On the Oscillatory Behavior of Solutions of Second Order Nonlinear Neutral Delay Differential Equations

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Abstract: - In this paper, the oscillatory behavior of solutions of a general class of nonlinear neutral delay differential equations is discussed. New criteria are established. Illustrative examples are also given to support the validity of the method

Key-Words: - Oscillation, Second order, Non-linear neutral delay differential equations.

1 Introduction

In this paper, we are concerned with the oscillation of solutions of the second-order nonlinear neutral delay differential equation

$$\left(r(t)\psi\left(x(t)\right)|z'(t)|^{\alpha-1}z'(t)\right)' + f\left(t,x\left(\sigma(t)\right)\right)=0, \quad (1)$$

where $t\in I=[t_0,\infty), t_0>0, z(t)=x(t)-p(t)x\left(\tau(t)\right)$, and $\alpha>0$ is the ratio of two odd integers. Throughout the paper, we assume the following:

(A_1) $r,p\in C(I,\mathbb{R}), r(t)>0,$ and $\int_{t_0}^{\infty}\frac{ds}{r^{1/\alpha}(s)}=\infty$;

(A_2) $\tau(t),\sigma(t)\in C(I,\mathbb{R}), \tau(t)\le t, \sigma(t)\le t, \sigma'(t)>0,$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

(A_3) $\psi\in C^1(\mathbb{R},\mathbb{R}), \psi(x)>0,$ and there exists a positive constant K such that $\psi(x)\le K^{-1}$ for all $x\ne 0$;

(A_4) $f(t,x)\in C(I\times\mathbb{R},\mathbb{R})$, and there exists a function $q(t)\in C(I,[0,\infty))$ such that $f(t,x)sgnx\ge q(t)|x|^{\alpha}$ for $x\ne 0$ and $t\ge t_0$.

By a solution of equation (1), we mean a continuous function x defined on an interval $[t_x,\infty)$ such that $r(t)\psi(x)|z'|^{\alpha-1}z'$ is continuously differentiable and x satisfies (1) for $t\in[t_x,\infty)$. Throughout the paper, we consider only solutions which satisfy $sup\{|x(t)|: t\ge T\ge t_x\}>0$. As usual, a solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory. Equation (1) is termed oscillatory if all its cotinuable solutions oscillate. In recent years, there has been increasing interest in studying oscillation of solutions of different classes of differential equations due to the fact that they have numerous applications in natural sciences, physics, biology, economy and engineering. There is a lot of interest in obtaining sufficient conditions for the oscillation or non-oscillation of solutions of various types of differential equations, (see [1-24]). In particular, much works have been done on the following particular cases of (1): Wong et al. [22] studied the second order differential equation

$$\left(x(t)-px(t-\tau)\right)''+q(t)f\left(x(t-\sigma)\right)=0, \quad (2)$$

and established necessary and sufficient oscillation conditions in the case, $0<p<1$, and $q(t)\ge 0$. Yildiz et al. [23] studied the nonlinear equations

$$\left(x(t)\pm r(t)f\left(x(t-\tau)\right)\right)''+p(t)g\left(x(t-\sigma)\right)-q(t)g\left(x(t-\delta)\right)=s(t), \quad (3)$$

under the conditions $\tau\ge 0, \sigma\ge\delta\ge 0,$ $0<p(t)<\infty$, and $q(t)\ge 0$. Mohamed et al. [17] studied the asympototic behavior of the nonlinear neutral differential equation

$$\left(x(t)+\delta p(t)x\left(\tau(t)\right)\right)''+q(t)f\left(x\left(\sigma(t)\right)\right)=0, \quad (4)$$

when $\delta=\pm1, q(t)>0$, and $p(t)>0$. Han et al. [11] studied the quasilinear neutral delay differential equation

$$\left(r(t)\left|\left(x(t)-p(t)x\left(\tau(t)\right)\right)'\right|^{\alpha-1}\left(x(t)-p(t)x\left(\tau(t)\right)\right)'\right)'+q(t)f\left(x\left(\sigma(t)\right)\right)=0, \quad (5)$$

when $0\le p(t)\le p<1$, and $\lim_{t\to\infty}p(t)=p_1<1$. More recently Li et al. [16] and Arul et al. [1] studied a class of second order nonlinear delay differential equations with non-positive neutral coefficients of the type

