A fixed point approach to the stability of a nonic functional Equation in modular spaces

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Abstract: - In this paper, we present a fixed point results which was proved by Khamsi [9] in modular function spaces to prove the generalized Hyers-Ulam stability of a nonic functional equation :

$$f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!\, f(y),$$

where $9! = 362880$ in modular spaces.

Key-Words: - Generalized Hyers-Ulam-Rassias stability, Modular spaces, Nonic functional equations

1 Introduction

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative answer to the said famous question of Ulam for Banach spaces. Hyer's theorem was generalized by Aoki [1] for additive mappings. In 1978, Rassias [23] generalized Hyers theorem by obtaining a unique linear mapping near an approximate additive mapping. The paper of Rassias has provided a lot of influence in the development of what we call the generalized Hyers-Ulam-Rassias stability of functional equations. Recently, Cho et al. [3], El-Fassi and Kabbaj [5], Gordji et al. [6] and Sadeghi [24] proved the generalized Hyers-Ulam-Rassias stability of several functional equations in modular spaces. Many authors proved the stability of various functional equations in various spaces [2, 4, 8, 10, 11, 15, 18, 22, 27].

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were initiated by Nakano [19], and refined and generalized by Musielak and Orlicz [17] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. In the present time the theory of modulars and modular spaces have been studied and extensively applied in various parts of analysis. Furthermore the most complete development of these theories was due to Krbec [13], Luxemburg [14], Musielak [16], Orlicz [20, Turpin [25] and Yamamuro [28].

We recall some basic facts and preliminary results concerning modular spaces [12].

Definition 1.1. Let X be a linear space. A function $\rho : X \to [0,\infty]$ is called a modular if for all $x, y \in X$

(M1) $\rho(x) = 0$ if and only if $x = 0$;

(M2) $\rho(\alpha x) = \rho(x)$ for all scalars α with $|\alpha| = 1$;

(M3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

then we say that ρ is a convex modular.

If (M3) is replaced by

$(M3')$ $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$

for all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and $x, y \in X$, then we say that ρ is a convex modular.

Remark 1.2.
(i) For a fixed $x \in X_\rho$, the valuation $\gamma \in K \mapsto \rho(\gamma x)$ is increasing.
(ii) $\rho(x) \le \delta\, \rho(\frac{1}{\delta} x)$ for all $x \in X_\rho$ provided ρ is a convex modular and $0 < \delta \le 1$;
(iii) Every norm defined on X is a modular on X. In general, the modular ρ does not behave as norm or distance because it is not subadditive. But one can associate the F-norm to a modular (see [16]).

A modular ρ defines a corresponding modular space X_ρ. The space X_ρ is given by

$$X_\rho = \{x \in X: \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$||x||_\rho = \inf \{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\}.$$

Example 1.3. [9] The Orlicz's modular is defined by the formula

$$\rho(f) = \int_R \varphi(|f(t)|)dm(t),$$

for every measurable real function f where m denotes the Lebesgue measure in R and $\varphi: R \to [0,\infty)$ is continuous. We also assume that $\varphi(0) = 0$ if and only if $\varphi(t) \to \infty$ as $t \to \infty$. The modular space induced by the Orlicz's modular ρ_φ is a modular function space, which is called the Orlicz space L^φ .

Definition 1.4. Let X_ρ be a modular space and $\{x_n\}$ be a sequence in X_ρ. Then
(i) $\{x_n\}$ is called $\rho-$convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is called $\rho-$Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) a subset S of X_ρ is called $\rho-$complete if every $\rho-$Cauchy sequence is $\rho-$convergent to a point in S.
(vi) the modular ρ has the Fatou property if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever $\{x_n\}$ is $\rho-$convergent to x.
(v) a modular ρ is said to satisfy the Δ_2-type condition if there exists $\kappa > 0$ such that $\rho(2x) \le \kappa\, \rho(x)$ for all $x \in X_\rho$.

The following fixed point theorem will play an important role in proving our main theorem:

Theorem 1.5. [9] Let (X,ρ) be a modular space such that ρ satisfies the Fatou property. Let C be a $\rho-$complete nonempty subset of X_ρ and $T: C \to C$ be quasicontraction and $x \in C$. If $(\omega - T(\omega)) < \infty$ *and* $(x - T(\omega)) < \infty$, *then* the $\rho-$limit ω of $\{T^n(x)\}$ is a fixed point of T.

