\omega(x) \geq x$. *If* $u \in C(R_0^+ \times R_0^+, R_0^+)$ *satisfies*

$$u(x,y) \leq a(x,y) + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\omega(u(s,t))dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)u(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{48}$$

for $(x,y) \in R_0^+ \times R_0^+$, *then*

$$u(x,y) \leq a(x,y)W^{-1}\Bigg[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg], \tag{49}$$

for all $(x,y) \in R_0^+ \times R_0^+$, *where* W^{-1} *is the inverse function of*

$$W(r) = \int_{r}^{\infty}\frac{ds}{\omega(s)}, \quad r \geq 0, \tag{50}$$

and $(x_1,y_1) \in R_0^+ \times R_0^+$ *is chosen so that* $\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt \in Dom(W^{-1})$, *for all* (x,y) *lying in* $[0,x_1]\times[0,y_1]$.

Proof: Since $a(x,y)$ is positive and non-decreasing, from (48) we have

$$\frac{u(x,y)}{a(x,y)} \leq 1 + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} h(s,t)\omega\Big(\frac{u(s,t)}{a(s,t)}\Big)dsdt + \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty}\int_{t}^{\infty} g(\tau,\sigma)\frac{u(\tau,\sigma)}{a(\tau,\sigma)}d\tau d\sigma\Big)dsdt.$$

Define a function $z(x,y)$ by the right side of the above inequality, then $z(x,y) > 0$, $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, and

$$\frac{\partial}{\partial x}\, z(x,y) \geq -\alpha'(x)\omega(z(x,y))\int_{\beta(y)}^{\infty} h(x,t)dt - \alpha'(x)z(x,y)\int_{\beta(y)}^{\infty} f(x,t)\Big(\int_{\alpha(x)}^{\infty}\int_{t}^{\infty} g(s,\sigma)dsd\sigma\Big)dt,$$

hence

$$\frac{\frac{\partial}{\partial x} z(x,y)}{\omega(z(x,y))} \geq -\alpha'(x) \int_{\beta(y)}^{\infty} h(x,t)dt - \alpha'(x) \int_{\beta(y)}^{\infty} f(x,t)\Big(\int_{\alpha(x)}^{\infty} \int_{t}^{\infty} g(s,\sigma)dsd\sigma\Big)dt.$$

By integration and using (50), we have

$$z(x,y) \leq W^{-1}\Bigg[\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} h(s,t)dsdt + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} f(s,t)\Big(\int_{s}^{\infty} \int_{t}^{\infty} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg].$$

Since $\frac{u(x,y)}{a(x,y)} \leq z(x,y)$, we obtain the result (49).

Remark 34 *(i) If we replace* $\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty}$ *with* $\int_{0}^{\alpha(x)}$ *and the functions with one-variable functions in Theorem* 33, *we obtain Proposition 22 in* [2].
(ii) If we take $\omega(u(x,y)) = u(x,y)$, *then Theorem* 33 *reduces to Theorem* 20.

3 APPLICATION

In this section, we present an application of the inequality (15) given in Corollary 11 to study the boundedness of solutions of certain non-linear hyperbolic partial differential equations with delay of the form

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} u^p(x,y) = m(x,y,u(x,y)) + f(x,y) \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} k(s,t,u(s,t))dsdt, \tag{51}$$

$$u(x,0) = \sigma_1(x);\ u(0,y) = \sigma_2(y);\ u(0,0) = k, \tag{52}$$

for all $(x,y) \in R_0^+ \times R_0^+$, where $\alpha,\ \beta \in C^1(R_0^+, R_0^+)$, $m,k \in C((R_0^+)^3, R^+)$, $f \in C(R_0^+ \times R_0^+, R_0^+)$ $\sigma_1, \sigma_2 \in C(R_0^+, R^+)$, and $k,\ q > p > 0$ are constants. We assume that those functions are defined and continuous on their respective domains of definitions.

The following example deals with the estimation of the solution of the partial differential equation (51) with the condition (52).

Example: Suppose that

$$\mid m(x,y,u) \mid \leq h(x,y)u^p, \tag{53}$$

$$\mid k(x,y,u) \mid \leq g(x,y)u^q, \tag{54}$$

$$\mid \sigma_1(x) + \sigma_2(y) - k \mid \leq r, \quad r > 0\ (constant), \tag{55}$$

where h, g are defined in Corollary 11.

If $u(x,y)$ is any solution of (51), then

$$u(x,y) \leq r^{\frac{1}{p}} C(x,y) \exp\frac{1}{p}\Bigg[\int_{0}^{x} \int_{0}^{y} h(s,t)dsdt + \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} m_3 f(s,t)\Big(\int_{0}^{s} \int_{0}^{t} g(\tau,\sigma)d\tau d\sigma\Big)dsdt\Bigg], \tag{56}$$

for all $(x,y) \in R_0^+ \times R_0^+$, where $C(x,y)$ is defined in (14).

Proof: It is clear that the solution $u(x,y)$ of (51) with the condition (52) satisfies the equivalent integral equation

$$u^p(x,y) = \sigma_1(x)+\sigma_2(y)-k + \int_{0}^{x} \int_{0}^{y} m(s,t,u(s,t))dsdt + \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} f(s,t)\Big(\int_{0}^{s} \int_{0}^{t} k(\tau,\sigma,u(\tau,\sigma))d\tau d\sigma\Big)dsdt.$$

From (53), (54), and (55) we have

$$u^p(x,y) \leq r + \int_{0}^{x} \int_{0}^{y} h(s,t)u^p(s,t)dsdt + \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} f(s,t) \Big(\int_{0}^{s} \int_{0}^{t} g(\tau,\sigma)u^q(\tau,\sigma)d\tau d\sigma\Big)dsdt, \tag{57}$$

for $(x,y) \in R_0^+ \times R_0^+$. Now a suitable application of Corollary 11 to (57), with $c(x,y) = r\ (constant)$, yields the inequality (56).

Remark 35 *In the special case where* $p = q = 1$ *in the boundary value problem* (51) *with the conditions* (52), (53), (54), *and* (55) *in the above example, we can obtain the inequality* (2) *in Theorem* 2.

4 Conclusion

In this paper, we worked on some retarded integral inequalities for functions of two independent variables. Also, we generalized those inequalities by integration over infinite intervals. An example was given as an application of the main results.

Acknowledgements: The authors are very grateful to the reviewers for their valuable feedback.

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