A Note on Symplectic J-SVD Like Decoposition

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Abstract: This paper presents a symplectic J-SVD like decomposition of $2n$-by-$2m$ rectangular real matrix based on symplectic reflectors The idea for this approach was to use symplectic reflector to firs reduce the matrix to J-bidiagonal form and then transform it to a diagonal form by using sequence of symplectic similarity transformations. This was done in parallel with the Golub-Kahan-Reinsch method. This method allowed us to compute eigenvalues for the skew-Hamiltonian matrix $A^J A$.

Key–Words: Singular value decomposition (SVD), Hamiltonian matrix, Skew-Hamiltonian matrix, Symplectic matrix, Symplectic reflector

1 Introduction

Singular Value Decomposition has been used in many field of scientifi computing such as data compression, signal processing, automatic control working on applied linear algebra, signal and image processing[14, 15]. This paper makes the main contribution to this area of research. Which is computation of a J-SVD like decomposition by applying symplectic reflector to columns and rows to obtain a J-bidiagonal matrix. By the use of sequences of symplectic reflectors we transform a J-bidiagonal matrix to a diagonal matrix, in parallel with Golub-Kahan-Reinsch method [9, 10]. This approach allowed us to compute eigenvalues for structured matrices such as the Hamiltonian matrix JA^TA and the skew-Hamiltonian matrix A^JA. Most eigenvalue problems that arise in practice are known to be structured. Therefore, preserving the structure can help preserve physically relevant symmetries in the eigenvalues of the matrix and may improve the accuracy and efficien y of eigenvalue computation. Hamiltonian and skew-Hamiltonian eigenvalue problems arise from a number of applications, particularly in systems and control theory [8, 13, 16].

The paper is organized as follows: section 2 introduce some notation and some basic result; a symplectic J-SVD like decomposition method is proposed in section 3; and in section 4, numerical results are given to demonstrate the effectiveness of the proposed algorithms.

2 Terminology, notation and some basic facts

An ubiquitous matrix in this work is the skew-symmetric matrix $J_{2n} = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$, where I_n and O_n are the $n \times n$ identity and zero matrix respectively. Note that $J_{2n}^{-1} = J_{2n}^T = -J_{2n}$. In the following, we omit the subscript n and $2n$ whenever the dimension of corresponding matrix is clear from its context. The J-transpose of any $2n$-by-$2p$ matrix M is define by $M^J = J_{2p}^T M^T J_{2n} \in \mathbb{R}^{2p\times 2n}$. Hamiltonian matrix $M \in \mathbb{R}^{2n\times 2n}$ has the explicit block structure $M = \begin{pmatrix} A & R \\ G & -A^T \end{pmatrix}$, where A, G, R are real $n \times n$ matrices and $G = G^T$, $R = R^T$. By straightforward algebraic manipulation, we can show that a Hamiltonian matrix M is equivalently define by the property $M^J = -M$. Likewise, a matrix M is skew-Hamiltonian if and only if $M^J = -M$, it has the explicit block structure $W = \begin{pmatrix} A & R \\ G & A^T \end{pmatrix}$, where A, G, R are real $n\times n$ matrices and $G = -G^T$, $R = -R^T$. Any matrix $S \in \mathbb{R}^{2n\times 2p}$ that satisfie this property $S^T J_{2n} S = J_{2p}$ ($S^J S = I_{2p}$) is called symplectic matrix. This property is also called J-orthogonality. The symplectic similarity transformations preserve Hamiltonian and skew-Hamiltonian structures.

