Optimal Control for Systems Described by Semi-linear Parabolic Equations

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Abstract: We study the well-posedness of an optimal control problem described by semi-linear parabolic equation. The control functions are represented by the coefficients $\lambda(u, v)$ and $\beta(u, v)$ which appear in the nonlinear part of the state problem and inside the source strength, respectively. These coefficients depend on the control function v. Then, we obtain some necessary optimality conditions for this problem.

Key-Words: Optimal control, Quasi-linear parabolic equation, Existence, Uniqueness theorems and necessary optimality conditions.

1 Introduction

Optimal control problems for systems with distributed parameters are often encountered in various applications. These problems for parabolic equations are of great practical importance, which occur in optimization problems of thermal and plasma physics, diffusion, filtering etc., and also in solving coefficient-wise inverse problems for parabolic equations in variational formulations [6].

The precise mathematical formulations of these problems depend, in general, on where the control functions occur [1]. The problems can be divided into two groups. The first group includes problems where the control functions occur in free coefficients of the state equations of boundary conditions. Currently, these problems have received most attention. The second group contains problems where the control functions occur in the state equation coefficients, including coefficients of higher order derivatives. These problems have been studied as a little.

In the present paper, we study a problem where the control functions are represented by the coefficients $\lambda(u, v)$ and $\beta(u(x, t; v), v)$ which appear in the nonlinear part of the state problem and inside the source strength, respectively. These coefficients depend on the control function v. This will help us to solve a large amount of problems in this field of the optimal control problems.

Abdelhamid, et. al. [16, 17, 18, 19, 20, 21] have computed the gradient formulas in the optimizations problems for estimating the unknown parameters. Furthermore, the authors studied the differentiability results for the objective functions. In these problems, we assumed that the coefficients of the control and their generalized first order derivatives are essentially bounded functions. Also, the well-posedness of the problem, the existence and uniqueness are investigated. Finally, we prove the differentiability of the objective functional to obtain a formula for its gradient, and establish the necessary optimality conditions.

2 Mathematical formulation

Let $\Omega = (0, l)$ be a bounded domain of E_n, $Q_T = \{(x, t) : x \in \Omega, t \in (0, T)\}$, and $V = \{v : v = (v_1, v_2, \dots, v_n) \in l_2, \|v\|_{l_2} \leq R\}$, where R and T are a fixed numbers, $Q_T = \Omega \times (0, T]$.

Let a control process be described in Q_T by the following initial boundary value problem for a parabolic equation with control coefficients $\lambda(u, v)$ and $\beta(u, v)$ depending on the solution of the state

$u(x,t)$ and the control v.

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x})=f(x,t,\beta(u,v)),(x,t)\in Q_T, \tag{2.1}$$

with initial and boundary conditions

$$u(x,0)=\phi(x), 0<x<l,$$

$$\lambda(u,v)u_x|_{x=0}=\psi_0(t), \lambda(u,v)u_x|_{x=l}=\psi_1(t), \tag{2.2}$$

where $\phi(x)\in L_2(0,T)$, $\psi_0(t)$ and $\psi_1(t)\in L_2(0,T)$ for any $T>0$ are given functions. The functions $\lambda(u,v)$ and $\beta(u,v)$ are continuous on $(u,v)\in[r_1,r_2]\times l_2$ and have continuous derivatives in u, $\forall(u,v)\in[r_1,r_2]\times l_2$ satisfy a Lipschitz condition. Here ν_0,ν_1,ρ_0, and ρ_1 are given numbers. Besides the above conditions, we use the additional restrictions

$$\nu_0\le\lambda(u,v)\le\nu_1, \rho_0\le\beta(u,v)\le\rho_1. \tag{2.3}$$

We consider a generalized solution of the problem (2.1)-(2.3) from the energetic class, i.e., the function $u(x,t)\in V_2^{1,0}(Q_T)$, where $Q_T=(0,l)\times(0,T)$ (see [10]).

We define some spaces and inequalities we need them later.

