Exponential Stability of Nonautonomous Infinite Dimensional Systems

MICHAEL GIL'
Ben Gurion University of the Negev
Department of Mathematics
Beer Sheva
ISRAEL
gilmi@bezeqint.net

Abstract: Let H be a Hilbert space with the unit operator I. We consider linear non-autonomous distributed parameter systems governed by the equation $dy/dt = S(t)y + B(t)y$ $(y = y(t), t > 0)$, where $S(t)$ is an unbounded operator, such that for some constant c, $S(t) + cI$ is dissipative; $B(t)$ is an operator uniformly bounded on $[0, \infty)$, having a uniformly bounded derivative and commuting with $S(t)$. Exponential stability conditions are established. An illustrative example is presented.

Key–Words: distributed parameter system, linear nonautonomous system, stability

1 Introduction and statement of the main result

In this paper, we investigate stability of linear nonautonomous distributed parameter systems governed by differential equations in a Hilbert space. The problem of stability analysis of various infinite dimensional systems continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems because of the absence of its complete solution. The literature on stability of linear distributed parameter systems is very rich, cf. [1, 2, 4, 9], but the time variant systems have been considered mainly in the case of equations with dissipative operators, cf. [1, 3]. Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein. Below we consider equations with operators which are non-dissipative in general.

Besides we considerably generalize the main result from [8] for systems with concentrated parameters and refine the the stability conditions from the paper [5] in which the leading operator is assumed to be constant.

Let H be a complex Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. All the considered operators are assumed to be linear. For an operator A, A^* is the adjoint one, $\sigma(A)$ is the spectrum. In addition, $\alpha(A) = \sup\ \Re\ \sigma(A)$, and $Dom(A)$ is the domain.

Our main object is the equation

$$\dot{y} = S(t)y + B(t)y \quad (y = y(t), t > 0; \dot{y} = dy/dt), \tag{1.1}$$

where $S(t)$ for each $t \geq 0$ is a closed operator in H with a dense constant domain

$$Dom\ (S(t)) \equiv D_0\ \ (t \geq 0),$$

continuous on D_0 and

$$\Lambda(S(t)) := \sup_{w \in D_0, \|w\|=1} \Re\ (S(t)w, w) < \infty, \tag{1.2}$$

$B(t)$ is an operator uniformly bounded on $[0, \infty)$, having a strong derivative $B'(t)$ which is also uniformly bounded on $[0, \infty)$ and

$$S(t)B(s)h = B(s)S(t)h\ \ (h \in D_0;\ t, s \geq 0). \tag{1.3}$$

So $B(t)$ maps D_0 into itself. Under the conditions below the quantity $\Lambda(S(t))$ is uniformly bounded on $[0, \infty)$.

A solution to (1.1) for given $y_0 \in D_0$ is a function $y : [0, \infty) \to D_0$ having a strong derivative and satisfying $y(0) = y_0$. The existence and uniqueness of solutions under considerations is assumed. For various existence results see for instance [1].

Equation (1.1) is said to be exponentially stable, if there are positive constants M, ϵ, such that $\|y(t)\| \leq M exp\ [-\epsilon t] \|y(0)\|$ $(t \geq 0)$ for any solution of (1.1).

It is assumed that

$$\gamma(S) := \sup_{t\geq 0} \frac{1}{t}\int_0^t \Lambda(S(s))ds < \infty$$

and for each $\tau \geq 0$, the operator $B(\tau) + \gamma(S)I$ is Hurwitzian, namely,

$$q(\tau) := 2\int_0^\infty \|e^{(B(\tau)+\gamma(S)I)s}\|^2 ds < \infty. \quad (1.4)$$

Now we are in a position to formulate the main result of this paper.

Theorem 1 *Let the conditions (1.2)-(1.4) and*

$$\sup_{t\geq 0} q^2(t)\|B'(t)\| < 2 \quad (1.5)$$

hold. Then equation (1.1) is exponentially stable.

This theorem is proved in the next section.