$$\left(r(t)\left(\left(x(t)-p(t)x\left(\tau(t)\right)\right)'\right)^{\alpha}\right)'+q(t)f\left(x\left(\sigma(t)\right)\right)=0, \quad (6)$$

under the conditions $0\le p(t)\le p<1$, and $q(t)\ge 0$. They established sufficient conditions guarantee that every solution oscillates or tends to zero. For other related contribution when $z(t) = x(t) - p(t)x(\tau(t))$ in the case of higher order equations, we mention the work of Baculîková et al. [2] and Džurina et al. [5] who investigated the asymptotic properties of solutions of the couple third order neutral differential equations

$$\left(r(t)\left(\left[x(t) \pm p(t)x(\tau(t))\right]''\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \tag{7}$$

when $0 \le p(t) < p < 1$, and $\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \infty$. Meanwhile Soliman et al. [20] studied couple of third order neutral differential equations of the type

$$\left[r(t)\left(\left(x(t) \pm \sum_{i=1}^{n} p_i(t)x(\tau_i(t))\right)''\right)^{\alpha}\right]' + \sum_{j=1}^{m} f_j\left(t, x\left(\sigma_j(t)\right)\right) = 0, \tag{8}$$

under the conditions $-\mu \le p_i(t) \le 1$ for all $i = 1,2,\dots\dots,n, \mu \in (0,1)$, and $\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} = \infty$. Also Rath et al. [18] studied the nonlinear neutral delay differential equation

$$\left(x(t) - p(t)x(t-\tau)\right)^{(n)} + \alpha q(t)G\left(y(t-\sigma)\right) = f(t), \tag{9}$$

under the conditions $p(t) = 1$ or $p(t) \le 0, \alpha = \pm 1$, and $q(t) > 0$.

The aim of this paper is to improve and extend some of the results given in [9, 13, 14, 16, 19]. Using a generalized Riccati substitution, and integral averaging techniques, we establish new oscillation criteria for the general Eq. (1). One may note that Eq. (1) is different from those discussed in [15], and [19].

2 Main Results

For simplicity, we define

$$Q(t) = q(t)\left(1 + p(\sigma(t))\right)^{\alpha}, \mathcal{Q}(t) = min\{q(t), q(\tau(t))\},$$
$$R(l,t) = \left(\int_{l}^{\sigma(t)} r^{-1/\alpha}(s)ds\right)\left(\int_{l}^{t} r^{-1/\alpha}(s)ds\right)^{-1}, and \tag{10}$$
$$\xi(t) = r^{1/\alpha}(t)\int_{t_1}^{t} r^{-1/\alpha}(s)ds$$

where $t_1 \ge t_0$ is sufficiently large.

Lemma 1 Let the conditions (A_1)-(A_4) be satisfied, and $0 \le p(t) < 1$. Suppose that $\alpha > 0$, and x is a positive solution of (1). Then $z'(t) > 0, \left(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t)\right)' \le 0$ and either

(C_1) $z(t) > 0$, or (C_2) $z(t) < 0$, for $t \ge t_1$, where $t_1 \ge t_0$ is sufficiently large.

Proof. The proof is similar to that given in [16] and so it is omitted.

Theorem 2 Let the conditions (A_1)-(A_4) hold, and $0 \le p(t) < 1$. Suppose that there exists a positive function $\rho \in C^1(I,\mathbb{R})$ such that, for all sufficiently large $t_1 \ge t_0$, we have

$$\int^{\infty}\left[\rho(t)q(t) - \frac{\rho'_{+}(t)r(\sigma(t))}{K\xi^{\alpha}(\sigma(t))}\right]dt = \infty, \tag{11}$$

where $\xi(\sigma(t))$ be as defined in (10), and $\rho'_{+}(t) = max\{0, \rho'(t)\}$, then every solution $x(t)$ of (1) either oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that there exists a $t_1 \ge t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$. It follows from (1) that

$$\left(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t)\right)' \le -q(t)x^{\alpha}(\sigma(t)) \le 0. \tag{12}$$

Therefore by Lemma 1, we have either (C_1) or (C_2). Suppose that the case (C_1) holds. Then from the definition of z, we get

$$x(t) = z(t) + p(t)x(\tau(t)) \ge z(t). \tag{13}$$

Using (12) and the fact that $\sigma(t) \le t$, it follows that

$$r(t)\psi(x(t))\left(z'(t)\right)^{\alpha} \le r(\sigma(t))\psi\left(x(\sigma(t))\right)\left(z'(\sigma(t))\right)^{\alpha}. \tag{14}$$