(a) $V_2^{1,0}(Q_T)$ is a Banach space consisting of elements from $W_2^{1,0}(Q_T)$ having a finite norm

$$|u(x,t)|_{Q_T}=\text{ess}\sup_{0<t<T}\|u(x,t)\|_{2,(0,l)}+\|u_x(x,t)\|_{2,Q_T},$$

and traces from $L_2(0,l)$ on the sections of $(0,l)$ continuously varying in $t\in[0,T]$.

(b) The space which consisting of all the convergence number series $\zeta_1,\zeta_2,\cdots,\zeta_i,\cdots$ is the Hilbert space l_2 with

$$\langle\beta,\eta\rangle_{l_2}=\sum_{i=1}^{n}\beta_i\eta_i, \qquad \|\zeta\|_{l_2}=[\langle\zeta,\zeta\rangle_{l_2}]^{\frac{1}{2}}.$$

(c) Cauchy's inequality with ϵ takes the form

$$|\,ab\,|\le\frac{\epsilon}{2}\,|\,a\,|^2+\frac{1}{2\epsilon}\,|\,b\,|^2,$$

which holds for all $\epsilon>0$ and for arbitrary a and b.

(d) For the space $L_2(D)$, Cauchy Bunyakoviskii inequality takes the form

$$|\int_D uv dx\,|\le(\int_D u^2dx)^{\frac{1}{2}}(\int_D v^2dx)^{\frac{1}{2}}.$$

Here and in what follows, we use the notation

$$\|u(x,t)\|_{2,(0,l)}=(\int_0^l u^2(x,t)dx)^{1/2},$$

$$\|u_x(x,t)\|_{2,Q_T}=(\int_{Q_T}u_x^2(x,t)dxdt)^{1/2}.$$

Consider the following problem: for an arbitrary $c\in(0,T)$ and on the solution $u(x,t)=u(x,t;v)$ of problem (2.1)-(2.4) corresponding to all admissible controls $v\in V$, minimize the functional

$$J[v]=\alpha_0\int_0^l(u(x,c;v)-z(x))^2dx$$
$$+\alpha_1\int_0^l f(x,T,\beta(u,v))dx+\alpha\|v-\omega\|_{l_2}^2. \tag{2.4}$$

where $u(x,c;v)$ and $z(x)$ are given functions and α_0, $\alpha_1\ge0$ and $\alpha_0+\alpha_1\neq0$ and $\alpha\ge0$. Hence, $\omega\in l_2$ is given such that $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$.

Definition 2.1: The problem of finding a function $u=u(x,t)\in V_2^{1,0}(Q_T)$ from conditions (2.1)-(2.4) given $v\in V$ is called reduced problem (see [3]).

Definition 2.2: A solution of the boundary value problem (2.1)-(2.4) corresponding to a control $v\in V$ is defined as a function $u=u(x,t;v)$ in $V_2^{1,0}(Q_T)$ satisfying the integral identity

$$\int_0^l\int_0^T[u\eta_t-\lambda u_x\eta_x+\eta f(x,t,\beta(u,v))]dxdt$$
$$=\int_0^T\eta(0,t)\psi_0(t)dt-\int_0^T\eta(l,t)\psi_1(t)dt \tag{2.5}$$

for all $\eta(x,t)\in W_2^{1,1}(Q_T)$ equal to zero at $t=T$. Let V be a closed and bounded subset of l_2. The function $f(x,t,\beta(u,v))$ is given continuous function for almost all $(x,t)\in Q_T$, bounded and measurable in $(x,t)\in Q_T$.

Under the above assumptions [6], the boundary value problem (2.1)-(2.4) be exist and has a unique solution in $V_2^{1,0}(Q_T)$ for each $v\in V$ and $\|u_x\|\le C_0$, for all $(x,t)\in Q_T$ and C_0 is a certain constant.