2 Proof of Theorem 1

Put $A(t) = B(t) + \gamma(S)I$.

Lemma 2 *Let conditions (1.2)-(1.4) hold. Then equation (1.1) is exponentially stable, provided the equation*

$$\dot{u}(t) = A(t)u(t) \ \ (t \geq 0), \quad (2.1)$$

is exponentially stable.

Proof: Let $V(t)$ be the Cauchy operator to the equation (2.1), that is, $V(t)$ is a bounded operator satisfying $V(t)u(0) = u(t)$ for any solution $u(t)$ of (2.1). Let $W(t)$ be the Cauchy operator to the equation

$$\dot{v}(t) = S_0(t)v(t) \ \ (t \geq 0), \quad (2.2)$$

where $S_0(t) = S(t) - \gamma(S)I$. Put

$$y(t) = W(t)V(t)y_0 \ \ (y_0 \in D_0).$$

According to (1.3) $W(t)V(t) = W(t)V(t)$. Taking into account that $dW(t)/dt=$ $S_0(t)W(t)$ and $dV(t)/dt = A(t)V(t)$, we have

$$\dot{y} = (S_0(t) + A(t))y = (S(t) + B(t))y.$$

So $W(t)V(t)y_0$ is a solution to (1.1). From (2.2) and (1.2) it follows

$$\frac{d}{dt}(v(t), v(t)) = (S_0(t)v(t), v(t)) + (v(t), S_0(t)v(t))$$

$$\leq 2(\Lambda(S(t)) - \gamma(S))(v(t), v(t)) \ \ (t \geq 0).$$

Thus,

$$\|W(t)\| \leq exp\,[\int_0^t \Lambda(S(s_1))ds_1 - \gamma(S)t]$$

$$\leq exp\,[t(\frac{1}{t}\int_0^t \Lambda(S(s_1))ds_1 - \gamma(S))] \leq 1.$$

Hence $\|y(t)\| = \|W(t)V(t)y_0\| \leq \|V(t)y_0\|$. This proves the required result. Q.E.D.

Furthermore, recall that the equation

$$A_0^*Y + YA_0 = E \quad (2.3)$$

with a constant bounded stable operator A_0 (i.e. $\alpha(A_0) < 0$) and a constant bounded operator E has a solution Y which is represented as

$$Y = -\int_0^\infty e^{A_0^* s} E e^{A_0 s} ds, \quad (2.4)$$

cf. [3, Section I.5]. Consequently, the operator

$$Q(t) := 2\int_0^\infty e^{A^*(t)s} e^{A(t)s} ds$$

is a unique solution of the equation

$$A^*(t)Q(t) + Q(t)A(t) = -2I \ \ (t \geq 0) \quad (2.5)$$

Lemma 3 *Let condition (1.4) hold and $A(t)$ is differentiable. Then*

$$\|Q(t)\| \leq q(t), \quad (2.6)$$

$Q(t)$ also is differentiable and $\|Q'(t)\| \leq q^2(t)\|A'(t)\|$.

Proof: Inequality (2.6) is due to (1.4). Differentiating (2.5) we have

$$A^*(t)Q'(t) + Q'(t)A(t)$$

$$= -((A^*(t))'Q(t) + Q(t)A'(t)) \ \ (t \geq 0).$$

Hence due to (2.4)

$$Q'(t) = \int_0^\infty e^{A^*(t)s}((A^*(t))'Q(t)$$

$$+Q(t)A'(t))e^{A(t)s}ds.$$

Thus,

$$\|Q'(t)\| \leq \frac{1}{2}q(t)\|(A^*(t))'Q(t) + Q(t)A'(t)\|$$

$$\leq (t)\|Q(t)\|\|A'(t)\|. \quad (2.7)$$

Now (2.6) yields the result. Q.E.D.