Thus

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{\left(r(s)\psi(x(s))\left(z'(s)\right)^{\alpha}\right)^{1/\alpha}}{r^{1/\alpha}(s)\psi^{1/\alpha}(x(s))}ds \ge K^{1/\alpha}z'(t)\psi^{1/\alpha}(x(t))\xi(t), \tag{15}$$

Now define

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))\left(z'(t)\right)^{\alpha}}{z^{\alpha}(\sigma(t))}, t \ge t_1.$$

Then $\omega(t) > 0$ for $t \ge t_1$ and

$$\omega'(t) \le \rho'_{+}(t)\frac{r(\sigma(t))}{K\xi^{\alpha}(\sigma(t))} - \rho(t)q(t). \tag{16}$$

Then by integrating from t_2 to t, we get

$$\int_{t_2}^{t}\left[\rho(s)q(s) - \frac{\rho'_{+}(s)r(\sigma(s))}{K\xi^{\alpha}(\sigma(s))}\right]ds \le \omega(t_2).$$

This contradicts (11). Now if z satisfies (C_2), then by [16, Lemma 2.2] $\lim_{t\to\infty} x(t) = 0$. Thus the proof is completed.

Theorem 3 Let $\alpha \ge 1$. Assume that the conditions (A_1)-(A_4) hold, and $0 \le p(t) < 1$. Suppose that for all large $t_1 \ge t_0$ there exists a positive function $\rho \in C^1(I,\mathbb{R})$ such that

$$\int^{\infty}\left[\rho(t)q(t) - \frac{\left(\rho'_{+}(t)\right)^2 r(\sigma(t))}{4\alpha K\rho(t)\sigma'(t)\xi^{\alpha-1}(\sigma(t))}\right]dt = \infty, \tag{17}$$

where $\xi(\sigma(t))$ be as defined in (10), and $\rho'_{+}(t) = max\{0, \rho'(t)\}$, then every solution $x(t)$ of (1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof follows the lines of Theorem 2 on view of the technique of [16], and so it is omitted.

The following result not only extends those of [9], [13], [14], and [19] but also it differs from them in using the notation $z(t) = x(t) - p(t)x(\tau(t))$.

Theorem 4 Assume that the conditions (A_1)-(A_4) hold. Suppose that $\alpha \geq 1$, and $p(t)$ is an odd function for all $t \geq t_0$ such that $0 \leq p(t) < 1$. Suppose further that there exists a positive function $\rho \in C^1(I, \mathbb{R})$ such that

$$\int^{\infty} \left[\rho(s)Q(s) - \frac{1}{K(\alpha+1)^{\alpha+1}} \frac{\left(\rho'_+(s)\right)^{\alpha+1} r(\sigma(s))}{\rho^{\alpha}(s)\left(\sigma'(s)\right)^{\alpha}}\right] ds = \infty, \tag{18}$$

where $Q(t)$ be as defined in (10), and $\rho'_+(t) = max\{0, \rho'(t)\}$. Then every solution $x(t)$ of (1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then from (1), we have (12). Now we conclude from Lemma 1, that z satisfies one of the two cases (C_1) and (C_2). Suppose first that case (C_1) holds. Then by the definition of z, we have

$$x(t) = z(t) + p(t)x(\tau(t)) \geq z(t) + p(t)z(\tau(t)) = z(t) - p(-t)z(\tau(t)) \geq z(t) - p(-t)z(t) = (1 + p(t))z(t). \tag{19}$$

This with (12) leads to

$$\left(r(t)\psi(x(t))\left(z'(t)\right)^{\alpha}\right)' + Q(t)z^{\alpha}(\sigma(t)) \leq 0, t \geq t_0. \tag{20}$$

Now define ω as in Theorem 2. Then $\omega(t) > 0$, and in view of [19], we arrive

$$\omega'(t) \leq -\rho(t)Q(t) + \frac{1}{K(\alpha+1)^{\alpha+1}} \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\rho^{\alpha}(t)\left(\sigma'(t)\right)^{\alpha}}.$$

This by integrating from t_0 to t, leads to

$$0 < \omega(t) \leq \omega(t_0) - \int_{t_0}^{t} \left[\rho(s)Q(s) - \frac{1}{K(\alpha+1)^{\alpha+1}} \frac{\left(\rho'_+(s)\right)^{\alpha+1} r(\sigma(s))}{\rho^{\alpha}(s)\left(\sigma'(s)\right)^{\alpha}}\right] ds.$$

Letting $t \to \infty$ in the above inequality, we get a contradiction with (18).