3 The existence and uniqueness theorems

Optimal control problems of the coefficients of differential equations do not always have solution [8]. Examples in [10] and elsewhere of problems of the type (2.1)-(2.4) having no solution for $\alpha = 0$. A problem of minimization of a functional is said to be unstable, when a minimizing sequence does not converge to an element minimizing the functional [6]. To prove the existence we need the following theorem:

Theorem 3.1 Under the above assumptions for every solution of the reduced problem (2.1)-(2.4) the following estimate is valid:

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \leq C[\mu_0\|\delta\lambda u_x\|^2_{L_2(Q_T)}+\mu_1\|\delta f\|^2_{L_2(Q_T)}]^{1/2}, \tag{3.6}$$

where C, μ_0 and μ_1 are positive constants independent on the control function v.

Proof

Set $\delta u(x,t) = u(x,t,v+\delta v) - u(x,t;v), u \equiv u(x,t;v)$. From (2.5) it follows the function $\delta u(x,t)$ satisfies the identity

$$\int_0^l\int_0^T[-\eta_t\delta u + \frac{\partial\lambda(u+\theta_1\delta u, v+\delta v)}{\partial u}\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x}\delta u$$

$$+\delta\lambda\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x} + \lambda^{'}\frac{\partial \delta u}{\partial x}\eta_x - \eta\delta f$$

$$-\frac{\partial f(x,t,\beta)}{\partial\beta}\frac{\partial\beta(u+\theta_2\delta u, v+\delta v)}{\partial u}\eta\delta u]dxdt = 0, \tag{3.7}$$

for all $\eta = \eta(x,t) \in W_2^{1,1}(Q_T)$ and $\eta(x,T) = 0$. Here $\theta_1, \theta_2 \in (0,1)$ are some positive numbers, and

$$\delta f = f(x,t,\beta(u,v+\delta v)) - f(x,t,\beta(u,v))$$

$$\lambda^{'} = \lambda(u+\delta u, v+\delta v), \delta\lambda = \lambda(u,v+\delta v) - \lambda(u,v),$$

$$\delta u = u(x,t;v+\delta v) - u(x,t;v),$$

Let us consider the function

$$\eta(x,t) = \begin{cases} 0 & t\in[t_1,T] \\ \int_t^T \overline{\eta}(x,\tau)d\tau & t\in[0,t_1] \end{cases}$$

where $\eta(x,t) \in W_2^{1,1}(Q_T)$ and it has the generalized derivatives

$$\eta_t = -\overline{\eta}(x,t),$$

and

$$\eta_x = \int_t^T \overline{\eta}_x(x,\tau)d\tau.$$

Put δu instead of $\overline{\eta}(x,t)$ for $(x,t) \in Q_t$ and $\eta(x,T) = 0$.

Following the method of [11], we obtain

$$\int_\Omega(\delta u(x,t_1))^2dx + \int_{Q_t}[\frac{\partial\lambda(u+\theta_1\delta u, v+\delta v)}{\partial u}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\delta u$$

$$+\delta\lambda\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + \lambda^{'}\frac{\partial \delta u}{\partial x}\frac{\partial \delta u}{\partial x}]dxdt$$

$$-\int_{Q_t}[\frac{\partial f(x,t,\beta)}{\partial\beta}\frac{\partial\beta(u+\theta_2\delta u, v+\delta v)}{\partial u}\delta u\delta u$$

$$+\delta u\delta f]dxdt = 0 \tag{3.8}$$

Hence, from the above assumptions and applying Cauchy Bunyakoviskii inequality, we obtain

$$\int_\Omega(\delta u(x,t_1))^2dx + \nu_0\int_{Q_t}(\tfrac{\partial\delta u}{\partial x})^2dxdt$$

$$\leq C_1(\int_{Q_t}(\tfrac{\partial\delta u}{\partial x})^2dxdt)^{1/2}(\int_{Q_t}(\delta u(x,t))^2dxdt)^{1/2}$$

$$+C_2(\int_{Q_t}(\delta\lambda\tfrac{\partial u}{\partial x})^2dxdt)^{1/2}(\int_{Q_t}(\tfrac{\partial\delta u}{\partial x})^2dxdt)^{1/2}$$

$$+(\int_{Q_t}(\delta f)^2dxdt)^{1/2}(\int_{Q_t}(\delta u)^2dxdt)^{1/2}$$

$$+C_3\int_{Q_t}(\delta u)^2dxdt, \tag{3.9}$$

where C_1, C_2, and C_3 are positive constants independent of δv.