Now, we consider a functional equation $f: X \to Y$ defined by
(1.1)

$$f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!\, f(y)$$

is called nonic functional equation since the function $f(x) = x^9$ is its solution. Every solution of the nonic functional equation is said to be a nonic mapping.
Note that (see, [21]) the above functional equation has the following properties
(i) $f(0) = 0$,
(ii) $f(-x) = -f(x)$ and
(iii) $f(2x) = 2^9 f(x)$.

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of a nonic functional equation (1.1) for mappings from linear spaces into modular spaces with the Δ_2 −condition using fixed point theorem which was prove by Khamsi [9] in modular spaces.
For a given mapping $f: X \to Y$, we defined a difference operator:

$$Df(x,y) = f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) - 9!\, f(y).$$

2 Main Results

In this section, we assume that ρ is a convex modular on X with the Fatou property such that satisfies the Δ_2 − condition with $0 < \kappa < 2$.

Theorem 2.1. Let X be a linear space and X_ρ be a $\rho-$complete modular space. Suppose $f: X \to X$ is a mapping with $f(0) = 0$ for which there exists $\varphi: X \times X \to [0,\infty)$ such that
(2.1)

$$\rho(Df(x,y)) \le \varphi(x,y)$$

for all $x, y \in X$. If there exists $0 << 1$ *such that*
(2.2)

$$\varphi(2x,2x) \le 2^9 L\varphi(x,x)$$

and

(2.3)
$$\lim_{\{n\to\infty\}} \{\frac{\varphi(2^n x, 2^n y)}{2^{\{9n\}}}\} = 0$$
for all $x, y \in X$, then there exists a unique nonic mapping $N : X \to X_\rho$ such that

(2.4)
$$\rho(N(x) - f(x)) \leq \left\{\frac{1}{2^9(1-L)}\right\} \Phi(x)$$
for all $x \in X$, where
$$\Phi(x) = \left\{\frac{1}{9!}\right\} [\varphi(0,2x) + \varphi(5x, x) + 9\varphi(4x, x) + 37\varphi(3x, x) + 93\varphi(2x, x) + 162\varphi(x, x) + 210\varphi(0, x)]$$

Proof. Let ρ be convex and satisfy the Δ_2-condition.

Replacing (x, y) with $(0, 2x)$ in (2.1), we get

(2.5)
$$\rho(f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - 362838f(2x)) \leq \varphi(0,2x)$$
for all $x \in X$. Replacing (x, y) with $(5x, x)$ in (2.1), we get

(2.6)
$$\rho(f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) + 84f(4x) - 36f(3x) + 9f(2x) - 362881f(x)) \leq \varphi(5x, x)$$
for all $x \in X$. Subtracting (2.5) and (2.6), then

(2.7)
$$\rho(9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) + 36f(3x) - 362847f(2x) + 362881f(x)) \leq \left\{\frac{\kappa}{2}\right\}(\varphi(0,2x) + \varphi(5x, x)) \leq \varphi(0,2x) + \varphi(5x, x)$$
for all $x \in X$. Replacing (x, y) with $(4x, x)$ in (2.1), we get

(2.8)
$$\rho(9f(9x) - 81f(8x) + 324f(7x) - 756f(6x) + 1134f(5x) - 1134f(4x) + 756f(3x) - 324f(2x) - 3265839f(x)) \leq 9\varphi(4x, x)$$
for all $x \in X$. Subtracting (2.7) and (2.8), we get

(2.9)
$$\rho(37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) - 720f(3x) - 362523f(2x) + 3628720f(x)) \leq \varphi(0,2x) + \varphi(5x, x) + 9\varphi(4x, x)$$
for all $x \in X$. Replacing (x, y) with $(3x,x)$ in (2.1), we get

(2.10)
$$\rho(37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x) + 4662f(4x) - 4662f(3x) + 3108f(2x) - 13427855f(x)) \leq 37\varphi(3x, x)$$

for all $x \in X$. Subtracting (2.9) and (2.10), we find

(2.11)
$$\rho(93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) + 3942f(3x) - 365631f(2x) + 17056575f(x)) \leq \varphi(0,2x) + \varphi(5x, x)) + 9\varphi(4x, x) + 37\varphi(3x, x)$$
for all $x \in X$. Replacing (x, y) with $(2x, x)$ in (2.1), we get