Remark 1. *An augmented matrix*

$$S = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & P_{11} & 0 & P_{12} \\ 0 & 0 & I & 0 \\ 0 & P_{21} & 0 & P_{22} \end{pmatrix}$$

is symplectic if and only if $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ *is also symplectic too.*

We obtained some useful results with this matrix. Setting $E_i = [e_i \; e_{n+i}] \in \mathbb{R}^{2n\times 2}$ for $i = 1, \cdots, n$, we obtain

$$E_i^J = E_i^T \text{ and } E_i^J E_j = \delta_{ij} I_2 \text{ where}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proposition 2. *Let* $U = [u_1 \;\; u_2]$ *be a* $2n$*-by-*2 *real matrix, where* $u_1 = \sum_{i=1}^{2n} u_i^{(1)} e_i$ *and* $u_2 = \sum_{j=1}^{2n} u_j^{(2)} e_j$. *Then,* U *is written uniquely as linear combination of* $(E_i)_{1\le i\le n}$ *on the ring* $\mathbb{R}^{2\times 2}$.

$$U = \sum_{i=1}^{n} E_i M_i \text{ where } M_i = \begin{pmatrix} u_i^{(1)} & u_i^{(2)} \\ u_{n+i}^{(1)} & u_{n+i}^{(2)} \end{pmatrix}$$

Proposition 3. *Let* M *be a* $2n$*-by-*$2n$ *real matrix. Then,* M *is expressed uniquely as* $M = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ij} E_j^T$ *where* $M_{ij} \in \mathbb{R}^{2s\times 2s}$ *is given by,*

$$\left(\begin{array}{c|c} m_{i,j} & m_{i,n+j} \\ \hline m_{n+i,j} & m_{n+i,n+j} \end{array}\right)$$

Proposition 4. *With the notations of the previous proposition, a matrix* $M \in \mathbb{R}^{2n\times 2n}$ *is Hamiltonian (or skew-Hamiltonian) if* $M_{ij}^J = -M_{ji}$ *(or* $M_{ij}^J = M_{ji}$*).*

Proof. The result is obvious, as $M^J = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ji}^J E_j^T$ and by definitio $M^J = -M$. □

Definition 5. *A matrix* $M = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ij} E_j^T \in \mathbb{R}^{2n\times 2n}$ *is called in upper* J*-bidiagonal form if* $M_{ij} = 0_2$ *for* $j \notin \{i, i+1\}$ *and, in addition,* M_{ii} *and* M_{ii+1} *are diagonal.*

2.1 Symplectic reflectors

The symplectic reflecto [2, 3] in $\mathbb{R}^{2n\times 2}$ is define in parallel with elementary reflectors

Proposition 6. *[3] Let* U *and* V *be two* $2n$*-by-*2 *real matrices that satisfy* $U^J U = V^J V = I_2$. *If the* 2*-by-*2 *matrix* $C = I_2 + V^J U$ *is nonsingular, the transformation* $S = (U+V)C^{-1}(U+V)^J - I_{2n}$ *is symplectic and takes* U *to* V. *This is called a symplectic reflector. Additionally, if* $U^J = U^T$ *and* $V^J = V^T$, *then* S *is orthogonal and symplectic.*

Remark 7. *The proposition above remains true only if* $U^J U = V^J V$ *. In this case,* $C = U^J U + V^J U$.

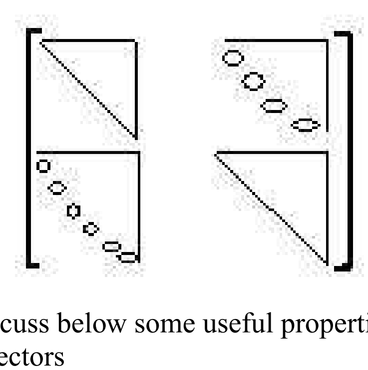
Lemma 8. *Let* $U = [u_1 \;\; u_2] \in \mathbb{R}^{2n\times 2}$ *be a non-isotropic matrix (*$U^J U \neq 0_2$*) and* $V = Uq(U)^{-1}$ *its normalized matrix. Then, there is a symplectic reflector* S *takes* V *to* E_1 *and therefore* U *to* $E_1 q(U)$, *which in turn takes the following form:*

$$SU = \begin{array}{c} \begin{pmatrix} * & \mathbf{0} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \mathbf{0} & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \swarrow n+1 \\ \\ \\ \\ \end{array}$$

where

$$q(U) = \begin{cases} \sqrt{\alpha} I_2 & \text{if } \alpha > 0 \\ \sqrt{-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \alpha < 0 \end{cases}$$
$$\alpha = u_1^H J u_2.$$