Applying Cauchy's inequality with ε and combine similar terms, then multiply both sides by two, we get

$$\|\delta u(x,t_1)\|^2_{L_2(\Omega)} + \nu_0\|\tfrac{\partial\delta u}{\partial x}\|^2_{L_2(Q_t)} \leq \tfrac{C_1\nu}{2}\|\tfrac{\partial\delta u}{\partial x}\|^2_{L_2(Q_t)}$$

$$+\tfrac{C_1}{2\nu}\|\delta u\|^2_{L_2(Q_t)} + \tfrac{C_2\nu}{2}\|\delta\lambda u_x\|^2_{L_2(Q_t)} + \tfrac{C_2}{2\nu}\|\tfrac{\partial\delta u}{\partial x}\|^2_{L_2(Q_t)}$$

$$+\tfrac{\nu}{2}\|\delta f\|^2_{L_2(Q_t)} + \tfrac{1}{2\nu}\|\delta u\|^2_{L_2(Q_t)}$$

$$+C_3\|\delta u\|^2_{L_2(Q_t)}, \tag{3.10}$$

Combining the similar terms, we get

$$\|\delta u(x,t_1)\|^2_{L_2(\Omega)} + \nu_0\|\frac{\partial\delta u}{\partial x}\|^2_{L_2(Q_t)} \leq C_4\|\frac{\partial\delta u}{\partial x}\|^2_{L_2(Q_t)}$$

$$+C_5\|\delta u\|^2_{L_2(Q_t)} + \mu_0\|\delta\lambda u_x\|^2_{L_2(Q_t)} + \mu_1\|\delta f\|^2_{L_2(Q_t)}, \tag{3.11}$$

where $C_4 = (\frac{C_1\nu}{2} + \frac{C_2}{2\nu})$, $C_5 = (\frac{C_1}{2\nu} + \frac{1}{2\nu} + C_3)$, $\mu_0 = \frac{C_2\nu}{2}$, $\mu_1 = \frac{\nu}{2}$. C_4, C_5, μ_0, and μ_1 are positive constants not depending on δv.

Now, we replace $\|\delta u\|^2_{L_2(Q_t)} = ty^2(t)$, where

$$y(t) \equiv \max_{0\le\tau\le t} \|\delta u(x,t)\|_{L_2(\Omega)},$$

$\|\delta u(x,0)\|^2_{L_2(\Omega)} = y(t)\|\delta u(x,0)\|_{L_2(\Omega)}$, and let

$$j(t) = \mu_0\|\delta\lambda u_x\|^2_{L_2(Q_t)} + \mu_1\|\delta f\|^2_{L_2(Q_t)},$$

Then, we obtain

$$\|\delta u(x,t_1)\|^2_{L_2(\Omega)} + \nu_0\|\frac{\partial \delta u}{\partial x}\|^2_{L_2(Q_t)} \le C_4\|\frac{\partial \delta u}{\partial x}\|^2_{L_2(Q_t)} + C_5 t y^2(t) + j(t). \quad (3.12)$$