Now if z satisfies (C_2), then by Lemma 2.2 [16], we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Now we consider the continuous Kamenev-Philos function $H: D \to [0, \infty)$ which satisfies

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for $(t,s) \in D_0$;

(ii) H has a nonpositive continuous partial derivative $\frac{\partial H}{\partial s}$ with respect to the second variable satisfying

$$-\frac{\partial}{\partial s}[H(t,s)v_2(s)] + 2H(t,s)v_2(s)\phi(s)\left(1 - \frac{\alpha K \sigma'(s)\xi^{\alpha-1}(\sigma(s))r(s)}{r(\sigma(s))}\right) = h(t,s)\sqrt{H(t,s)v_2(s)}.$$

for a locally integrable function $h \in L_{loc}(D, \mathbb{R})$, and a function $v_2 \in C^1(I, (0, \infty))$.

The following result improves a criterion of Erbe et al. [9].

Theorem 5 Assume that the conditions (A_1)-(A_4) hold, $p(t)$ is an odd function such that $0 \leq p(t) < 1$, and there exists $H: D \to [0, \infty)$ such that (i), (ii) be satsfied. Further assume that for all sufficiently large $T, T \geq t_0$

$$\lim_{t\to\infty} sup \frac{1}{H(t,T)} \int_T^t \left[H(t,s)v_2(s)\psi_2(s) - \frac{r((s))\Phi(s)h^2(t,s)}{4\alpha K \sigma'(s)\xi^{\alpha-1}(\sigma(s))}\right] = \infty, \tag{21}$$

where

$$\Phi(t) = exp\left(-2\int^t \phi(s)ds\right), \tag{22}$$

and

$$\psi_2(t) = \Phi(t)\left\{Q(t) - \left(r(t)\phi(t)\right)' + \frac{\alpha K \sigma'(t)\xi^{\alpha-1}(\sigma(t))\left(r(t)\phi(t)\right)^2}{r(\sigma(t))}\right\}. \tag{23}$$

Then every solution $x(t)$ of Eq. (1) is either oscillatory for all $\alpha > 1$or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Without loss of generality, suppose that $x(t) > 0$ for all $t \geq t_0$.

Then we have (12). From Lemma 1, z satisfies one of the two cases (C_1) and (C_2). By considering each of the two cases separately, and going through as in [9], we arrive (20).

Now define

$$u_2(t) = \Phi(t)r(t)\left(\frac{\psi(x(t))\left(z'(t)\right)^{\alpha}}{z^{\alpha}(\sigma(t))} + \phi(t)\right). \tag{24}$$

This with (20) leads to

$$u_2'(t) \leq -2\phi(t)u_2(t) + \Phi(t)\left\{-Q(t) + \left(r(t)\phi(t)\right)' - \frac{\alpha r(t)\sigma'(t)\left(z'(t)\right)^{\alpha} z'(\sigma(t))\psi(x(t))}{z^{\alpha+1}(\sigma(t))}\right\}. \tag{25}$$

On the other hand, we obtain

$$\frac{z'(\sigma(t))}{z(\sigma(t))} = \frac{1}{r(\sigma(t))\psi(x(\sigma(t)))} \times \frac{r(\sigma(t))\psi(x(\sigma(t)))(z'(\sigma(t)))^{\alpha}}{z^{\alpha}(\sigma(t))}\left(\frac{z(\sigma(t))}{z'(\sigma(t))}\right)^{\alpha-1} \geq \frac{r(t)\psi(x(t))(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))}\frac{\xi^{\alpha-1}(\sigma(t))}{r(\sigma(t))}K, for\ t \geq t_1, \tag{26}$$

This with (15) and (25), leads to

$$u_2'(t) \leq -2\phi(t)u_2(t) + \Phi(t)\left\{-Q(t) + (r(t)\phi(t))' - \frac{\alpha K\sigma'(t)\xi^{\alpha-1}(\sigma(t))r^2(t)}{r(\sigma(t))}\left(\frac{\psi(x(t))(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))}\right)^2\right\}. \tag{27}$$