Lemma 4 *Let*

$$\sup_{t\geq 0} \|Q'(t)\| < 2. \tag{2.8}$$

Then

$$(Q(t)u(t), u(t)) \leq (Q(0)u(0), u(0)) \quad (t \geq 0).$$

Proof: Multiplying equation (2.1) by $Q(t)$ and doing the scalar product, we can write

$$(Q(t)u'(t), u(t)) = (Q(t)A(t)u(t), u(t)).$$

Since

$$\frac{d}{dt}(Q(t)u(t), u(t)) = (Q(t)u'(t), u(t))$$

$$+(u(t), Q(t)u'(t)) + (Q'(t)u(t), u(t)),$$

it can be written

$$\frac{d}{dt}(Q(t)u(t), u(t)) = (Q(t)A(t)u(t), u(t))$$

$$+(u(t), Q(t)A(t)u(t)) + (Q'(t)u(t), u(t))$$

$$= ((Q(t)A(t) + A^*(t)Q(t))u(t), u(t))$$

$$+(Q'(t)u(t), u(t)) = -2(u(t), u(t))$$

$$+(Q'(t)u(t), u(t)).$$

Hence, condition (2.8) implies

$$\frac{d}{dt}(Q(t)u(t), u(t)) \leq (-2+\|Q'(t)\|)(u(t), u(t)) < 0.$$

This proves the result. Q.E.D.

Furthermore, for a stable operator A_0 put $y_1(t) = e^{A_0 t}v$ ($v \in H$). Then $\dot{y}_1(t) = A_0 y_1$, and

$$\frac{d(y_1(t), y_1(t))}{dt} = ((A_0 + A_0^*)y_1(t), y_1(t)). \tag{2.9}$$

Hence

$$\frac{d(y_1(t), y_1(t))}{dt} \geq \lambda(A_0 + A_0^*)(y_1(t), y_1(t))$$

and therefore

$$\|e^{A_0 t}v\|^2 \geq e^{t\lambda(A_0+A_0^*)}\|v\|^2,$$

where $\lambda(A_0+A_0^*)$ is the smallest eigenvalues of $A_0+A_0^*$. Recall that A_0 is stable, so $\lambda(A_0 + A_0^*) < 0$. Put

$$Q_0 = 2\int_0^\infty e^{A_0^* s}e^{A_0 s}ds.$$

Then due to (2.9)

$$(Q_0 h, h) = 2\int_0^\infty (e^{A_0^* s}e^{A_0 s}h, h)ds$$

$$\geq 2\int_0^\infty e^{\lambda(A_0+A_0^*)s}ds\ \|h\|^2 = 2\|h\|^2|\lambda(A_0+A_0^*)|^{-1}$$

($h \in H$). Hence, for any continuous function $u_1 : [0,\infty) \to H$ we have

$$(Q(t)u_1(t), u_1(t)) \geq 2\|u_1(t)\|^2|\lambda(A(t) + A^*(t))|^{-1}.$$

Now the previous lemma implies.

$$(u(t), u(t)) \leq |\lambda(A(t) + A^*(t))|(Q(0)u(0), u(0)).$$

But $|\lambda(A(t)+A^*(t))|$ is uniformly bounded and therefore all the solutions of (2.1) are uniformly bounded (i.e. (2.1) is Lyapunov stable). Furthermore, substitute into (2.1)

$$u(t) = u_\epsilon(t)e^{-\epsilon t} \quad (\epsilon > 0). \tag{2.10}$$

Then

$$\dot{u}_\epsilon(t) = (A(t) + \epsilon I)u_\epsilon(t). \tag{2.11}$$

Applying our above arguments to (2.11) can assert that equation (2.10) with small enough $\epsilon > 0$ is Lyapunov stable. So due to (2.10) equation (2.1) is exponentially stable, provided (2.8) holds. Now Lemma 3 implies

Lemma 5 *Let*

$$\sup_{t\geq 0} q^2(t)\|A'(t)\| < 2.$$

Then (2.1) is exponentially stable.

The assertion of Theorem 1 follows from Lemmas 2 and 5, and the equality $A'(t) = B'(t)$. Q.E.D.