Remark 9. *Using symplectic reflectors with a matrix* $A \in \mathbb{R}^{2n\times 2n}$, *we obtain the factorization* $A = SR$, *where* $S \in \mathbb{R}^{2n\times 2n}$ *is symplectic and* $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$. R *is* J*-triangular and, in addition,* R_{12} *is a strictly* n*-by-*n *upper triangular matrix.* R *is as follows:*



We discuss below some useful properties of symplectic reflectors

Proposition 10. *Let* S *be a* $2n$*-by-*$2n$ *real symplectic matrix. There is then a sequence of symplectic reflectors* $S_1, S_2, \cdots, S_n$, *such that* $S = S_1 S_2 \cdots S_n$.

Proof. **Step 1:**

Set $U_1 = [q_1, q_{n+1}] \in \mathbb{R}^{2n\times 2}$. As S is symplectic, then $U_1^J U_1 = I_2$. Then, the symplectic reflecto $P_1 = (U_1 + E_1)(I_2 + E_1^J U_1)^{-1}(U_1 + E_1)^J - I_{2n}$ verifie $P_1 U_1 = E_1$. The $(n+1)^{th}$-component of both $(P_1 q_k)$ and $(P_1 q_{n+k})$ is equal to zero for $k = 2, 3, \dots n$. On the one hand, $(P_1 q_1)^T J (P_1 q_k) = q_1^T J q_k = 0$, and on the other hand, $(P_1 q_1)^T J (P_1 q_k) = e_1^T J (P_1 q_k) = e_{n+1}^T (P_1 q_k)$ is simply the $(n+1)^{th}$-component of $(P_1 q_k)$. Likewise, the firs component of both $(P_1 q_k)$ and $(P_1 q_{n+k})$ disppears. Finally, we obtain

$$P_1 S = \begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{1\;0\;\cdots\;0}}^{n} & \overbrace{\hphantom{0\;0\;\cdots\;0}}^{n} \end{array} \\ \begin{array}{c} n \\ \\ \\ n \end{array} & \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * \end{array}\right) \end{array}$$

Thereafter, we continue to update the value of q_i: $q_i \longleftarrow P_1 q_i$ by varying i from 1 to $2n$. Note that now we have $q_1 = e_1$ and $q_{n+1} = e_{n+1}$.

Step 2:

Set $U_1 = [q_1, q_{n+1}] \in \mathbb{R}^{2n\times 2}$. As S is symplectic, then $U_1^J U_1 = I_2$ and the symplectic reflecto allows us to set $U_2 = [q_2, q_{n+2}] \in \mathbb{R}^{2n\times 2}$. As $P_1 S$ is still symplectic, U_2 verifie $U_2^J U_2 = I_2$, and the symplectic reflecto $P_2 = (U_2 + E_2)(I_2 + E_2^J U_2)^{-1}(U_2 + E_2)^J - I_{2n}$ has the following form:

$$P_2 = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & * & * & \vdots & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ \hline 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & * & * & * & 0 & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{array}\right)$$

and verifie $P_2 U_2 = E_2$. As in step 1, we obtain

$$P_2 P_1 S = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & * & 0 & 0 & * & \dots & * \\ \hline 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \end{array}\right)$$

We thereby obtain $P_n \cdots P_2 P_1 S = I_{2n}$, and then $S = S_1 S_2 \cdots S_n$ where $S_k = P_k^J$, which achieves the desired result. □

Remark 11. *In lemma* 2.1*, by using* $U = [u \ \ -Ju]$*, where* $u \in \mathbb{R}^{2n}$ *with* $\|u\| \neq 0$*, we obtain* S *that orthogonal and symplectic.*