Multiplying both sides by $H(t,s)v_2(s)$ and integrating from T to t, we have, for all $t \geq T \geq t_1$,

$$\begin{aligned}&\int_T^t H(t,s)v_2(s)\psi_2(s)ds \\ &\leq H(t,T)v_2(T)u_2(T) \\ &- \int_T^t \left[\sqrt{\frac{\alpha K H(t,s)v_2(s)\sigma'(s)\xi^{\alpha-1}(\sigma(s))}{r(\sigma(s))\Phi(s)}}u_2(s) \right.\\ &\left.+ \frac{1}{2}\sqrt{\frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}}h(t,s)\right]^2 ds \\ &+ \frac{1}{4}\int_T^t \frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}h^2(t,s)ds.\end{aligned}$$

Hence

$$\int_T^t \left[H(t,s)v_2(s)\psi_2(s) - \frac{1}{4}\frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}h^2(t,s)\right]ds \leq H(t,T)v_2(T)u_2(T) - \int_T^t \left[\sqrt{\frac{\alpha K H(t,s)v_2(s)\sigma'(s)\xi^{\alpha-1}(\sigma(s))}{r(\sigma(s))\Phi(s)}}u_2(s) + \frac{1}{2}\sqrt{\frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}}h(t,s)\right]^2 ds. \tag{28}$$

Consequently, we have, for $t > T$

$$\int_T^t \left[H(t,s)v_2(s)\psi_2(s) - \frac{1}{4}\frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}h^2(t,s)\right]ds \leq H(t,T)v_2(T)|u_2(T)|. \tag{29}$$

Hence

$$\begin{aligned}&\lim_{t\to\infty} sup \frac{1}{H(t,T)}\int_T^t \left[H(t,s)v_2(s)\psi_2(s) \right.\\ &\left.- \frac{1}{4}\frac{r(\sigma(s))\Phi(s)}{\alpha K\sigma'(s)\xi^{\alpha-1}(\sigma(s))}h^2(t,s)\right]ds \\ &\leq v_2(T)|u_2(T)|.\end{aligned}$$

Which contradicts (21). Therefore Equation (1) is oscillatory. Now if z satisfies (C_2), then by [16], $\lim_{t\to\infty} x(t) = 0$, and so the proof is completed.

Now we are going to consider the case $0 \leq p(t) \leq p_0 < \infty$. We start by the following result which depends and extends Theorem 4 of [14]

Theorem 6 Let $0 < \alpha \leq 1$, and $0 \leq p(t) \leq p_0 < \infty, \tau \in C^1(I,\mathbb{R})$, $\tau'(t) \geq \tau_0$, and $\tau \circ \sigma = \sigma \circ \tau$. Assume further that the conditions (A_1)-(A_4) hold. If there exists a function $\rho \in C^1(I,(0,\infty))$ such that

$$\int_{t_{**}}^{\infty}\left[\rho(t)Q(t)R^{\alpha}(t_*,t) - \frac{(\rho'_+(t))^{\alpha+1}}{K(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}\left(r(t) + \frac{p_0^{\alpha}r(\tau(t))}{\tau_0^{\alpha+1}}\right)\right]dt = \infty, \tag{30}$$

for all sufficiently large t_1 and for some $t_2 \geq t_1 \geq t_0$, where $Q(t)$, and $R(t_1,t)$ are as defined in (10), then (1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive non-oscillatory solution of (1); Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \geq t_1$, and we get (12). Using the condition $\int_{t_0}^{\infty}\frac{ds}{r^{1/\alpha}(s)} = \infty$, we conclude that there exists a $t_2 \geq t_1$ such that $z'(t) > 0$, for all $t \geq t_2$, the inequality (12) leads to

$$\left(r(t)\psi(x(t))(z'(t))^{\alpha}\right)' \leq -q(t)x^{\alpha}(\sigma(t)) \leq 0. \tag{31}$$

Thus by the assumption $\tau'(t) \geq \tau_0 > 0$, there exists a $t_3 \geq t_2$ such that, for all $t \geq t_3$

$$\frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\psi\left(x(\tau(t))\right)(z'(t))^{\alpha}\right)' \leq -p_0^{\alpha}q(\tau(t))x^{\alpha}\left(\sigma(\tau(t))\right). \tag{32}$$