3 A particular case

Let $\mathbf{C}^n$ be the complex n-dimensional Euclidean space with a scalar product $(.,.)_n$, the Euclidean norm $\|.\|_n = \sqrt{(.,.)_n}$ and the unit matrix I_n. For $n \times n$-matrix, $\|A\|_n = \sup_{x\in\mathbf{C}^n} \|Ax\|_n/\|x\|_n$ is the spectral (operator) norm, A^* is the adjoint operator, $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A: $N_2(A) = \sqrt{trace\ AA^*}$; $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues with their multiplicities. So $\alpha(A) = \max_k\ Re\ \lambda_k(A)$.

Furthermore, let Ω be a bounded domain of the real Euclidean space with a finite Lebesgues measure. In this section $H = L^2(\Omega, \mathbf{C}^n)$ is a Hilbert space of functions f, h defined on Ω with values in $\mathbf{C}^n$ and the scalar product

$$(f,h)_{L^2} = \int_\Omega (f(x), h(x))_n dx.$$

Consider the equation

$$\dot u(t,x) = S(t)u(t,x) + b(t)u(t,x) \quad (x \in \Omega, t \geq 0), \tag{3.1}$$

where $b(t)$ is a differentiable in t matrix independent of x, and $S(t)$ is a linear operator in $L^2(\Omega, \mathbf{C}^n)$, commuting with $b(t)$ and satisfying (1.2).

Introduce the quantity

$$g(A) = (N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2)^{1/2}$$

for an $n \times n$-matrix A plays an essential role hereafter. The following relations are checked in [6, Section 1.5]: $g^2(A) \leq N_2^2(A) - |Trace\ A^2|$,

$$g(A) \leq \frac{1}{\sqrt{2}} N_2(A - A^*) \tag{3.2a}$$

and

$$g(e^{i\tau}A + zI_n) = g(A) \quad (\tau \in \mathbf{R}, z \in \mathbf{C}); \tag{3.2b}$$

if A is a normal matrix: $A^*A = AA^*$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \leq g(A_1) + g(A_2)$. In addition, by the inequality between the geometric and arithmetic mean values,

$$(\frac{1}{n}\sum_{k=1}^n |\lambda_k(A)|^2)^n \geq (\prod_{k=1}^n |\lambda_k(A)|)^2.$$

Hence $g^2(A) \leq N_2^2(A) - n|det\ A|^{2/n}$. For a constant Hurwitz matrix A_0, due to [6, Lemma 1.9.2],

$$2\int_0^\infty \|e^{A_0 s}\|_n^2 ds \leq \sum_{j,k=0}^{n-1} \frac{(k+j)!g^{k+j}(A_0)}{2^{k+j}|\alpha(A_0)|^{k+j+1}(k!j!)^{3/2}}. \tag{3.3}$$

Define $B(t)$ by the multiplication by matrix $b(t)$ and take into account that

$$\|e^{B(t)s}\|_{L^2} \leq \|e^{b(t)s}\|_n \quad (t, s \geq 0).$$

In addition, from (3.2b) it follows that $g(b(t) + \gamma(S)I_n) = g(b(t))$. Assume that $b(t) + \gamma(S)I_n$ is Hurwitzian. Then (3.3) implies

$$q(t) = 2\int_0^\infty \|e^{(b(t)+\gamma(S)I_n)s}\|_n^2 ds \leq \mu(t) \quad (t \geq 0), \tag{3.4}$$

where

$$\mu(t) := \sum_{j,k=0}^{n-1} \frac{(k+j)!g^{k+j}(b(t))}{2^{k+j}|\alpha(b(t)) + \gamma(S)|^{k+j+1}(k!j!)^{3/2}}.$$

Now Theorem 1 yields

Corollary 6 *Let the conditions (1.2) , $\alpha(b(t)) + \gamma(S) < 0$ $(t \geq 0)$ and*

$$\sup_{t \geq 0} \mu^2(t)\|b'(t)\| < 2 \tag{3.5}$$

hold. Then equation (3.1) is exponentially stable.