(2.12)
$$\rho(93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x) + 11718f(3x) - 11625f(2x) - 33740865f(x)) \leq 93\varphi(2x, x)$$
for all $x \in X$. Subtracting (2.11) and (2.12) we obtain

(2.13)
$$\rho(162f(6x) - 1248f(5x) + 4152f(4x) - 7776f(3x) - 354006f(2x) + 50797440f(x)) \leq \varphi(0,2x) + \varphi(5x, x) + 9\varphi(4x, x) + 37\varphi(3x, x) + 93\varphi(2x, x)$$
for all $x \in X$. Replacing (x, y) with (x, x) in (2.1), we get

(2.14)
$$\rho(162f(6x) - 1458f(5x) + 5832f(4x) - 13446f(3x) + 18954f(2x) - 58801140f(x)) \leq 162\varphi(x, x)$$
for all $x \in X$. Subtracting (2.13) and (2.14), we find

(2.15)
$$\rho(210f(5x) - 1680f(4x) + 5670f(3x) - 372960f(2x) + 109598580f(x)) \leq \varphi(0,2x) + \varphi(5x, x) + 9\varphi(4x, x) + 37\varphi(3x, x) + 93\varphi(2x, x) + 162\varphi(x, x)$$
for all $x \in X$ Replacing (x, y) with $(0, x)$ in (2.1), we get

(2.16)
$$\rho(210f(5x) - 1680f(4x) + 5670f(3x) - 10080f(2x) - 76195980f(x)) \leq 210\varphi(0, x)$$
for all $x \in X$. Subtracting (2.15) and (2.16), we get

(2.17)
$$\rho(-362880f(2x) + 185794560f(x)) \leq \varphi(0,2x) + \varphi(5x, x)) + 9\varphi(4x, x) + 37\varphi(3x, x) + 93\varphi(2x, x) + 162\varphi(x, x) + 210\varphi(0, x)$$
for all $x \in X$. Thus, we deduce that

(2.18)

$$\rho(f(2x) - 2^9 f(x)) \le \frac{1}{9!}[\varphi(0,2x) + \varphi(5x,x) + 9\varphi(4x,x) + 37\varphi(3x,x) + 93\varphi(2x,x) + 162\varphi(x,x) + 210\varphi(0,x)] \equiv \Phi(x)$$

for all $x \in X$. Now, we consider that the modular $\tilde{\rho}$ on the set $\Omega = \{g : X \to X_\rho,\ g(0) = 0\}$ as follows:

$$\tilde{\rho}(g) = \inf\{c > 0 : \rho(g(x)) \le c\,\Phi(x), \text{for all } x \in X\}.$$

Now, by sever steps, we show that the conditions of Theorem 1.5 holds.

Step 1: $\tilde{\rho}$ is a convex modular

It is sufficient to show that $\tilde{\rho}$ satisfies the condition (M3') of Definition 1.1.

Let $g, h \in \Omega$ and $\epsilon > 0$ be given. Then there exist $c_1 > 0$ *and* $c_2 > 0$ such that

$$c_1 \le \tilde{\rho}(g) + \epsilon \text{ and } c_2 \le \tilde{\rho}(h) + \epsilon$$

and so

$$\rho(g(x)) \le c_1\,\Phi(x) \text{ and}\}\ \rho(h(x)) \le c_2\,\Phi(x)$$

for all $x \in X$.

If $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$, then we get

$$\rho(\alpha\, g(x) + \beta\, h(x)) \le \alpha\,\rho(g(x)) + \beta\,\rho(h(x)) \le (\alpha\, c_1 + \beta\, c_2)\Phi(x),$$

which implies

$$\tilde{\rho}(\alpha\, g + \beta\, h) \le \alpha\,\tilde{\rho}(g) + \beta\,\tilde{\rho}(h) + (\alpha + \beta)\epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\tilde{\rho}(\alpha\, g + \beta\, h) \le \alpha\,\tilde{\rho}(g) + \beta\,\tilde{\rho}(h)$$

for all $g, h \in \Omega$. Thus, $\tilde{\rho}$ is a convex modular on Ω.