Lemma 12. *Let* $u \in \mathbb{R}^{2s}$ *be a nonzero* $2s$*-component real vector. The orthogonal symplectic reflector* $S = (U + \sqrt{\alpha}E_1)(\alpha I_2 + \sqrt{\alpha}E_1^J U)^{-1}(U + \sqrt{\alpha}E_1)^J - I_{2s}$*, where* $U = [u \ \ -Ju]$ *verifies* $Su = \sqrt{\alpha}e_1$ *with* $\alpha = u^T u = \|u\|_2^2$*.*

Proof. As $U^J U = \alpha I_2$ with $\alpha = u^T u = \|u\|_2^2 > 0$, then a simple calculation gives the result. □

2.2 Symplectic Givens rotations

In the following, we defin the rotation on $\mathbb{R}^{2n\times 2}$ seen as a free $\mathbb{K}$-module structure on $\mathbb{K} = \mathbb{R}^{2\times 2}$. For more information on symplectic rotations see [7].

Definition 13. *A rotation in the* (E_i, E_j) *plane is defined by* $R(i,j,C,S) = I_{2n} + E_i\,(C - I_2)\,E_i^T + E_i S E_j^T - E_j S^J E_i^T + E_j\left(C^J - I_2\right)E_j^T$ *where* C *and* S *are* 2*−by−*2 *matrix (recall that* $E_i = [e_i \ e_{n+i}] \in \mathbb{R}^{2n\times 2}$*).*

Let us now examine the condition in which the rotation $R(i,j,C,S)$ is symplecic.

Proposition 14. *[7] The rotation* $R(i,j,C,S)$ *in the* (E_i, E_j) *plane is symplectic if and only if* $det(S) + det(C) = 1$ *and* $CS = SC$*. It is also orthogonal if* S *and* C *are in the following form* $S = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$*,* $C = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$*.*

Let $U \in R^{2n\times 2}$, such that $U = \sum_{i=1}^{n} E_i M_i$. If $M_1 M_2 = M_2 M_1$, then, by taking $S = \frac{1}{\sqrt{\alpha}} M_2^J$, $C = \frac{1}{\sqrt{\alpha}} M_1^J$ and $P = R(1, 2, C, S)$(where $\alpha = det(M_1) + det(M_2) > 0$), the 2-by-2 second component is zero and $W = PU$ is in following form:

$$W = \begin{array}{c} \begin{pmatrix} \sqrt{\alpha} & 0 \\ \mathbf{0} & \mathbf{0} \\ * & * \\ \vdots & \vdots \\ * & * \\ 0 & \sqrt{\alpha} \\ \mathbf{0} & \mathbf{0} \\ * & * \\ \vdots & \vdots \\ * & * \end{pmatrix} \end{array} \begin{array}{l} \\ \longleftarrow 2 \\ \\ \\ \\ \\ \longleftarrow n+2 \\ \\ \\ \\ \end{array}$$

Remark 15. *The symplectic rotation defined above is simply a symplectic reflector* $S = (U + E_i)\left(I_2 + E_i^J U\right)^{-1}(U + E_i)^J - I_{2n}$ *and takes* U *to* V

3 The J-SVD decomposition

Golub, Kahan and Reinsch [9, 10] presented an effective, widely used method to fin the SVD of an arbitrary rectangular real matrix A. The method is based on computing a bidiagonal matrix for two unitary matrices constructed from the product of a sequence of Householder transformations. The second phase consists in transforming the obtained bidiagonal matrix to a diagonal one by a variant of the QR iteration. Our purpose was to describe J-SVD decomposition of a $2n$-by-$2m$ rectangular matrix A on the basis of J-bidiagonalization with symplectic reflectors The proposed method is define in parallel with the Golub-Kahan-Reinsch approach. It allows us to compute the eigenvalues of skew-Hamiltonian matrix $A^J A$ without computing the product of the full matrix. We obtained the following result for $rank(A^T JA) = rank(A)$,

$$PAQ = \left(\begin{array}{cc|cc} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \Sigma_p & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

where P, Q are symplectic matrices and $\Sigma_p = \sigma_1, \cdots, \sigma_p$, $2p = rank(A)$.