Combining (31) and (32), using the condition $\tau \circ \sigma = \sigma \circ \tau$ and [4, Lemma 2], we conclude that

$$\begin{aligned}&\left(r(t)\psi(x(t))(z'(t))^{\alpha}\right)' \\ &+ \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\psi\left(x(\tau(t))\right)\left(z'(\tau(t))\right)^{\alpha}\right)' \\ &\leq -\left[q(t)x^{\alpha}(\sigma(t)) + p_0^{\alpha}q(\tau(t))x^{\alpha}\left(\tau(\sigma(t))\right)\right] \\ &\leq -min\{q(t),q(\tau(t))\}\left[x^{\alpha}(\sigma(t)) \right.\\ &\left.- p_0^{\alpha}x^{\alpha}\left(\tau(\sigma(t))\right) + 2p_0^{\alpha}x^{\alpha}\left(\tau(\sigma(t))\right)\right] \\ &< -min\{q(t),q(\tau(t))\}\left[x^{\alpha}(\sigma(t)) \right.\\ &\left.- p_0^{\alpha}x^{\alpha}\left(\tau(\sigma(t))\right)\right] < -Q(t)z^{\alpha}(\sigma(t)).\end{aligned}$$

Although z here is different from that used by Li et al. [14], however the remaining part of the proof can go similar to their proof.

The following result depends and extends Theorem 2 of [13] and so the proof is omitted.

Theorem 7 Assume that the conditions (A_1)-(A_4) hold, and $\int_{t_0}^{\infty} \frac{ds}{r^{1/\alpha}(s)} < \infty$. Suppose that $p(t)$ is an odd function, with $0 \le p(t) < \infty$. Suppose further that there exists a function $\rho_1(t) \in C^1(I, \mathbb{R})$ such that, for some ≥ 1 , there exist continuous functions $H, h: D \to \mathbb{R}$ such that (i), (ii) be satisfied with

$$\frac{\partial}{\partial s} H(t,s) = -h(t,s)\big(H(t,s)\big)^{\frac{\alpha}{\alpha+1}}, \tag{33}$$

we have

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Bigg[H(t,s)\psi_1(s) - \frac{\beta^\alpha}{(\alpha+1)^{\alpha+1}} \frac{v_1(s) r(\sigma(s))}{K(\sigma'(s))^\alpha} h^{\alpha+1}(t,s)\Bigg] ds = \infty, \tag{34}$$

where

$$v_1(t) = exp\left[-(\alpha+1)\int^t \sigma'(s)\left(\frac{Kr(s)\rho_1(s)}{r(\sigma(s))}\right)^{1/\alpha} ds\right], \tag{35}$$

and

$$\psi_1(t) = v_1(t)\left[Q(t) + \sigma'(t)\left(\frac{K^{1/\alpha+1}\rho_1(t) r(t)}{\big(r(\sigma(t))\big)^{1/\alpha+1}}\right)^{\frac{\alpha+1}{\alpha}} - \big(r(t)\rho_1(t)\big)'\right]. \tag{36}$$

Then equation (1) is oscillatory for all $\alpha \ge 1$.

3 Examples

Example 8: Consider the D.E.

$$\left(\frac{t^\alpha}{1+x^2(t)}|z'|^{\alpha-1} z'(t)\right)' + t x^\alpha(t) = 0, t \ge 1, \tag{37}$$

where $\alpha > 0$ is the ratio of two odd integers. Here $p(t) = \frac{1}{2}, r(t) = t^\alpha, \tau(t) = t-1, \psi(x) = \frac{1}{1+x^2}, f\Big(t, x\big(\sigma(t)\big)\Big) = t x^\alpha(t)$. It is clear that $K = 1, q(t) = t, \sigma(t) = t$, and $\xi\big(\sigma(t)\big) = tlnt$. Choosing $\rho(t) = 1$, therefore

$$\int^\infty \left[\rho(t)q(t) - \frac{\rho'_+(t) r\big(\sigma(t)\big)}{K\xi^\alpha\big(\sigma(t)\big)}\right] dt = \int^\infty t dt = \infty.$$

Then by Theorem 2, Eq. (37) is oscillatory.