4 Example

Consider the problem

$$\dot u(t,x) = \frac{\partial}{\partial x} a(t,x) \frac{\partial u(t,x)}{\partial x} + b(t)u(t,x)$$
$$(t > 0;\ 0 < x < 1), \tag{4.1}$$

with the boundary condition

$$u(t,0) = u(t,1) = 0, \tag{4.2}$$

where $b(t) = (b_{jk}(t))$ is a real differentiable $2 \times 2-$ matrix independent of x, $a(t,x)$ is a positive scalar function, differentiable in x and continuous in t.

Take $H = L^2([0,1], \mathbf{C}^2)$. In the considered case the operator $S(t) = \frac{d}{dx} a(t,x) \frac{d}{dx}$ with the domain

$$D_0 = \{h \in L^2([0,1], \mathbf{C}^2) : h'' \in L^2([0,1], \mathbf{C}^2);$$
$$h(0) = u(1) = 0\}$$

is selfadjoint. Besides,

$$-(S(t)h, h) = -(a(t,x)h', h') \geq -a_0(t)(h', h'),$$

where $a_0(t) = \inf_x a(t,x) > 0$. Simple calculations show that $\Lambda(S(t)) = -\pi^2 a_0(t)$ and therefore,

$$\gamma(S) = -\pi^2 \inf_{t \geq 0} \frac{1}{t} \int_0^t a_0(s) ds. \tag{4.3}$$

Since $n = 2$, the eigenvalues of $b(t)$ are simply calculated. In addition, due to (3.2a) $g(b(t)) \leq |b_{12}(t) - b_{21}(t)|$, and

$$\mu(t) := \frac{1}{\rho(t)} + \frac{g(b(t))}{\rho^2(t)} + \frac{g^2(b(t))}{2\rho^3(t)} \tag{4.4}$$

with $\rho(t) = |\alpha(b(t)) + \gamma(S)|$, provided $\alpha(b(t)) + \gamma(S) < 0$, that is, $b(t) + \gamma(S)I_2$ is a Hurwitz matrix. Now we can directly apply Corollary 6.

5 Conclusion

We have established the exponential stability conditions for equation (1.1). Besides, we do not require that $S(t) + B(t)$ is dissipative. As the example shows, our test can be effectively applied, provided the norm of the derivative of operator $B(t)$ is sufficiently small.

References:

[1] Chicone, C. and Yu. Latushkin, *Evolution Semigrous in Dynamical Systems and Differential Equations*, Amer. Math. Soc., Math. Surv. and Monographs, v. 70, 1999.

[2] R. Curtain, and H. Zwart, *Introduction to Infinite-Dimensional Systems Theory*, Springer, N.Y. 1995.

[3] Daleckii, Yu L. and Krein, M. G., *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R.I. 1974.

[4] Fattorini, H.O. *Infinite Dimensional Linear Control Systems: The Time Optimal and Norm Optimal Problems*, North-Holland Math. Stud., vol. 201, Elsevier Science B.V., Amsterdam, 2005.

[5] Gil', M.I. Freezing method for evolution equations. *Communications in Applied Analysis, 1*, (1996) No. 2, 245-256.

[6] Gil', M. I. *Explicit Stability Conditions for Continuous Systems*, Lectures Notes In Control and Information Sciences, Vol. 314, Springer Verlag, Berlin, 2005.

[7] Gil', M.I. Integrally small perturbations of semigroups and stability of partial differential equations, *International Journal of Partial Differential Equations, Vol. 2013* (2013), Article ID 207581, 5 pages

[8] Gil', M. I. Stability of linear nonautonomous multivariable systems with differentiable matrices, *Systems & Control Letters, Vol. 81*, (2015), 31-33

[9] Oostveen, J., *Strongly Stabilizable Distributed Parameter Systems*, SIAM, Frontiers in Applied Mathematics, Philadelphia, 2000.