3.1 The J-Bidiagonalization method

We present here two approaches for computing the J-bidiagonal form of $2n$-by-$2m$ rectangular real matrix. The firs uses a sequence of symplectic reflector applied alternately from the left and the right to the zero parts of the matrix. The second is based on an symplectic Lanczos J-bidiagonalization.

3.1.1 First approach

Let A be a $2n$-by-$2m$ rectangular real matrix. For the algorithm, we used symplectic reflector to compute a J-bidiagonal form B, such that $A = PBQ$, where $P \in \mathbb{R}^{2n\times 2n}$, $Q \in \mathbb{R}^{2m\times 2m}$ are symplectic matrices. We illustrate the method for $n = 4$, $m = 3$ as follows:

$$\begin{matrix}\begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} \\ \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix}\end{matrix} \xrightarrow[P_1 \text{ applied from the left}]{\substack{\text{the fir t step} \\ \text{is to zero} \\ \text{the}(2{:}8{,}1),\ (1{:}4{,}5) \\ \text{and } (6{:}8{,}5) \text{ positions}}} \begin{matrix}\begin{bmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} \\ \begin{bmatrix} 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix}\end{matrix}$$

$$\begin{matrix}\begin{bmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} \\ \begin{bmatrix} 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix}\end{matrix} \xrightarrow[Q_1 \text{applied from the right}]{\substack{\text{the second step is} \\ \text{to zero the } (1{,}3) \\ (1{,}5{:}6),\ (5{,}2{:}3) \\ \text{and } (5{,}6) \text{ positions}}} \begin{matrix}\begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix}\end{matrix}$$

$$\begin{matrix}\begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix}\end{matrix} \xrightarrow[P_2 \text{ applied from the left}]{\substack{\text{the third step} \\ \text{is to zero the } (3{:}4{,}2), \\ (6{:}8{,}2) \\ \text{positions}}} \begin{matrix}\begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & 0 & \ast \\ 0 & 0 & \ast \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ast \\ 0 & 0 & \ast \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ast \\ 0 & 0 & \ast \\ 0 & 0 & \ast \end{bmatrix} & \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & 0 & \ast \\ 0 & 0 & \ast \end{bmatrix}\end{matrix}.$$

The last step is to zero the $(4,3)$, $(5:8,3)$, $(2:3,6)$, $(8,6)$ positions, applying the symplectic reflecto P_3 from the left to obtain the desired J-bidiagonal form:

$$\begin{pmatrix}\begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & 0 & \ast \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ 0 & 0 & \ast \\ 0 & 0 & 0 \end{bmatrix}\end{pmatrix}.$$

Algorithm 3.1: J-Bidiagonalization Algorithm

Input : Matrix $A \in \mathbb{R}^{2n\times 2m}$
Output: Symplectic matrix $P \in \mathbb{R}^{2n\times 2n}$ and symplectic matrix $Q \in \mathbb{R}^{2m\times 2m}$ and the J-Bidiagonal matrix B so that $PAQ = B$.
P and Q are products of symplectic reflector
1. For $k = 1, 2, \cdots, m$
• Set $U_k = [u\ \ v] = AE_k$ where $E_k = [e_k\ e_{n+k}]$
• **For** $i = 1, \cdots, k-1$
$$\begin{cases} u\,(i) \longleftarrow 0 & , \quad u\,(n+i) \longleftarrow 0 \\ v\,(i) \longleftarrow 0 & , \quad v\,(n+i) \longleftarrow 0 \end{cases}$$
• **EndFor**
• Compute the symplectic reflecto P_k associated to U_k and
Update $A \longleftarrow P_k A$
• Set $V_k = [u\ \ v] = AE_k^T$
• **For** $i = 1, \cdots, k-1$
$$\begin{cases} u\,(i) \longleftarrow 0 & , \quad u\,(n+i) \longleftarrow 0 \\ v\,(i) \longleftarrow 0 & , \quad v\,(n+i) \longleftarrow 0 \end{cases}$$
• **EndFor**
• Compute the symplectic reflecto Q_k associated to V_k and
Update $A \longleftarrow AQ_k^T$
2. EndFor k
3. $B \longleftarrow A$.