Step 2: $M_{\tilde{\rho}}$ is $\tilde{\rho}$-complete.

Let $\{g_n\}$ be a $\tilde{\rho}$-Cauchy sequence in $\Omega_{\tilde{\rho}}$ and $\epsilon > 0$ be given. Then there exists $N \in Z^+$ such that

$$\tilde{\rho}(g_n - g_m) \le \epsilon$$

for all $n, m \ge N$, and then, we get

$$\rho(g_n(x) - g_m(x)) \le \epsilon\,\Phi(x)$$

for all $x \in X$ and $n, m \ge N$. If x is any given element of X, then $\{g_n(x)\}$ is a ρ-Cauchy sequence in X_ρ. Since X_ρ is $\rho-$ complete, $\{g_n(x)\}$ is $\rho-$convergent in X_ρ for all $x \in X$. So, we can define a mapping $g: X \to X_\rho$ by $g(x) = \lim_{\{m\to\infty\}} g_m(x)$ for any $x \in X$. Letting $m \to \infty$ in the above inequality, we obtain

$$\tilde{\rho}(g_n - g) \le \epsilon$$

for all $n \ge N$, since ρ has the Fatou property.

Then $\{g_n\}$ is a $\tilde{\rho}$-convergent sequence in $M_{\tilde{\rho}}$. Thus, $M_{\tilde{\rho}}$ is $\tilde{\rho}$-complete.

Step 3: T is a $\tilde{\rho}$-strict contractive. We consider the mapping

$$T : \Omega_{\tilde{\rho}} \to \Omega_{\tilde{\rho}} \text{ by}$$

$$T\,g(x) = \left\{\frac{g(2x)}{2^9}\right\}$$

for all $x \in X$. Let $g, h \in \Omega_{\tilde{\rho}}$ and let $c \in [0,\infty)$ be a constant with $\tilde{\rho}(g - h) \le c$. Then, we have

$$\rho(g(x) - h(x)) \le c\,\Phi(x)$$

for all $x \in X$. Thus, we get

$$\rho(Tg(x) - Th(x)) \le \left\{\frac{1}{2^9}\right\}\rho(g(2x) - h(2x)) \le \left\{\frac{1}{2^9}\right\} c\,\Phi(2x) \le cL\Phi(x)$$

for all $x \in X$, which implies

$$\tilde{\rho}(Tg - Th) \le L\,\tilde{\rho}(g - h)$$

for all $g, h \in \Omega_{\tilde{\rho}}$, that is, T is a $\tilde{\rho}$-strict contractive on $\$\,\Omega_{\tilde{\rho}}$.

Step 4:

$$\delta_{\tilde{\rho}}(f) = \sup\{\tilde{\rho}(T^n(f) - T^m(f)) : n, m \in \backslash Z^+\} < \infty.$$

It follows from (2.18) that

$$\rho\left(\left\{\frac{f(2x)}{2^9}\right\} - f(x)\right) \le \frac{\Phi(x)}{2^9}$$

and

(2.19)

$$\rho\left(\left\{\frac{f(2^2x)}{2^9}\right\} - f(2x)\right) \le \frac{\Phi(2x)}{2^9}$$

for all $x \in X$. It follows from (2.19), ρ is convex modular and satisfies the Δ_2 −condition that

$$\rho\left(\frac{f(2^2x)}{(2^9)^2} - f(x)\right) \le \rho\left(\left\{\frac{2}{2\cdot 2^9}\right\}\left(\frac{f(2^2x)}{2^9} - f(2x)\right) + \frac{2}{2}\left(\left\{\frac{f(2x)}{2^9}\right\} - f(x)\right)\right)$$

$$\le \left\{\frac{\kappa}{2\cdot 2^9}\right\}\left\{\frac{\Phi(2x)}{2^9}\right\} + \left\{\frac{\kappa}{2}\right\}\left\{\frac{\Phi(x)}{2^9}\right\} \le \left\{\frac{1}{2^9}\right\}\left(\left\{\frac{\kappa}{2\cdot 2^9}\right\}\Phi(2x) + \left\{\frac{\kappa}{2}\right\}\Phi(x)\right)$$