This follows the two inequalities

$$\|\frac{\partial \delta u}{\partial x}\|^2_{L_2(Q_t)} \le \nu_0^{-1} j(t), \quad (3.13)$$

and

$$y^2(t) \le j(t). \quad (3.14)$$

We take the square root of both sides of (3.12), (3.13), and add together the resulting inequalities and majorize the right hand side in the following way [12]

$$y(t) + \|\frac{\partial \delta u}{\partial x}\|_{L_2(Q_t)} \le (1+\nu_0^{-1/2})j^{1/2}(t), \quad (3.15)$$

then we obtain

$$\|\delta u\|_{V_2^{0,1}(Q_t)} = \max_{0\le t\le t_1} \|\delta u(x,t)\|_{L_2(\Omega)} + \|\frac{\partial \delta u}{\partial x}\|_{L_2(Q_t)}, \quad (3.16)$$

and

$$\|\delta u\|_{V_2^{0,1}(Q_t)} \le C j^{1/2}(t), \quad (3.17)$$

where $C = (1+\nu_0^{-1/2})$ is positive constant not depending on δv. Theorem 3.1 is proved.

Lemma 3.1: ([10]) Under the above assumptions, the boundary value problem (2.1)-(2.2) has a unique solution in $V_2^{1,0}(Q_T)$ for each $v \in V$, and this solution belongs to $W_2^{1,1}(Q_T)$ and admits the following estimate

$$\|u\|_{2,Q_T} \le M_1[\|\psi_0\|_{2,(0,T)} + \|\psi_1\|_{2,(0,T)}]. \quad (3.18)$$

Here and in the following $M_i, i = 1,2,\ldots$ are positive constants independent of the quantities to be estimated and admissible controls. It follows from the estimate (3.18) that the functional (2.4) is defined on V and takes finite values.

Note that the functional (2.4) is nonlinear, and it is difficult to analyze its convexity.

Corollary 3.1 Under the above assumptions [7], the right part of estimate (3.6) converges to zero at $\|\delta v\|_{l_2} \to 0$, therefore

$$\|\delta u\|_{V_2^{1,0}(Q_T)} \to 0 \text{ a } \|\delta v\|_{l_2} \to 0. \quad (3.19)$$

Hence from the theorem on trace [13] we get

$$\|\delta u(0,t)\|_{L_2(0,T)} \to 0, \|\delta u(l,T)\|_{L_2(0,T)} \to 0 \text{ as} \|\delta v\|_{l_2} \to 0. \quad (3.20)$$

Now we consider the functional $J_0(v)$ of the form

$$J_0(v) = \alpha_0 \int_0^l (u(x,c;v) - z(x))^2 dx + \alpha_1 \int_0^l f(x,T,\beta(u,v))dx. \quad (3.21)$$

Lemma 3.2 The functional $J_0(v)$ is continuous on V.

proof

Let $\delta v = (\delta v_0, \delta v_1, \ldots, \delta v_n)$ be an increment of control on an element $v \in V$ such that $v + \delta v \in V$. For the increment of $J_0(v)$ we have

$$\delta J_0(v) = J_0(v+\delta v) - J_0(v) = 2\alpha_0 \int_0^l (u(x,c;v) - z(x))\delta u(x,c;v)dx + \alpha_0 \int_0^l (\delta u(x,c;v))^2 dx + \alpha_1 \int_0^l \frac{\partial f(x,T,\beta(u+\theta_3\delta u, v+\theta_3\delta v))}{\partial u}\delta u(x,T,\beta)dx \quad (3.22)$$

Applying the Cauchy-Bunyakovskii inequality, we obtain

$$\|\delta J_0(v)\| \le 2\alpha_0\|u(t,c;v) - z(x)\|_{L_2(0,l)}\|\delta u(x,c;v)\|_{L_2(0,x)} + \alpha_0\|\delta u(x,c;v)\|^2_{L_2(0,l)} + \alpha_1\|\frac{\partial f(x,T,\beta(u+\theta_3\delta u, v+\theta_3\delta v))}{\partial u}\|_{L_2(0,l)}\|\delta u(x,c;v)\|^2_{L_2(0,l)}. \quad (3.23)$$

An application of the Corollary 3.1 completes the proof.