3.1.2 Second approach

The Lanczos bidiagonalization technique can be approached from several equivalent perspectives. We started by setting up the notation. Consider , $A = P^J BQ$ where $P = [p_1, p_2, \cdots, p_{2n}]$, $Q = [q_1, q_2, \cdots, q_{2n}]$ are symplectic matrices, and B is J-bidiagonal matrix, as follows:

$$B = P^J AQ = \left(\begin{array}{cccc|cccc} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \beta_{m-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \gamma_1 & \delta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_2 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \delta_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Using the above result, we constructed the following algorithm to obtain the J-bidiagonal form and symplectic matrices P and Q.

Algorithm 3.2: Symplectic Lanczos $J-$Bidiagonalization

Input: Matrix $A \in \mathbb{R}^{2n\times 2m}$ $(n \geq m)$ and a symplectic matrix $V_1 = [\mathbf{q}_1\ \mathbf{q}_{m+1}] \in \mathbb{R}^{2m\times 2}$
Output: Symplectic matrix $P \in \mathbb{R}^{2n\times 2n}$ and symplectic matrix $Q \in \mathbb{R}^{2m\times 2m}$ such that $P^J AQ$ is $J-$bidiagonalization.
1. Set $C_0 = 0_{2\times 2}$ and $U_0 = 0_{2m\times 2}$
2. For $i = 1, 2, \cdots, m$
- $W = AV_i - U_{i-1}C_{i-1}$
$(AV_i = U_{i-1}C_{i-1} + U_iN_i)$
- Compute a diagonal 2-by-2 real matrix N_i such that $N_i^J N_i = W^J W$
(see, $J-$Normalization above)
- Set $\alpha_i = N_i(1,1)$ and $\gamma_i = N_i(2,2)$ and $U_i = [\mathbf{p}_i\ \mathbf{p}_{n+i}] = WN_i^{-1}$
- $W = A^J U_i - V_i N_i^J$
$\left(A^J U_i = V_i N_i^J + V_{i+1}C_i^J\right)$
- Using $J-$Normalization above, compute a diagonal 2-by-2 real matrix C_i such that $C_i^J C_i = W^J W$
-Set $\beta_i = C_i(1,1)$ and $\delta_i = C_i(2,2)$ and $V_{i+1} = [\mathbf{q}_{i+1}\ \mathbf{q}_{m+i+1}] = WC_i^{-J}$
3. End
4.$B = P^J AQ =$

$$\left(\begin{array}{cccc|cccc}
\alpha_1 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & \beta_{m-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \gamma_1 & \delta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_2 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & \delta_{m-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_m \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

is the desired form.

3.2 The J-SVD decomposition via a symplectic Golub-Kahan-Reinsch method

This method consists of two phases. In the firs phase, finit sequences of symplectic reflector are constructed as described above to obtain the desired J-bidiagonal matrix (see algorithm 4.1). $B = PAQ^J$ is J-bidiagonal where P and Q are symplectic matrices. They can also be obtained by symplectic Lanczos $J-$bidiagonalization (see algorithm 4.2). The second phase consists to iterative diagonalization of J-bidiagonal matrix B by a symplectic QR-like method using the symplectic Givens rotations described in paragraph 2.2.

$$B = B^{(0)} \longrightarrow B^{(1)} = U^{(1)}B^{(0)}V^{(1)} \cdots \longrightarrow$$

$$B^{(k)} = U^{(k)}B^{(k-1)}V^{(k)} \cdots \longrightarrow \widetilde{\Sigma} = \left(\begin{array}{cc|cc}
\Sigma_p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & \Sigma_p & 0 \\
0 & 0 & 0 & 0
\end{array}\right)$$

where $U^{(k)}$, $V^{(k)}$ are product of symplectic Givens rotations and $\Sigma_p =$**diag**$(\sigma_1, \cdots, \sigma_p)$, $2p = rank(A)$ and $(\sigma_i > 0)_{1\leq i\leq p}$.