Example 9: Consider the D. E.

$$\left(\frac{t}{1+x^2(t)} z'(t)\right)' + t x(\lambda t) = 0, t \ge 1. \tag{38}$$

Here $\alpha = 1, 0 < \lambda \le 1, r(t) = t, p(t) = \frac{1}{2}, \tau(t) = t-1, \psi(x) = \frac{1}{1+x^2}$, and $f\Big(t, x\big(\sigma(t)\big)\Big) = t x(\lambda t)$. Thus $K = 1, q(t) = t, \sigma(t) = \lambda t, \sigma'(t) = \lambda$, and $\xi(t) = tlnt, \int_1^\infty \frac{ds}{r^{1/\alpha}(s)} = \infty$. Choosing $\rho(t) = \frac{1}{t}$, therefore

$$\int^\infty \left[\rho(t)q(t) - \frac{\big(\rho'_+(t)\big)^2 r\big(\sigma(t)\big)}{4\alpha K\rho(t)\sigma'(t)\xi^{\alpha-1}\big(\sigma(t)\big)}\right] dt = \int_1^\infty dt = \infty,$$

Then by Theorem 3, Eq. (38) is oscillatory.

Remark 10 We may note that the results of [16] do not work for the two equations (37) and (38). So our criteria (11) and (17) are new.

Example 11: Consider the D. E.

$$\left(\frac{t^2}{1+x^2(t)}|z'(t)| z'(t)\right)' + t^3 x^3(\lambda t) = 0, t \ge 1. \tag{39}$$

Here $\alpha = 2, 0 < \lambda \le 1, r(t) = t^2, p(t) = \frac{1}{t}, \tau(t) = t-1, \psi(x) = \frac{1}{1+x^2}$, and $f\Big(t, x\big(\sigma(t)\big)\Big) = t^3 x^3(\lambda t)$. So $K = 1, q(t) = t^3, \sigma(t) = \lambda t, \sigma'(t) = \lambda, Q(t) = t^3\left(1 + \frac{2}{\lambda t} + \frac{1}{(\lambda t)^2}\right)$, and $\int_1^\infty \frac{ds}{r^{1/\alpha}(s)} = \infty$. Let $\rho(t) = \frac{1}{t^3}$. Therefore

$$\int^\infty \left[\rho(s)Q(s) - \frac{1}{K(\alpha+1)^{\alpha+1}} \frac{\big(\rho'_+(s)\big)^{\alpha+1} r\big(\sigma(s)\big)}{\rho^\alpha(s)\big(\sigma'(s)\big)^\alpha}\right] ds = \int^\infty \left[1 + \frac{2}{\lambda s} + \frac{1}{(\lambda s)^2}\right] ds = \left[s + \frac{2}{\lambda}\ln s - \frac{1}{\lambda^2 s}\right]_1^\infty.$$

Then by Theorem 4, Eq. (39) is oscillatory.

Remark 12 One may note that the criterion of [19] do not apply for Eq. (39). So our criterion (18) improves that given in [19].

Example 13: Consider the D. E.

$$\left(\frac{\sqrt{t}}{1+x^2(t)} z'(t)\right)' + \frac{\vartheta}{4t} x(t) = 0, t \ge 1, \tag{40}$$

where $\vartheta > 0, \tau > 0$. Hence $\alpha = 1, r(t) = \sqrt{t}, 0 \le p(t) = \frac{3}{2} \le p_0 = \frac{3}{2}, q(t) = \frac{\vartheta}{4t}, K = 1, \tau(t) = t - \tau, \sigma(t) = t, \tau'(t) = 1 = \tau_0$, and $\tau \circ \sigma = \sigma \circ \tau$. Choosing $\rho(t) = 1$. Note that $Q(t) = min\{q(t), q\big(\tau(t)\big)\} = \frac{\vartheta}{4t}, R(l,t) = \left(\int_l^{\sigma(t)} r^{-1/\alpha}(s) ds\right)\left(\int_l^t r^{-1/\alpha}(s) ds\right)^{-1} = 1$, then

$$\int_{t_{**}}^\infty \left[\rho(t)Q(t)R^\alpha(t_*, t) - \frac{\big(\rho'_+(t)\big)^{\alpha+1}}{K(\alpha+1)^{\alpha+1}\rho^\alpha(t)}\left(r(t) + \frac{p_0^\alpha r\big(\tau(t)\big)}{\tau_0^{\alpha+1}}\right)\right] dt = \int_{t_{**}}^\infty \left[\frac{\vartheta}{4t}\right] dt = \left[\frac{\vartheta}{4}\ln t\right]_{t_{**}}^\infty = \infty.$$

Then by Theorem 6, Eq. (40) is oscillatory.

Remark 14 We claim that the results of [14] may do not work for Eq. (40).

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