Lemma 3.3 (Weierstrass theorem) Let V_0 be a non empty compact subset of H and let f a real-valued continuous function defined on V_0. Then f assumes its maximum and minimum values on V_0, i.e. there exist points $\overline{x} \in V_0$ and $\widetilde{x} \in V_0$ such that.

$$f(\overline{x}) = \max\{f(x) : x \in V_0\}.$$

and

$$f(\widetilde{x}) = \min\{f(x) : x \in V_0\}.$$

Theorem 3.2 For any $\alpha \geq 0$, the problem (2.1)-(2.4) has at least one solution.
proof The set of V is closed and bounded in l_2. Since $J_0(v)$ is continuous on V by Lemma 3.2, so

$$J_\alpha(v) = J_0(v) + \alpha\|v - \omega\|_{l_2}^2. \tag{3.24}$$

Then from the Weierstrass theorem [15] it follows that the problem (2.1)-(2.4) has at least one solution. This completes the proof of Theorem 3.2.
According to the above discussions, we can easily obtain a theorem concerning the uniqueness solution for the considering optimal control problem (2.1)-(2.4).
Theorem 3.3 Let $\omega \in H$ be a given element, then there exists a dense subset V_0 of the space H such that for any $\omega \in V_0$ with $\alpha > 0$, the optimal control problem (2.1)-(2.4) has a unique solution.
proof (A corollary of the Goebel theorem [14]) Assume that H is a uniformly convex space and V is a bounded and closed subset of H. A functional $J_0(v)$ is lower semicontinuous and bounded from below on V, and $\alpha > 0$ is a given number. Then there exists a dense subset $V_0 \subset H$ such that for any $\omega \in V_0$ the functional

$$J_\alpha(v) = J_0(v) + \alpha\|v - \omega\|_{l_2}.$$

Then the optimal control problem (2.1)-(2.4) has a unique solution, and this completes the proof of the theorem.

4 The differentiability of the cost functional and necessary optimality conditions

Now let us study the differentiability of the functional and establish the necessary optimality condition in problem (2.1)-(2.4). We introduce the conjugate problem that implies the definition of functions $\Theta = \Theta(x, t, v)$ as the solution of the problem.
The lagrangian function $L(x, t, u, v, \Theta)$ is defined by

$$L(x, t, u, v, \Theta) = J_\alpha(v)$$
$$+ \int_0^l \int_0^T \Theta[\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\lambda(u, v) - f(x, t, \beta(u, v))]dxdt, \tag{4.25}$$

where $\Theta(x, t) \in V_2^{1,0}$ is the generalized solution of the boundary value problem conjugated to 2.1-2.4 as

$$\frac{\partial \Theta}{\partial t} - \lambda_u \frac{\partial \Theta}{\partial x}\frac{\partial u}{\partial x} + \lambda' \frac{\partial \Theta}{\partial x} + \Theta \frac{\partial f}{\partial u} = 2\alpha_0(u(x, c; v) - z(x)), \tag{4.26}$$

with initial and boundary conditions

$$\Theta(x, t)|_{t=T} = -\alpha_1 \tfrac{\partial}{\partial u} f(x, T, \beta(u, v)),$$

$$\lambda'\Theta_x|_{x=0} = 0, \quad \lambda'\Theta_x|_{x=1} = 0, \tag{4.27}$$

where $u = u(x, t, v)$ is the solution of the problem (2.1)-(2.2). A solution of the boundary value problem (4.26)-(4.27) corresponding to the control $v \in V$ is defined as a function $\Theta = \Theta(x, t, v)$ in $V_2^{1,0}(Q_T)$ satisfying the integral identity

$$\int_0^l \int_0^T [-\Theta\eta_t - \lambda_u \tfrac{\partial \Theta}{\partial x}\tfrac{\partial u}{\partial x}\eta + \lambda' \tfrac{\partial \Theta}{\partial x}\eta + \Theta \tfrac{\partial f}{\partial u}\eta]dxdt$$

$$= -2\alpha_0 \int_0^l (u(x, c; v) - z(x))\eta dx$$

$$-\alpha_1 \int_0^l \tfrac{\partial f(x,T,\beta(u,v))}{\partial u}\eta dx, \tag{4.28}$$