Numerical examples

We report here the results of numerical tests in which we compared our method for computing the eigenvalues of skew-Hamiltonian matrix $A^J A$ with the Matlab method. We calculated the error in J-SVD decomposition for rectangular matrix A and the relative errors of computed eigenvalues of $A^J A$.

Example:

Let A be a rectangular matrix of order 16×12 define as follows:

$$A = P\left(\begin{array}{ccc|ccc}
\Sigma & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \Sigma & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) Q^T$$

where P is a 16×16 random orthogonal symplectic matrix, and Q is a 12×12 random orthogonal symplectic matrix.

• $\Sigma = diag(9, 8, 5, 4, 2, 1,)$, the error for J-SVD decomposition was $3.8352e - 015$. The relative errors by our method and that of the Matlab method for nonzero eigenvalues are shown in the table below:

eigenvalue	J-SVD	Matlab 7.8.0
± 81	$2.2808e-015$	$5.2633e-015$
± 64	$5.1070e-015$	$9.9920e-015$
± 25	0	$5.6843e-016$
± 16	$1.3323e-015$	$5.2458e-015$
± 4	0	$2.9976e-015$
± 1	$6.6613e-016$	$4.7740e-015$

• $\Sigma = diag\left(10^2, 10, 1, 10^{-1}, 10^{-2}, 10^{-4}\right)$, the error for J-SVD decomposition was $8.5520e-008$. The relative errors by our method and that of the Matlab method for nonzero eigenvalues are shown in the table below:

eigenvalue	J-SVD	Matlab 7.8.0
$\pm 10^4$	$3.6380e-016$	$8.4254e-016$
$\pm 10^2$	$7.1054e-016$	$4.2633e-016$
± 1	$1.7764e-015$	$5.8842e-015$
$\pm 10^{-2}$	$2.7756e-015$	$5.1963e-011$
$\pm 10^{-4}$	$4.0251e-014$	$1.3693e-009$
$\pm 10^{-8}$	$1.1465e-011$	$2.4381e-006$

• In this case a matrix A is of order 24×20 and $\Sigma = diag\left(10^3, 10^2, 10, 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\right)$, the error for J-SVD decomposition was $4.6566e-016$. The relative errors by our method and that of the Matlab method for nonzero eigenvalues are shown in the table below:

eigenvalue	J-SVD	Matlab 7.8.0
$\pm 10^6$	0	$1.8190e-016$
$\pm 10^4$	$2.8422e-016$	$1.4381e-013$
$\pm 10^2$	$6.6613e-016$	$1.6211e-012$
± 1	$2.2551e-014$	$1.4152e-009$
$\pm 10^{-2}$	$5.8005e-014$	$1.5873e-008$
$\pm 10^{-4}$	$2.4882e-013$	$9.0538e-006$
$\pm 10^{-6}$	$2.5295e-010$	0.0012
$\pm 10^{-8}$	$1.8854e-009$	0.3243
$\pm 10^{-10}$	$9.4335e-010$	22.3130
$\pm 10^{-12}$	$4.6566e-008$	$1.0000e+05$

4 Conclusion

We have presented a numerical method for computing symplectic J-SVD like decomposition. This method was inspired by the Golub-Kahan-Reinsch method. Our approach here was based on the use of symplectic reflectors The structured matrices such as skew-symmetric matrix AJA^T, the Hamiltonian matrix JA^TA and the skew-Hamiltonian matrix A^JA can be derived from such a decomposition. The numerical examples presented show the effectiveness of proposed algorithm.

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