for all $x \in X$. By mathematical induction, we can deduce that

(2.20)

$$\rho\left(\frac{f(2^n x)}{2^{9n}} - f(x)\right) \le \left\{\frac{1}{2^9}\right\}\cdot \sum_{i=1}^{n-1}\frac{\kappa^i}{2^i}\,\frac{\Phi(2^{\{n-i\}}x)}{(2^9)^{\{n-i\}}} + \left(\left\{\frac{\kappa}{2}\right\}\right)^{\{n-1\}}\frac{\Phi(x)}{2^9} \le \left\{\frac{1}{2^9(1-L)}\right\}\Phi(x)$$

for all $x \in X$. Hence, it follows from (2.20) that

$$\rho\left(\frac{f(2^n x)}{2^{\{9n\}}}-\frac{f(2^m x)}{2^{\{9m\}}}\right)$$
$$\leq \left\{\frac{\kappa}{2}\right\}\left(\rho\left(\frac{f(2^n x)}{2^{\{9n\}}}-f(x)\right)\right.$$
$$\left.+\rho\left(\frac{f(2^m x)}{2^{\{9m\}}}-f(x)\right)\right)$$
$$\leq \left\{\frac{\kappa}{2^9(1-L)}\right\}\Phi(x)$$

for all $x \in X$ and $n, m \in \backslash Z^+$, which implies that

$$\tilde{\rho}\left(T^n(f)-T^m(f)\right) \leq \frac{\kappa}{2^9(1-L)}$$

for all $n, m \in Z^+$. By the definition of $\delta_{\tilde{\rho}}(f)$, we have $\delta_{\tilde{\rho}}(f) < \infty$.

Step 5: $\tilde{\rho}$ satisfies the Fatou property. Let $\{g_n\}$ be $\tilde{\rho}$-convergent to a point $g \in M_{\tilde{\rho}}$. Suppose that $\lim\inf_{\{n\to\infty\}} \tilde{\rho}(g_n) < \tilde{\rho}(g)$ and $l = \lim\inf_{\{n\to\infty\}} \tilde{\rho}(g_n) < \infty$. Then there exists a subsequence $\{g_{n_i}\}$ in $\{g_n\}$ such that $\lim_{\{i\to\infty\}} \tilde{\rho}(g_{n_i}) = l$. Let $\epsilon > 0$ be given. Then there exists $i_0 \in N$ such that $\tilde{\rho}\left(g_{n_i}\right) \leq \epsilon + l$ for all $i \geq i_0$, and so

$$\rho\left(g_{n_i}(x)\right) \leq (\epsilon + l)\Phi(x)$$

for all $i \geq i_0$ and $x \in X$. Thus it follows that

$$\lim_{\{i\to\infty\}} \tilde{\rho}\left(g_{n_i}(x)\right) \leq l\,\Phi(x)$$

for all $x \in X$. On the other hands, we have $\lim_{\{i\to\infty\}} \rho(g_{n_i}(x)-g(x)) = 0$ for all $x \in X$. Since ρ satisfies the Fatou property, we have

$$\rho\left(g(x)\right) \leq \lim\inf_{\{i\to\infty\}} \tilde{\rho}\left(g_{n_i}\right) \leq l\,\Phi(x)$$

for all $x \in X$. This shows that $\tilde{\rho}(g) \leq l$, which is a contradiction. Thus, $\tilde{\rho}(g) \leq \lim\inf_{\{n\to\infty\}} \tilde{\rho}(g_n)$.

Step 6: $\tilde{\rho}(T\,N-f) < \infty$ and $\tilde{\rho}(T\,N - N) < \infty$. Since $\delta_{\tilde{\rho}}(f) < \infty$, $\{T^n(f)\}$ is $\tilde{\rho}-$convergent to $N \in M_{\tilde{\rho}}$ (see, Lemma 3.3 of [9]) .
Also, since ρ has the Fatou property,

$$\tilde{\rho}(T\,N-f) < \infty.$$

On the other hand, if we replace x by $2^n x$ in (2.19), then we obtain

$$\rho\left(\left\{\frac{f\left(2^{\{n+1\}x}\right)}{2^9}\right\}-f(2^n\,x)\right) \leq \left\{\frac{1}{2^9}\right\}\Phi(2^n x)$$