for any function $\eta \in W_2^{1,1}(Q_T)$ that is zero for t=0. It follows from the results of the monograph [10] that, for each $v \in V$, the problem (4.26)-(4.27) has a unique solution in $V_2^{1,0}(Q_T)$. This solution belongs to $W_2^{1,1}(Q_T)$, satisfies (4.27) for almost all $(x, t) \in Q_T$, and admits the estimate

$$\|\Theta\|_{2,Q_T} \leq M_2[\alpha_0\|u(x, c; v) - z(x)\|_{2,(0,l)}$$
$$+\alpha_1\|f_u(x, T, \beta(u, v))\|_{2,(0,l)}]. \tag{4.29}$$

By taking into account inequality (3.9) and the estimates (3.8), we obtain the estimate

$$\|\Theta\|_{2,Q_T} \leq M_3[\|\psi_0\|_{2,(0,T)} + \|\psi_1\|_{2,(0,T)}$$

$$+ \alpha_0\|z(x)\|_{2,(0,l)} + \alpha_1\|f_u(x, T, \beta(u, v))\|_{2,(0,l)}]. \tag{4.30}$$

The Gradient formula for the modified function: The sufficient differentiability conditions of function (3.22) and its gradient for formula will be obtained by defining the Hamiltonian function [3] $H(u, \Theta, v)$ as in the following theorem
Theorem 4.1: Suppose that the above assumptions holds. Then, the gradient of the functional $J(v)$ at an arbitrary $v \in V$ defined by the first derivative of the Hamltonian function is $\frac{\partial J(v)}{\partial v} \equiv \frac{-\partial H(u,v,\Theta)}{\partial v}$.
proof. Suppose that $v = (v_1, v_2, \ldots, v_n) \in l_2, \delta v = (\delta v_1, \delta v_2, \ldots, \delta v_n)$ $\delta v \in l_2$, $v + \delta v \in l_2$ $\delta u =$

$u(x,t;v+\delta v)-u(x,t;v)$.
The increment of the functional $J(v)$ is

$$\delta J(v)=J(v+\delta v)-J(v)=$$
$$=2\alpha_0\int_0^l(u(x,c;v)-z(x))\delta u(x,c;v)dx$$
$$+\alpha_1\int_0^l\frac{\partial f(x,T,\beta)}{\partial\beta}\frac{\partial\beta}{\partial u}\delta u(x,T,\beta)dx$$
$$+\alpha_1\int_0^l\frac{\partial f(x,T,\beta(u+\theta_3\delta u,v+\theta_3\delta v))}{\partial\beta}\frac{\partial\beta}{\partial u}\delta u(x,T,\beta)dx$$
$$-\alpha_1\int_0^l\frac{\partial f(x,T,\beta)}{\partial\beta}\frac{\partial\beta}{\partial u}\delta u(x,T,\beta)dx$$
$$+2\alpha\langle v-\omega;\delta v\rangle_{l_2}+\alpha\|\delta v\|_{l_2}^2$$
$$+\alpha_0\|\delta u\|_{2,(0,l)}^2, \quad (4.31)$$

where

$$\delta J(v)=2\alpha_0\int_0^l(u(x,c;v)-z(x))\delta u(x,c;v)dx$$
$$+2\alpha\langle v-\omega;\delta v\rangle_{l_2}$$
$$+\alpha_1\int_0^l\frac{\partial f(x,T,\beta)}{\partial\beta}\frac{\partial\beta}{\partial u}\delta u(x,T,\beta)dx+R_1(\delta v), \quad (4.32)$$

and

$$R_1(\delta v)=\alpha_1\int_0^l[\tfrac{\partial f(x,T,\beta(u+\theta_3\delta u,v+\theta_3\delta v))}{\partial\beta}\tfrac{\partial\beta}{\partial u}$$
$$-\tfrac{\partial f(x,T,\beta)}{\partial\beta}\tfrac{\partial\beta}{\partial u}]\delta u(x,T,\beta)dx$$
$$+\alpha_0\|\delta u\|_{2,(0,l)}^2+\alpha\|\delta v\|_{l_2}^2. \quad (4.33)$$

Using the obtained estimation (3.7), the inequality $|R_1(\delta v)|\le C_7\|\delta v\|_{l_2}$ can be verified where C_7 is a constant not depend on δv.
If we put $\delta u(x,t)=\eta(x,t)$ in identity (4.28), $\eta(x,t)=\Theta(x,t)$ in (3.7) and add together we obtain

$$2\alpha_0\int_0^l(u(x,c;v)-z(x))\delta u(x,c;v)dx$$
$$+\alpha_1\int_0^l\frac{\partial f(x,T,\beta)}{\partial\beta}\frac{\partial\beta}{\partial u}\delta u(x,T)dx$$
$$=\int_0^l\int_0^T[\delta f\Theta-\delta\lambda u_x\Theta_x]dxdt+R_2(\delta v), \quad (4.34)$$

where

$$R_2(\delta v)=\int_0^l\int_0^T[[\tfrac{\partial f(x,T,\beta(u+\theta_3\delta u,v+\theta_3\delta v))}{\partial\beta}\tfrac{\partial\beta}{\partial u}$$
$$-\tfrac{\partial f(x,T,\beta)}{\partial\beta}\tfrac{\partial\beta}{\partial u}]\delta u(x,T,\beta)$$
$$-(\tfrac{\partial\lambda(u+\theta_1\delta u,v+\theta_1\delta v)}{\partial u}-\tfrac{\partial\lambda(u,v)}{\partial u})\tfrac{\partial u}{\partial x}\tfrac{\partial\Theta}{\partial x}\delta u(x,t)$$
$$-(\tfrac{\partial\delta u}{\partial x}+\delta u(x,t))\lambda(u+\theta_1\delta u,v+\theta_1\delta v)\tfrac{\partial\Theta}{\partial x}$$
$$+(\Theta\delta u)_t]dxdt, \quad (4.35)$$

where $\theta_i, i=1,2,\dots$ are positive numbers, $R_2(\delta v)$ is estimated as $|R_1(\delta v)|\le C_8\|\delta v\|_{l_2}$, and C_8 is a constant not depend on δv. Using the above assumptions, we have

$$\delta\lambda=\langle\lambda_v(u,v),\delta v\rangle_{l_2}+O(\|\delta v\|_{l_2}),$$
$$\delta f=\langle f_v(x,t,\beta(u,v)),\delta v\rangle_{l_2}+O(\|\delta v\|_{l_2}).$$

Then we obtain

$$\delta J(v)=\int_0^l\int_0^T\langle f_v(x,t,\beta(u,v))\Theta$$
$$-\lambda_v(u,v)u_x\Theta_x,\delta v\rangle_{l_2}+2\alpha\langle v-\omega,\delta v\rangle_{l_2}+R_3(\delta v), \quad (4.36)$$

where

$$R_3(\delta v)=R_1(\delta v)+R_2(\delta v)+O(\|\delta v\|_{l_2}).$$

From the formula of $R_3(\delta v)$, we have $|R_3(\delta v)|\le C_9\|\delta v\|_{l_2}$, and C_9 is a constant not depend on δv.
From (4.35)-(4.36) and using the function $H(u,\Theta,v)$ [7], we have

$$\delta J(v)=\langle-\frac{\partial H(u,\Theta,v)}{\partial v},\delta v\rangle_{l_2}+O(\|\delta v\|_{l_2}), \quad (4.37)$$

which shows the differentiability of the functional $J(v)$ and also gives the gradient formula of the functional $J(v)$ as

$$\frac{\partial J(v)}{\partial v}=-\frac{\partial H(u,\Theta,v)}{\partial v}.$$

Hence, the theorem is proved.

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