for all $x \in X$. Thus, we have

$$\rho\left(\frac{f\left(2^{\{n+1\}}x\right)}{2^{\{9(n+1)\}}}-\frac{f\left(2^{\{n\}}x\right)}{2^{\{9n\}}}\right)$$
$$\leq \left\{\frac{1}{2^{\{9n\}}}\right\}\rho\left(\frac{f\left(2^{\{n+1\}}x\right)}{2^{\{9\}}}-f\left(2^{\{n\}}\right)\right)$$
$$\leq \left\{\frac{1}{2^{\{9(n+1)\}}}\right\}\Phi(2^{\{n\}}x)$$
$$\leq \left\{\frac{L^n}{2^9}\right\}\Phi(x) \leq \Phi(x)$$

for all $x \in X$, and so $\tilde{\rho}(T\,N-N) < \infty$.
Hence, all conditions of Theorem 1.5 are fulfilled. So, the $\tilde{\rho}-$limit N of $M_{\tilde{\rho}}$ is a fixed point of T.
It follows from (2.20) that

$$\tilde{\rho}(N - f) \leq \frac{1}{2^9(1-L)} .$$

Thus, the inequality (2.4) holds for all $x \in X$.
If we replace x by $2^n x$ and y by $2^n y$ in (2.1) then we obtain

$$\rho\left(\frac{Df(2^n x, 2^n\,y)}{2^{9n}}\right) \leq \left\{\frac{1}{2^{9n}}\right\}\rho(Df(2^n\,x, 2^n\,y))$$
$$\leq \left\{\frac{1}{2^{9n}}\right\}\varphi(2^n\,x, 2^n\,y)$$

for all $x, y \in X$. Taking as the limit $n \to \infty$ in the above inequality and using (2.3), $DN(x,y) = 0$ for all $x, y \in X$. Thus, a mapping N is nonic.
For the uniqueness of N, let N' be another nonic mapping satisfying (2.4). Then

$$\tilde{\rho}(N-N')$$
$$\leq \frac{\kappa}{2}\left(\tilde{\rho}(T\,N - f) + \frac{\kappa}{2}\,\tilde{\rho}(T\,N' - f)\right)$$
$$\leq \frac{\kappa}{2^9(1-L)} < \infty.$$

Since T is $\tilde{\rho}-$strict contraction, we obtain

$$\tilde{\rho}(N-N') = \tilde{\rho}(T\,N-T\,N') \leq L\,\tilde{\rho}(N - N'),$$

since $\tilde{\rho}(N - N') < \infty$.
Thus, $N = N'$. This completes the proof. □

Corollary 2.2. Let X be a normed space and Y be a Banach space. Suppose $f: X \to Y$ is a mapping with $f(0) = 0$ and there exist constants $\epsilon, \theta \geq 0$ such that

$$\left\|Df(x,y)\right\| \leq \epsilon + \theta(\|x\| + \|y\|)$$

for all $x, y \in X$. Then there exists a unique nonic mapping $N: X \to Y$ such that

$$\left\|N(x)-f(x)\right\| \leq \frac{513\epsilon}{9289280(1-L)} + \frac{507\,\theta\,\|x\|}{181440(1-L)}$$

for all $x, y \in X$.

Proof. It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and

$\kappa = 2$. Let $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \epsilon + \theta(\|x\| + \|y\|)$. Then, the proof follows from Theorem 2.1. □

Example 2.3. Let ρ be an Orlicz function and satisfy the $\Delta_2 -$condition with $0 < \kappa < 2$. Let $f: X \to L^{\rho}$ be a mapping satisfying

$$\int_R \varphi(|Df(x, y)|)dm(t) \leq \varphi(x, y)$$

for all $x, y \in X$ and $f(0) = 0$, where $\varphi: R \to [0, \infty)$ is a given function such that

$$\varphi(kx, ky) \leq k^2 L\varphi(x, y)$$

for all $x, y \in X$ and a constant $0 < L < 1$. Then there exists a unique nonic mapping $N : X \to L^{\rho}$ such that

$$\int_R \varphi(|N(x) - f(x)|)dm(t) \leq \frac{1}{2k^2(1 - L)} \varphi(x, 0)$$

for all $x \in X$.

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